

SOME FIXED POINT THEOREMS FOR EXPANSION MAPPINGS

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ABSTRACT

In the present paper, we consider three self maps f, g, h on a complete metric space (X, d) such that f, g are continuous and h is orbitally continuous and obtain fixed point theorem by using a generalized contraction.

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1. Introduction. Rawat and Sharma [2] considered 3 self continuous map f, g and h on a complete metric space (X, d) satisfying $fh = hf, gh = hg$ and obtained a unique common fixed point. Jain and Yadav [4] presented a common fixed point theorem for family of mapping by employing complete mappings in 2-metric space.

Let us define three surjective mappings f, g and h on a complete metric space (X, d) so that f, g are continuous and h is orbitally continuous satisfying

$$(1.1) \quad d(fhx, ghy) \geq \frac{\alpha [d(hx, fhx)d(hy, ghy) + d(hx, hy)^2 + d(hx, ghy)d(hy, fhx)]}{2d(hx, fhx) + d(hx, hy) + d(hy, ghy)}$$

Let $h(x_0) \in X$ there exists a point $hx_1 \in f^{-1}hx_0, hx_2 \in g^{-1}hx_1$, continuing this procedure, we have a sequence $\{hx_n\}$ with

$$(1.2) \quad hx_{2n+1} \in f^{-1}hx_{2n}, \quad hx_{2n+2} \in g^{-1}hx_{2n+1}.$$

2. Main Results.

Theorems 2.1 If there exist three self maps f, g and h on a complete metric space (X, d) satisfying (1.1) and (1.2) such that $fh = hf, gh = hg, f, g$, are continuous and h is orbitally continuous, then f, g have a unique common fixed point.

Proof. Since we have

$$(2.1) \quad d(hx_{2n}, hx_{2n+1}) = d(fhx_{2n+1}, ghx_{2n-2}),$$

which in view of (1.1) yields.

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$$(2.2) \quad d(hx_{2n}, hx_{2n+1}) \geq \frac{\alpha \left[d(hx_{2n}, hx_{2n+1})d(hx_{2n-1}, hx_{2n+2}) + d(hx_{2n+1}, hx_{2n+2})^2 \right] + d(hx_{2n+1}, ghx_{2n+2})d(hx_{2n+2}, fhx_{2n+1})}{2d(hx_{2n+1}, fhx_{2n+1}) + d(hx_{2n+1}, hx_{2n+2}) + d(hx_{2n+2}, ghx_{2n+2})}$$

using property of metric space, we get

$$d(hx_{2n}, hx_{2n+1}) \geq \frac{\alpha}{2} d(hx_{2n+1}, hx_{2n+2}),$$

or

$$d(hx_{2n+1}, hx_{2n+2}) \geq \frac{2}{\alpha} d(hx_{2n}, hx_{2n+1})$$

which reduce to

$$(2.3) \quad d(hx_{2n+1}, hx_{2n+2}) \leq \lambda d(hx_{2n}, hx_{2n+1}) \quad , \quad \text{where } \lambda = 2/\alpha .$$

Since

$$(2.4) \quad d(hx_{2n+1}, hx_{2n+2}) = d(ghx_{2n+1}, fhx_{2n+3})$$

therefore, from (2.1) and (2.4), we obtain

$$(2.5) \quad d(hx_{2n+1}, hx_{2n+2}) \geq \frac{\alpha \left[d(hx_{2n+3}, fhx_{2n+3})d(hx_{2n+2}, ghx_{2n+2}) + d(hx_{2n+3}, hx_{2n+2})^2 \right] + d(hx_{2n+3}, ghx_{2n+2})d(hx_{2n+2}, fhx_{2n+2})}{2d(hx_{2n+3}, fhx_{2n+3}) + d(hx_{2n+3}, hx_{2n+2}) + d(hx_{2n+2}, ghx_{2n+2})}$$

which with an appeal to properties on metric space, gives

$$(2.6) \quad d(hx_{2n+1}, hx_{2n+2}) \geq \frac{\alpha}{3} d(hx_{2n+2}, hx_{2n+3}).$$

Therefore, we can write

$$(2.7) \quad d(hx_{2n+2}, hx_{2n+3}) \leq \frac{3}{\alpha} d(hx_{2n+1}, hx_{2n+2}).$$

If λ is minimum of $(2/\alpha, 3/\alpha, \dots)$, we get

$$d(hx_{2n+2}, hx_{2n+3}) \leq \lambda d(hx_{2n+1}, hx_{2n+2}) \leq \lambda^2 d(hx_{2n}, hx_{2n+1}).$$

Repeating the procedure, finally we obtain

$$(2.8) \quad \begin{aligned} d(hx_{2n+1}, hx_{2n+2}) &\leq \lambda d(hx_{2n}, hx_{2n+1}) \\ &\leq \lambda^2 d(hx_{2n-1}, hx_{2n}) \\ &\dots \quad \dots \quad \dots \\ &\leq \lambda^{2n+1} d(hx_0, hx_1) \end{aligned}$$

Thus we have

$$(2.9) \quad d(hx_n, hx_{n+k}) \leq \frac{\lambda^n}{1-\lambda} d(hx_0, hx_1).$$

Since $0 \leq \lambda < 1$ and $n \rightarrow \infty$, we get

$$(2.10) \quad d(hx_n, hx_{n+k}) \rightarrow 0.$$

Thus we have $\{hx_n\}$ a Cauchy sequence by completeness of X . There exists $u \in X$ such that $\{hx_n\} \rightarrow 0$. We can say that $\{fx_{2n}\}$ and $\{hx_{2n+1}\}$ also converge to u .

Using continuity of h , we say that $f(hx_{2n}) = h(fx_{2n}) \rightarrow hu$.

$$(2.11) \quad g(hx_{2n+1}) = h(gx_{2n+1}) \rightarrow hu,$$

$$h(hx_{2n}) \rightarrow hu.$$

$$\text{Thus } f(hx_{2n}) = fu,$$

from which, we further obtain

$$(2.12) \quad fu = gu = hu.$$

Since h is orbitally continuous mapping of a complete metric space (X, d) , we have

$$u = \lim_{n \rightarrow \infty} h^n u = \lim_{n \rightarrow \infty} h(h^n u) = hu.$$

Hence u is a common fixed point of f, g and h .

That is

$$(2.13) \quad fu = gu = hu = u.$$

Further we assume that u, v are two common fixed points of f and g then from (2.13) and (1.1), we can write

$$d(u, v) = d(fhu, ghv)$$

and

$$(2.14) \quad d(u, v) \geq \frac{\alpha [d(hu, fhu)d(hv, ghv) + d(hu, hv)^2 + d(hu, ghv)d(hv, fhu)]}{2d(hu, fhu) + d(hu, hv) + d(hu, ghv)}.$$

In view of (2.13) and (2.14), we can again write

$$d(u, v) \geq \alpha \left[\frac{2d(u, v)^2}{d(u, v)} \right],$$

$$(2.15) \quad d(u, v)(1 - \alpha) \geq 0 \Rightarrow d(u, v) \geq 0$$

i.e. $u = v$

Thus f, g have a unique fixed point.

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