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ON RELATIVE INFORMATION GENERATING FUNCTION WITH UTILITIES

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ABSTRACT

In the present communication relative information generating function utilities has been defined and its particular and limiting cases have been studied. Interestingly, the derivatives of this information generating function at $t=0$ give various well -known measures of information . The relative useful informarion generating functions for uniform, geometric and exponential probability distributions have been defined. Some important properties of the proposed information generating function have also been studied.

Key words : Information generating function , Measure of 'useful' information Geometric and exponention probability distributions, Jensen' s inequality and Conjugate relation.

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1. Introduction. The concept of information genrating function was introduced by Golomb [3] who defined the information generating function

$$T(t)=\sum_{i\in N} p_i^t, \qquad t\geq 1,$$

where $\{p_i\}$ is a complete probability distribution with $i\in N$, N a discrete infinite sample space and t is a real or complex variable. It may be noted that

$$-\frac{\delta T(t)}{\delta t}\Big|_{t=1}=H(P), \qquad \text{... (1.2)}$$

where $H(P)$ is Shannon's entropy.

The quantity (1.2) measures the avearage information but does not take into account the effectiveness or importance of the events. In order to distinguish the events $E_1, E_2, \ldots, E_n$ with respect to a given qualitative characteristic of the physical system taken into account. Belis and Guiasu [1] introduced a utility distribution $U=(u_1,u_2,...,u_n)$, where $u_i>0$ accounts for the utility of i^{th} events and is indepenedent of its probability p_i, Analogous to (1.1) Hooda and Bhaker [6] defined the following useful information generating function :

$$M(P;U,t)=\sum_{i=1}^{n} u_i p_i^t, \qquad ...(1.3)$$

where P and U are respectively probability and utility distributions and t is a real or complex variable. Since $0 \le p_i \le 1$ for each i and $<u_i>$ is bounded for an experiment under consideration, therefore, the expression (1.3) is convergent for all $t \ge 1$. Since (1.3) being positive term convergent series is absolutely convergent series, therefore, it is uniformly convergent and hence each term of the series possesses continuous derivative. Consequently, we get the derivative of (1.3) at $t=1$ as given below :

$$-\frac{\delta M}{\delta t}(P;U,t)\Big|_{t=1} = -\sum_{i=1}^{n} u_i p_i \log p_i = H(P;U), \qquad ...(1.4)$$

which is Belis and Guiasu [1] measure of useful information. This measure has wide applications in economics, questionnaire theory, accountancy etc. In case utilities are ignored, (1.3) reduces to (1.1).

$$Let\ P=\left\{(p_1,p_2,...,p_n), 0 \le p_i \le 1, \sum_{i=1}^{n} p_i = 1\right\}$$

be a discrete probability distribution of a set of events $\xi=(E_1,E_2,...,E_n)$ of a discrete infinite sample space N on the basis of an experiment having predicted probability distribution :

$$Q=\left\{(q_1,q_2,...,q_n), 0 < q_i \le 1, \sum_{i=1}^{n} q_i = 1\right\}.$$

The Kullback's measure of relative information is defined as :

$$H(PQ)=\sum_{i=1}^{n} p_i \log(p_i/q_i). \qquad ...(1.5)$$

It may be noted that (1.5) vanishes when $p_i=q_i$ for each i and with assumption $0\ log = 0$.

To take effectiveness or usefulness of the events into account let us attach a utility scheme $U=(u_1,u_2,...,u_n)$, where $u_i>0$, the utility of i^{th} event E_i is independent of its probability of occurrence p_i and predicted probability q_i. u_i is only a utility or importance of the outcome E_i for an observation related to some specified goal refer Longo [9]. Thus we have two utility information schemes :

$$S = \begin{bmatrix} E_1 & E_2 & \dots & E_n \\ p_1 & p_2 & \dots & p_n \\ u_1 & u_2 & \dots & u_n \end{bmatrix} \qquad \dots(1.6)$$

of a set of n events after an experiment and

$$S^* = \begin{bmatrix} E_1 & E_2 & \dots & E_n \\ q_1 & q_2 & \dots & q_n \\ u_1 & u_2 & \dots & u_n \end{bmatrix} \qquad \dots(1.7)$$

of the same set of n events before the experiment.

A measure of useful relative information that the scheme (1.6) provides about the scheme (1.7) was introduced and characterized by the authors [2] and is given by

$$H(P \mid Q;U) = \sum_{i=1}^{n} u_i p_i \log(p_i / q_i) / \sum_{i=1}^{n} u_i p_i \,. \qquad \dots(1.8)$$

In case utilities are ignored or $u_i=1$ for each i, then (1.8) reduces to (1.5). An additive measure of useful relative information as given below was proposed by Kaur [7]:

$$H_\alpha^\beta(P \mid Q;U) = (\alpha-1)^{-1} \log\left(\sum u_i^\beta p_i^{\alpha+\beta-1} q_i^{1-\alpha} / \sum (u_i p_i)^\beta \right), \alpha > 0, \neq 1, \beta \geq 1, \qquad \dots(1.9)$$

In the present paper, in Section 2, we define a generalized useful relative information genrating function whose derivative at $t=0$ yields (1.9). Its particular cases are also studied. In Section 3 to 6, various results for the generalized useful relative information generating function have been established.

2. Generalized Useful Relative Information Generating Function .

Let $P=\{(p_1,p_2,\dots,p_n);p_i>0, \Sigma\, p_i=1\}$ be a discrete probability distributon whose predicted probability distribution is $Q=\{(q_1,q_2,\dots,q_n);q_i>0, \Sigma\, q_i=1\}$ and $U=\{(u_1,u_2,\dots,u_n);u_i>0\}$ be utility distiribution of a discrete infinite sample space N.

Let us consider

$$M(P \mid Q;U) = 2^{\sum_{i=1}^{n} u_i p_i \, log(p_i / q_i) / \sum_{i=1}^{n} u_i p_i} \qquad \dots(2.1)$$

then we define

$$M(P \mid Q;U,t) = [M(P \mid Q;U)]^t \,. \qquad \dots(2.2)$$

Differentiating (2.2) with respect to t at $t=0$, we have

$$\frac{\partial M}{\partial t}(P \mid Q;U,t)\big|_{t=0} = \sum_{i=1}^{n} u_i p_i \log(p_i / q_i) / \sum_{i=1}^{n} u_i p_i$$

$$= H(P|Q;U). \quad \text{... (2.3)}$$

Following Hardy, Littlewood and Polya [4], we have the following weighted mean of order α–1 of p_i, q_i and u_i :

$$M_{\alpha,\beta_i}(P|Q;U) = \left[\sum_{i=1}^{n} u_i^{\beta} p_i^{\alpha+\beta i} q_i^{1-\alpha} / \sum_{i=1}^{n} (u_i p_i)^{\beta_i}\right]^{1/\alpha-1}, \alpha > 0, \alpha \neq 1, \beta_i \geq 1 \quad \text{...(2.4)}$$

for which we have the generalized useful relative information generating function as given below:

$$M_{\alpha,\beta_i}(P|Q;U,t) = \left[M_{\alpha,\beta_i}(P|Q;U)\right]^t \quad \text{...(2.5)}$$

where t is a real or complex variable. Differentiating (2.5) with respect to t at t=0, we have

$$\frac{\partial M_{\alpha,\beta_i}}{\partial t}(P|Q;U,t)|_{t=0} = (\alpha-1)^{-1} \log\left[\sum_{i=1}^{n} u_i^{\beta_i} p_i^{\alpha+\beta_i-1} q_i^{1-\alpha} / \sum_{i=1}^{n} (u_i p_i)^{\beta_i}\right]$$

$$= H_{\alpha}^{\beta_i}(P|Q;U). \quad \text{...(2.6)}$$

We call the new measure (2.6) as the generalized useful relative information measure of order α and type $\{\beta_i\}$. When β_i = β for each i, (2.5) reduces to

$$M_{\alpha,\beta}(P|Q;U,t) = \left[\sum_{i=1}^{n} u_i^{\beta} p_i^{\alpha+\beta-1} q_i^{1-\alpha} / \sum_{i=1}^{n} (u_i p_i)^{\beta}\right]^{t/\alpha-1}, \quad \text{...(2.7)}$$

and

$$H_{\alpha}^{\beta}(P|Q;U,t) = (\alpha-1)^{-1} \log\left[\sum_{i=1}^{n} u_i^{\beta} p_i^{\alpha+\beta-1} q_i^{1-\alpha} / \sum_{i=1}^{n} (u_i p_i)^{\beta}\right] \quad \text{...(2.8)}$$

Particular Cases :

i) It utilities are ignored or u_i=1 for each i in (2.7), we have

$$M_{\alpha,\beta}(P|Q,t) = \left[\sum_{i=1}^{n} p_i^{\alpha+\beta-1} q_i^{1-\alpha} / \sum_{i=1}^{n} p_i^{\beta}\right]^{1/\alpha-1} \quad \text{...(2.9)}$$

Further, on differentiating (2.9), with respect to t at t=0, we get

$$\frac{\partial M_{\alpha,\beta}}{\partial t}(P|Q;t)|_{t=0} = (\alpha-1)^{-1} \log\left(\sum_{i=1}^{n} p_i^{\alpha+\beta-1} q_i^{1-\alpha} / \sum_{i=1}^{n} p_i^{\beta}\right)$$

$$= H_{\alpha}^{\beta}(P|Q) \quad \text{...(2.10)}$$

which is the generalized measure of relative information characterized and studied by Sharma [10].

ii) If we set β=1 in (2.9), we get

$$M_{\alpha}(P|Q;t) = \left(\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha}\right)^{t/(\alpha-1)}, \quad \text{...(2.11)}$$

which is the generalized relative information generating function of order α.

Further if we let $\alpha \to 1$ in (2.11), it reduces to the following information generating function for complete discrete distributions P and Q.

$$M(P|Q;t) = 2^{t\sum_{i=1}^{n} p_i \log(p_i/q_i)} \qquad \text{...(2.12)}$$

and

$$\frac{\partial M}{\partial t}(P|Q;t) = \sum_{i=1}^{n} p_i \log p_i/q_i \qquad \text{...(2.13)}$$

which is Kullback's [8] measure of relative information.

iii) In case $\beta = 1$, (2.7) reduces to

$$M_\alpha(P|Q;U,t) = \left[\sum_{i=1}^{n} u_i p_i^\alpha q_i^{1-\alpha} / \sum_{i=1}^{n} u_i p_i\right]^{t/(\alpha-1)}, \qquad \text{...(2.14)}$$

which is the useful relative information generating function of order α and gives

$$\frac{\partial M_\alpha}{\partial t}(P|Q;U,t)|_{t=0} = (\alpha-1)^{-1}\log\left(\sum_{i=1}^{n} u_i p_i^\alpha q_i^{1-\alpha} / \sum_{i=1}^{n} u_i p_i\right)$$

$$= H_\alpha(P|Q;U), \qquad \text{...(2.15)}$$

which is the generalized measure of useful relative information of order α. When $\alpha \to 1$, (2.15) gives

$$H(P|Q;U) = \sum_{i=1}^{n} u_i p_i \log(p_i/q_i) / \sum_{i=1}^{n} u_i p_i \qquad \text{...(2.16)}$$

which is (1.8).

As particular examples, we give below the useful information generating functions and the corresponding useful relative information measures derived from these in case of uniform, geometric and exponential distributions.

Examples.

(i) For the uniform, probability distribution $(1/N, 1/N, \ldots, 1/N)$ after experiment, predicted probability distribution $(1/M, 1/M, \ldots, 1/M)$ before experiment and the utility distribution $(u, u, \ldots, u)$ of an experiment after putting in (2.7) we have

$$M_{\alpha,\beta}(P|Q;U,t) = (M/N)^t, \qquad \text{... (2.17)}$$

and

$$\frac{\partial M_{\alpha,\beta}}{\partial t}(P|Q;U,t)_{t=0} = \log(M/N), \qquad \text{...(2.18)}$$

which is the same result as if utilities are ignored. Thus in case of uniform probability distribution, utilities play no role in the generalizd measure of useful relative infomation. Moreover. (2.18) is also independent of α and β.

(ii) Consider the geometric probability distributions $(p_1, p_1 p, p_1 p^2, \ldots)$, $p + p_1 = 1$,

$(q_1,q_1q,q_1q^2,\ldots)$, $q_1+q=1$ and geometric utility distribution $(u_1,u_1u,u_1u^2,\ldots)$. It is the general case when utility distribution is also geometric. Then (2.7) gives.

$$M_{\alpha,\beta}(P\,|\,Q;U,t)=\left[\frac{p_1^{\alpha-1}q_1^{1-\alpha}\left\{-(pu)^{\beta}\right\}}{1-u^{\beta}p^{\alpha+\beta-1}q^{1-\alpha}}\right]^{t/\alpha-1}, \text{ provided } pu<1, \quad \ldots(2.19)$$

$$\frac{\partial M_{\alpha,\beta}}{\partial t}(P\,|\,Q;U,t)|_{t=0}=(\alpha-1)^{-1}\log\left[p_1^{\alpha-1}\left\{1-(pu)^{\beta}\right\}/1-u^{\beta}p^{\alpha+\beta-1}q^{1-\alpha}\right],$$

$\alpha>0,\ \alpha\neq1$ and $\beta\geq1$. ...(2.20)

In case utilities are ignored in (2.19) and (2.20), we get respectively

$$M_{\alpha,\beta}(P\,|\,Q;U,t)=\left[p_1^{\alpha-1}q_1^{1-\alpha}\left(1-p^{\beta}\right)/1-p^{\alpha+\beta-1}q^{1-\alpha}\right]^{t/(\alpha-1)}, \quad \ldots(2.21)$$

$$\frac{\partial M_{\alpha,\beta}}{\partial t}(P\,|\,Q;U,t)|_{t=0}=(\alpha-1)^{-1}\log\left[p_1^{\alpha-1}q_1^{1-\alpha}\left(1-p^{\beta}\right)/1-p^{\alpha+\beta-1}q^{1-\alpha}\right],$$

$\alpha>0,\ \alpha\neq1$ and $\beta\geq1$. ...(2.22)

(iii) For the actual and predicted exponential probability distributions with mean $1/\lambda$ and $1/\mu$ respectively and utility distribution with mean $1/\lambda$, we have

$$p(x)=\lambda e^{-\lambda x},\lambda>0,\quad 0\leq x<\infty,$$

$$q(y)=\mu e^{-\mu x},\mu>0,\quad 0\leq y<\infty,$$

$$u(z)=\gamma e^{-\gamma z},\gamma>0,\quad 0<z<\infty,$$

On substitutin these in (2.7), we get

$$M_{\alpha,\beta}(p(x)/q(y);u(z),t)=\left[\lambda^{\alpha-1}\mu^{1-\alpha}\beta/\mu(1-\alpha)(\alpha+\beta-1)\right]^{1/(\alpha-1)}, \quad \ldots(2.23)$$

$$\frac{\partial M_{\alpha,\beta}}{\partial t}(p(x)/q(y);u(z);t)|_{t=0}=(\alpha-1)^{-1}\log\left[\lambda^{\alpha-1}\mu^{1-\alpha}\beta/\mu(1-\alpha)(\alpha+\beta-1)\right], \quad \ldots(2.24)$$

It may be noted that (2.23) comes out to be independent of utility distribution whatever form it may have. Setting $\beta=1$ in (2.23) and (2.24), we get

$$M_{\alpha}(p(x)/q(y);u(z),t)=\left[\lambda^{\alpha-1}\mu^{1-\alpha}/\mu(1-\alpha)\alpha\right]^{1/(\alpha-1)} \quad \ldots(2.25)$$

and

$$\frac{\partial M_{\alpha}}{\partial t}(p(x)/q(y);u(z);t)|_{t=0}=(\alpha-1)^{-1}\log\left[\lambda^{\alpha-1}\mu^{1-\alpha}/\mu(1-\alpha)\alpha\right], \quad \ldots(2.26)$$

which is the generalized measure of useful relative information of order α given by (2.15). Further, when $\alpha\to1$ in (2.25) and (2.26), we get the results obtained by Hooda and Bhaker [6].

3. Convergence. It is evident when the sample space is finite, the useful relative information generating function $M_{\alpha,\beta}(P|Q;U,t)$ is covergent for all values of α,β for which the function is defined and for finite values of t. When the sample space is infinite, for finite β and t and for $\alpha>\alpha'$ and $\alpha>\alpha''$ where α' depends on probability α" depends upon the utility distribution, then the generating function can be shown to be convergent.

4. The r^{th} Derivative. We have observed in Section 2, that the first derivative at $t=0$ of the generalized useful relative information generating function gives the useful relative information measures of order and type β denoted by $H_\alpha^\beta(P|Q;U)$. It can be easily proved that

$$\frac{\partial^r M_\alpha}{\partial t^r}(P|Q;U,t)|_{t=0} = \left[H_\alpha^\beta(P|Q;U)\right]^r. \qquad ...(4.1)$$

In particular (4.1) holds for the generalized useful information measure of order α and type β introduced by Hooda and Singh [6]. It is also true in case of Gurdial and Pessow's [2] useful information measure of order α and Belis and Guiasu [1] measure of useful information.

5. Joint Distributions. Let X and Y be two independent random variables having probability distributions P and Q after experiment and P' and Q' as the predicted probability distributions respectively and their corresponding utility distributions be U and V respectively.

Let Z be the random variable composed of cartesian product of X and Y, i.e. $Z=X\times Y$ having probability distribution $R=\{p_i q_j\}$, $i=1,2,...,n$ and $j=1,2,...,m$ after experiment and $R'=\{p'_i q'_j\}$, $i=1,2,...,n$ and $j=1,2,...,m$ before experiment and utility distribution $W=\{u_i v_j\}$, $i=1,2,...,n$ and $j=1,2,...,m$ of experiment.

If $M_{\alpha,\beta}(P|P';U,t)$ and $M_{\alpha,\beta}(Q|Q';V,t)$ are the generalized relative information generating functions of random variables X and Y respectively, then the generalized information generating function of Z is given by

$$M_{\alpha,\beta}(R|R';W,t) = M_{\alpha,\beta}(P|P';U,t)M_{\alpha,\beta}(Q|Q';V,t), \qquad ...(5.1)$$

and

$$\frac{\partial}{\partial t}M_{\alpha,\beta}(R|R';W,t)_{t=0} = H_\alpha^\beta(P|P';U) + H_\alpha^\beta(Q|Q';V). \qquad ...(5.2)$$

6. Conjugate Relation. Two real numbers α' and α" are said to be conjugate if

$$1/\alpha' + 1/\alpha'' = 1. \qquad ...(6.1)$$

Since we restrict our definition for useful relative information of positive orders only, so it is obvious that α'>1 and α">1.

Theorem 1. For two conjugate real numbers α' and α"

$$M_{1,\beta}(P|Q;U,t) < M'_{\alpha,\beta}(P|Q;U,a_1 t)M''_{\alpha,\beta}(P|Q;U,a_2 t), \qquad ...(6.2)$$

where $a_1 = (\alpha'-1)/\alpha'$ and $a_2 = (\alpha''-1)/\alpha''$ and $M_{1,\beta}(P|Q;U,t)$ is the limiting form of (2.7) for $\alpha\to 1$.

Proof. From Jensen's inequlity, we have

$$\frac{\sum (u_i p_i)^\beta \log(p_i / q_i)}{\sum (u_i p_i)^\beta} < \log\left(\sum u_i^\beta p_i^{\beta+1} q_i^{-1} / \sum (u_i p_i)^\beta\right), \quad ...(6.3)$$

Applying Holder's inequality in Right hand side of (6.3), we have

$$\log\left[\sum \left(u_i^\beta p_i^{\beta+1} q_i^{-1} / \sum (u_i p_i)\right)^\beta\right]$$

$$\log\left[\sum u_i^\beta p_i^{\alpha'+\beta+1} q_i^{-1} / \sum (u_i p_i)^\beta\right]^{1/\alpha'} \times \left[\sum u_i^\beta p_i^{\alpha''+\beta-1} q_i^{-1} / \sum (u_1 p_i)^\beta\right]^{1/\alpha''}.$$

Therefore $t\ \log\left[\sum u_i^\beta p_i^{\alpha'+\beta-1} q_i^{-1} / \sum (u_i p_i)^\beta\right]$

$$\leq \log\left[\sum u_i^\beta p_i^{\alpha'-1} q_i^{-1} / \sum (u_i p_i)^\beta\right]_1^{a_1\ t/\alpha'-1'} \times \left[\sum u_i^\beta p_i^{\alpha''+\beta-1} q_i^{-1} / \sum (u_i p_i)^\beta\right]_2^{a_2\ t/\alpha''-1} . ...(6.4)$$

From (6.3) and (6.4), we get

$$t\sum (u_i p_i)^\beta \log(p_i / q_i) / \sum (u_i p_i)^\beta < \log\left[\sum u_i^\beta p_i^{\alpha''+\beta-1} q_i^{-1} / \sum (u_i p_i)^\beta\right]_1^{a\ t/\alpha'-1}$$

$$\times \left[\sum u_i^\beta p_i^{\alpha''+\beta-1} q_i^{-1} / \sum (u_i p_i)^\beta\right]_2^{a_2\ t/\alpha''-1} \quad ...(6.5)$$

On simplifying (6.5), we get the required result (6.2) of the Theorem1.

In case utilities are ignored or $u_i = 1$ for each i, (6.2) reduces to

$$M_{1,\beta}(P \mid Q; t) < M_{\alpha',\beta}(P \mid Q; a_1 t) M_{\alpha'',\beta}(P \mid Q; a_2 t). \quad (6.6)$$

If we restrict t to be positive, then from (6.5) we have

$$H^\beta(P \mid Q; U) < a_1 H_\alpha^\beta(P \mid Q; U) + a_2 H_{\alpha''}^\beta(P \mid Q; U). \quad (6.7)$$

In case $\beta = 1$, (6.7) reduces to

$$H(P \mid Q; U) < a_1 H_{\alpha'}(P \mid Q; U) + a_2 H_{\alpha''}(P \mid Q; U). \quad ...(6.8)$$

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