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COMPLETE LIFT OF $F_\lambda(2v+3,1)$ STRUCTURE IN TANGENT BUNDLE

By

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ABSTRACT

Andreou [1] studied the structure defined by the (1,1) tensor field F Satisfying $F^5+F+r = 0$. Ishihara and Yano [2], defined integrability conditions of F-structure. Srivastava and Niwas [7], considered the integrability condition of $F_\lambda(2v+3,1)$ structure. The horizontal and complete lift from a differentiable manifold M^n of class C^∞ to its cotangent bundle ${}^CT(M^n)$ have been studied by Yano and Patterson [5]. Yano and Ishihara [9]. studied lift of F-structure in the tangent and cotangent bundle. Kim [3] dealt with the properties of F-structure. Srivastava [8], considered lift of $(1,1)$ tensor field F satisfying

$$F^{v+1} = \lambda^2 F^{v-1} \text{ and } F^v = (-1)^{v+3} F,$$

Niwas, Ali and Srivastava [4], studied some properties of lifts (horizontal and complete) for the structure satisfying

$$a^n F^n + a_{n-1} F^{n-1} + \ldots + a_1 F = 0.$$

Here we give some results for the (1,1) tensor field F satisfying $F^{2v+3}=-\lambda^2 F$, taking horizontal, complete and diagonal lift.

1. Preliminaries. Let M^n be n-dimensional differentiable manifold of class C^∞. If $F \neq 0$ be a (1,1) tensor field of class C^∞, rank r satisfying [4]

(1.1) $F^{2v+3}=-\lambda^2 F$

Here tensors ℓ and m are defined as follows :

(1.2) $\ell = -F^{2v+2}/\lambda^2, m = 1 + F^{2v+2}/\lambda^2$.

From (1.2), we have

(1.3) $\ell + m = 1, \ell^2 = \ell, m^2 = m, \ell m = m\ell = 0, F\ell = F, Fm = 0$.

Thus there exits in M^n two complementary distributions D_ℓ *and* D_m corresponding to the projection tensors ℓ and m respectively. If rank of f is r then D_ℓ is r dimensional and D_m is $(n-r)$ dimensional. $D_m M^n = n$.

2. The Complete Lift of F_λ Structure in the Tangent Bundle $T(M^n)$. Let (M^n) be an n-dimensional differentiable manifold of class C^∞ and $T_p(M^n)$ is the tangent space at a point p of M^n then

$$T(M^n) = \bigcup_{p \in M^n} T_p(M^n)$$

is the tangent bundle cover the manifold M^n. The tangent bundle $T(M^n)$ of M^n is a differentiable manifold of dimension $2n$.

Let T_s^r denote the set of tensor field of class C^∞ and type (r,s) in M^n and let $T_s^r[T(M^n)]$ denotes the corresponding set of tensor fields in $T(M^n)$. The complete lift F^c of an element F of $T_1^1(M^n)$ with local components F_1^h has components of the form Yano and Ishihara [9]

(2.1) $$F^c = \begin{pmatrix} F_i^h & 0 \\ \partial F_i^h & F_i^h \end{pmatrix}.$$

Let $F, G \in T_1^1(M^n)$ then we have [9]

(2.2) $(FG)^c = F^c G^c$

which yields on putting $F=G$

(2.3) $(F)^c = (F^c)^2$.

Similarly on putting $G=F^2$ in (4.3.2) we get

(2.4) $(F^3)^c = (F^c)^3$.

Continuing the above process of replacing G in equation (2.2) by some higher power of F, we get

(2.5) $(F^K)^c = (F^c)^K$, where K is any positive integer.

If G and H are tensors of the same type then

(2.6) $(G+H)^c = G^c + H^c$.

Taking complete lifts on both sides of (1.1), we get

(2.7) $(F^{2v+3} + \lambda^2 F)^c = 0$

$\Rightarrow (F^{2v+3})^c + (\lambda^2 F)^c = 0$ or $(F^c)^{2v+3} + \lambda^2 F^c = 0$.

The rank of F^c in $2r$ iff rank of F is r. Thus we have

Theorem 1. Let $F \in T_1^1(M^n)$, the F satisfies this structure equation (1.1), if F^c satisfies (1.4), further more F is of rank r iff F^c is of rank r. Let F be an $F_\lambda(2v+3,1)$ – structure of rank r in M^n. Then complete lift ℓ^c of ℓ and m^c of m are complementary projection tensors in $T(M^n)$. Thus there exist in $T(M^n)$ two complementary distributions D_ℓ^c and D_m^c are respectively complete lifts D_ℓ^c and D_m^c of D_ℓ and D_m.

3. Integrability Conditions of $F_\lambda(2v+3,1)$ Structure in Tangent Bundle. Let $F \in T_1^1(M^n)$ and F satisfies (1.1) then the Nijenhuis tensor N_F of F is a tensor field of type (1,2) given by Yano and Ishihara [9]

(3.1) $$N_F(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y].$$

Let N^c be the Nijenhuis tensor of F^c in $T(M^n)$, where F^c is the complete lift of F in M^n, then we have

(3.2) $$N^c(X^c,Y^c)=[F^cX^c,F^cY^c]-F^c[F^cX^c,Y^c]-F^c[X^c,F^cY^c]+(F^2)^c[X^c,Y^c].$$

$\forall\, X,Y \in I_0^1(M^n)$ and $F \in I_1^1(M^n)$, we have

(3.3) (a) $[X^c,Y^c]=[X,Y]^c$, (b) $[X,Y]^c = X^c+Y^c$, (c) $F^cX^c=(FX)^c$.

From (1.3) and (3.3) we have

(3.4) $F^c\ell^c=(F\ell)^c=F^c$ and $F^cm^c=(Fm)^c=0$,

$$(\ell^c)^2=(\ell^2)^c=\ell^c, \qquad (m^c)^2=(m^2)^c=m^c.$$

Theorem 2. In the $F^{2v+3}=-\lambda^2F$ structure, we have

(3.5) (a) $$\left(N^c(m^cX^c,m^cY^c)\right)^{(2v+3)/2} \neq \lambda^2F^c[m^cX^c,m^cY^c],$$

(b) $$m^cN^c(X^c,Y^c)=m^c[F^cX^c,F^cY^c],$$

(c) $$m^cN^c(\ell^cX^c,\ell^cY^c)=m^c[F^cX^c,F^cY^c],$$

(d) $$m^cN^c\left[(F^{2v+3}+\lambda^2F)^cX^c,(F^{2v+3}+\lambda^2F)^cY^c\right]=\lambda^2m^2N^c(\ell^cX^c,\ell^cY^c).$$

Proof. Since
$$\begin{aligned}N^c(m^cX^c,m^cY^c) &= [F^cm^cX^c,F^cm^cY^c]-F^c[F^cm^cX^c,m^cY^c]\\ &\quad -F^c[F^cm^cX^c,m^cY^c]-F^c[m^cX^c,F^cm^cY^c]+(F^2)^c[m^cX^c,m^cY^c]\\ &=(F^c)^2[m^cX^c,m^cY^c].\end{aligned}$$

Thus we get $N^c(m^cX^c,m^cY^c)=(F^c)^2[m^cX^c,m^cY^c]$.

Since $(F^c)^{2v+3}=-\lambda^2F^c$, therefore above equation reduces to

$$\left[N^c(m^cX^c,m^cY^c)\right]^{(2v+3)/2}=-\lambda^2F^c[m^cX^c,m^cY^c]$$

Thus we get (3.5)(a).

Similarly other results can easily be derived.

Theorem 3. For any X, $Y \in T_1^1(M^n)$ the following conditions are equivalent

(3.6) (a) $m^cN^c(X^c,Y^c)=0$, (b) $m^cN^c(\ell^cX^c,\ell^cY^c)=0$,

(c) $m^cN^c\left[(F^{2v+3}+\lambda^2F)^cX^c,(F^{2v+3}+\lambda^2F)^cY^c\right]=0$.

If we put $m^cN^c\left[(F^{2v+3}+\lambda^2F)^cX^c,(F^{2v+3}+\lambda^2F)^cY^c\right]=0$

in (3.5)(d), then we get $m^cN^c(\ell^cX^c,\ell^cY^c)=0$ i.e. we get (3.6)(b) and also from

(3.5)(c) and (3.6)(b) we see that if (3.6)(c) is satisfied then (3.6)(a)and (3.6)(b) are also satisfied.

Theorem 4. The complete lift D_m^c in $T(M^n)$ of a distribution D_m in M^n is integrable if D_m is integrable in M^n.

Proof. The distribution D_m is integrable iff Srivastava [7]

(3.7) $\ell[mX,mY]=0,\ \forall X,Y \in T_0^1(M^n)$ and $\ell = 1-m$.

Taking complete lift of both sides, we get

(3.8) $\ell^c[m^cX^c,m^cY^c]=0,\ \forall X,Y \in T_0^1(M^n)$,

where $\ell^c = (1-m)^c = 1-m^c$ is the projection tensor complementary to m^c. Thus the condition (3.7) implises (3.8).

Theorem 5. The complete lift D_m^c in $T(M^n)$ of a distribution D_m in M^n is integrable if $\ell^c N^c(m^cX^c,m^cY^c)=0$, or equivalently $N^c(m^cX^c,m^cY^c)=0 \quad \forall X,Y \in T_0^1(M^n)$.

Proof. The distribution D_m is integrable in M^n iff $N(mX,mY)=0 \quad \forall X,Y \in T_0^1(M^n)$.

Since $N^c(m^cX^c,m^cY^c)=(F^c)^2[m^cX^c,m^cY^c]$,

Therefore, by multiplying throughout by ℓ^c, we get

$\ell^c N^c(m^cX^c,m^cY^c)=(F^c)^2\ell^c[m^cX^c,m^cY^c]$,

but since $\ell^c[m^cX^c,m^cY^c]=0$, Therefore

(3.9) $\ell^c m^c(m^cX^c,m^cY^c)=0$ and also

(3.10) $m^c N^c(m^cX^c,m^cY^c)=0$.

Adding (3.9) and (3.10) we get

$$(\ell^c+m^c)N^c(m^cX^c,m^cY^c)=0 .$$

Since $\ell^c+m^c=I^c=1$, therefore we prove $N^c(m^cX^c,m^cY^c)=0$.

Theorem 6. Let the distribution D_ℓ be integrable in M^n, that is $mN(X,Y)=0$ $\forall X,Y \in T_0^1(M^n)$. Then the distribution D_ℓ^c is integrable in $T(M^n)$ iff any one of the conditions of theorem (2.2) is satisfied $\forall\ X,Y \in T_0^1(M^n)$.

Proof. Since the distribution D_ℓ is integrable in M^n iff $mN(\ell X, \ell Y)=0$.

Thus distribution D_ℓ^c is integrable is $T(M^n)$ iff

$m^c N^c(\ell^c X^c,\ell^c Y^c)=0$. Thus theorem follows.

Theorem 7. The complete lift F^c of an $F^{2v+3}+\lambda^2 F=0$ structure in M^n is partially integrable in $T(M^n)$ iff F is partially integrable in M^n.

Proof. $F^{2v+3}+\lambda^2F=0$ structure in M^n is partially integrable iff

$N(\ell X,\ell Y)=0,\ \forall X,Y\in T_0^1(M^n)$.

Keeping in view of (1.1) and (3.1) we obtain

(3.11) $N^c[\ell^c X^c,\ell^c Y^c]=[N(\ell X,\ell Y)]^c$

which implies

$$N^c(\ell^c X^c,\ell^c Y^c)=0 \text{ iff } N(\ell X,\ell Y)=0.$$

Also from theorem (3.5) $N^c(\ell^c X^c,\ell^c Y^c)=0\Rightarrow N(\ell X,\ell Y)=0$.

Hence the theorem follows.

4. The Horizontal Lift of Structure. The horizontal lift S^H of a tensor field S of arbitrary type in M^n to $T(M^n)$ is defined by Yano and Ishihara [9].

(4.1) $S^H=S^C-\nabla_\gamma S,$

where S is a tensor field defined by

$S=S^{i\ldots h}_{k\ldots j}\dfrac{\partial}{\partial x^i}\otimes\ldots\otimes\dfrac{\partial}{\partial x^h}\partial x^k\otimes\ldots\otimes\partial x^j$ in M^n with affine connection ∇ and $\nabla_\gamma S$ is a tensor field in $T(M^n)$ given by

(4.2) $\nabla_\gamma S=\left(\gamma^\ell\nabla_\ell S^{i\ldots h}_{k\ldots j}\right)\dfrac{\partial}{\partial y^i}\otimes\ldots\otimes\dfrac{\partial}{\partial y^h}\partial x^k\otimes\ldots\otimes\partial x^j$

with respect to induced coordinates (xh,yh) in $\pi^{-1}(U)$.

The horizontal lift F^H of a tensor field F of type (1,1) in M^n with local components F_i^h in M^n has components

(4.3) $F^H:\begin{pmatrix}F_i^h & 0\\ 0 & F_i^h\end{pmatrix}.$

Theorem 8. Let $F\in I_1^1(M^n)$ be an F_λ-structure in M^n then its horizontal lift F^H is also a F_λ-structure in $T(M^n)$.

Proof. If $P(t)$ is a polynomial in one variable t, then we have

(4.4) $(P(F))^H=(P(F^H))$

Also I be the identity tensor field of type (1,1) in M^n then

(4.5) $I^H=I.$

Thus

(4.6) $\left(F^{2v+3}\right)^H+\lambda^2(F)^H=0\Rightarrow\left(F^H\right)^{2v+3}+\lambda^2F^H=0.$

Now from the local components of F^H, we see that if F is of rank r, then F^H is of rank $2r$. Thus we have

Theorem 9. If F is structure of rank r in M^n then its horizontal lift F^H is also F_λ-structure of rank $2r$ in $T(M^n)$

Proof. Let m be a projection tensor field of type (1,1) in M^n defined by (1.2) and $m^2=m$ then $(m^H)^2=m^H$.

Thus m^H is also a projection tensor in $T(M^n)$. Hence there exists in $T(M^n)$ a distribution D^H corresponding to m^H, which is called the horizontal lift of the distribution D.

5. Diogonal Lift in Cotangent Bundle. If the diagonal lift of the tensor field F of type (1,1) in M^n is F^D in ${}^cT(M^n)$ then Yano and Ishihara [9]

(5.1) $$F^D : \widetilde{F}^{\alpha}_{\beta} = \begin{pmatrix} F^h_i & 0 \\ 0 & -F^i_h \end{pmatrix},$$

(5.2) $$F^D G^D + G^D F^D = (FG + GF)^H,$$

(5.3) $$F^D G^H + G^D F^H = F^H G^D + G^H F^D = (FG + GF)^D,$$

(5.4) $$F^H F^D = F^D F^H = (F^2)^D,$$

(5.5) $$(F^D)^{2p} = (F^{2p})^H, (F^D)^{2p+1} = (F^{2p+1})^D \quad \forall F \in I^1_1(M^n)$$

Theorem 10. The diagonal lift of an almost $F_\lambda(2v+3,\ 1)$ structure in M^n with symmetric affine connection is also an almost F_λ in ${}^cT(M^n)$.

Proof. Thus from (1.1) keeping in view of (5.4) and (5.5), we obtain

$$(F^{2v+3})^D = \lambda^2 (F)^D \Rightarrow (F^D)^{2v+3} = \lambda^2 F^D.$$

Theorem 11. If F is F_λ-structure of rank r in M^n then its diagonal lift F^D is also F_λ-structure of rank $2r$ in ${}^cT(M^n)$.

Proof. Since m be a projection tensor field of type (1,1) in M^n defined by (1.2) and $m^2=m$ then

$(m^2)^D=(m^D)^2=m^D$.

Thus m^D is also a projection tensor in ${}^cT(M^n)$. Hence there exists in ${}^cT(M^n)$ a distrubtion D corresponding to m^D which is called diagonal lift of the distribution D^D.

REFERENCES

[1] F.G. Andreou, On a structure defined by a tensor field of type (1,1) satisfying $F^5+F=0$, *Tensor N.S.*, **36** (1982), 76-89.

[2] S. Ishihara and K. Yano, On integrability conditions of a structure F satisfying $F^3+F=0$, *Quat. Jour. Math. Oxford*, 2nd Ser. **15** (1964), 217-222.

[3] J.B. Kim, Notes on F-manifold, *Tensor N.S.*, **29** (1975), 299-302.

[4] Ram Niwas, S. Ali and S.K. Srivastava, On horizontal and complete lift of (1,1) tensor field F satisfying $a_nF^n+a_{n-1}F^{n-1}+...+a_1 F=0$. *Demonstration Mathematics*. **35** (2002), 155-163.

[5] K. Yano and E.M. Patterson, Vertical and complete lifts from a manifold to its cotangent bundle, *Jour. Math. Soc. Japan*, **19** (1967), 91-113.

[6] K. Yano and E.M. Patterson, Horizontal lift from a manifold to its cotangent Bundle. *Jour. Math. Soc. Japan*, **19** (1967), 185-198.

[7] S.K. Srivastava and R. Niwas, On $F_\lambda(2v+3,1)$ structure manifolds and its integrability condition, *The Nepali Math. Soc. Report* **18** (1999-2000) 51-61.

[8] S.K. Srivastava, On horizontal and complete lifts of (1,1) tensor field F satisfying $F^{v+1}-\lambda^2 F^{v-1}=0$ and $F^v+(-1)^{v+1}F=0$, *Ph.D. Thesis, Lucknow University*, (1999).

[9] K. Yano and S. Ishihara, *Tangent and Cotangent Bundle*, Maicel Dekker Inc. New York (1973).

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COMMON FIXED POINT OF COMPATIBLE MAPPINGS OF TYPE (α)

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ABSTRACT

In this paper we prove a common fixed point theorem for six mappings in a complete metric space. Our theorem generalizes the entire results of Banach [1], Kannan [10], Fisher [6], Chatterjee [2] and Cho [3].

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1. Introduction. Sessa [14] defined a generalization of commutativity, which is called weak commutativity. Further, Jungck [7] introduced more generalized commutativity, so called compatibility, which is more general than that of weak commutativity del. The utility of compatibility in the context of fixed point theory was demonstrated by extending a theorem of Park and Bae [13]. Also, Jungck [8] extended the results of Khan and Imdad [11].

Kang et. al. [12] extended the results of Ding [5], Diviccaro and Sessa [4] and Jungck [8] by using one of the mappings continuous and employing compatible mappings.

Recently, Jungck-Murthy-Cho [9] introduced the concept of compatible mappings of type (α) in metric spaces.

In this paper, we prove a common fixed point theorem for six mappings under the condition of compatible mappings of type (α), in a complete metric space.

2. Compatible mappings of type (α). In this section, we give some definitions and the concept of compatible mappings of type (α) in metric spaces and some properties of these mappings for our main result.

Definition 1. Let (X,d) be a metric space and $\{x_n\}$ be a sequence in X.

(1) $\{x_n\}$ is said to be convergent to a point x in X, if

$$\lim_{n\to\infty} d(x_n,x) = 0$$

(2) $\{x_n\}$ in X is said to be Cauchy sequence, if $\lim_{m,n\to\infty} d(x_m, x_n) = 0$

(3) A metric spame (X,d) is said to be complete if every Cauchy sequence in X is convergent.

Definition 2 [7]. Let A and B be mappings from a metric space (X,d) into itself. Then A and B are said to be compatible, if

$$\lim_{n\to\infty} d(ABx_n, BAx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$Lim_{n\to\infty} Ax_n = Lim_{n\to\infty} Bx_n = z \text{ for some } z \in X.$$

Definition 3 [9]. Let A and B be mappings from a metric space (X,d) into itself. Then A and B are said to be compatible of type (α) if

$$\lim_{n\to\infty} d(ABx_n, BBx_n) = 0 \text{ and } \lim_{n\to\infty} d(BAx_n, AAx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Ax_n = z \text{ for some } z \in X.$$

Proposition 1 [9]. Let (X,d) be a metric space. Let A and B be continuous mappings from X into itself. Then A and B are compatible if and only if they are compatible of type (α).

Proposition 2 [9]. Let (X,d) be a metric space and A and B are mappings from X into itself. If A and B are compatible of type (α)
and $Az = Bz$ for some $z \in X$, then

$$ABz = BBz = BAz = AAz.$$

Proposition 3 [9]. Let (X,d) be a metric space and A and B are mappings from X into itself. If A and B compatible of type (α) and $\{x_n\}$ is sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \text{ for some } z \text{ in } X, \text{ then}$$

(i) $\lim_{n\to\infty} BAx_n = Az$, if A is continuous at z,

(ii) $ABz = BAz$ and $Az = Bz$, if A and B are continuous at z.

3. A Common Fixed Point Theorem. In this section, we prove a common fixed point theorem for six mappings satisfying some conditions.

Theorem. Let (X,d) be a complete metric space. Let A,B,S,T,P and Q be mappings from X into itself such that

(3.1) $P(X) \subset AB(X), Q(X) \subset ST(X)$,

(3.2) $AB=BA$, $ST = TS$, $PA = AP$, $PB=BP$, $QS=SQ$, $QT=TQ$,

(3.3) A,B,S and T are continuous,

(3.4) the pairs $\{P, AB\}$ and $\{Q, ST\}$ are compatible of type (α),

$$(3.5)\quad d(Px,Qy) \le \alpha_1\left[\frac{d(STy,Qy)d(ABx,ABy)+d(ABy,Qy)d(ABx,Px)}{d(ABx,Qy)+d(ABx,STy)}\right]$$
$$+\alpha_2[d(STy,Px)+d(STy,Qy)+d(ABx,STy)]$$
$$+\alpha_3[d(ABx,Qy)+d(ABy,Px)+d(STy,ABy)]$$
$$+\alpha_4[d(ABx,Px)+d(ABy,Qy)]$$

for all $x, y \in X$, where each $\alpha_i \ge 0$ and $2\alpha_1+2\alpha_2+2\alpha_3+\alpha_4<1$.

Then A, B, S, T, P and Q have a common fixed point in X.

Proof . By (3.1), Since $P(X) \subset AB(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $Px_0 = ABx_1$. Since $Q(X)\ ST \subset (X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Qx_1 = STx_2$. Inductively, we can define a sequence $\{y_n\}$ in X as follows.

$y_{2n} = Px_{2n} = ABx_{2n+1}$

$y_{2n+1} = Qx_{2n+1} = STx_{2n+2}$ for $n = 0,1,2,\ldots$

By Putting $x = x_{2n+1}$ and $y = x_{2n+2}$ in (3.5) we write

$d(Px_{2n+1}, Qx_{2n+2})$

$$\le \alpha_1\left[\frac{d(STx_{2n+2},Qx_{2n+1})d(ABx_{2n+1},ABx_{2n+2})+d(ABx_{2n+2},Qx_{2n+2})d(ABx_{2n+1},Px_{2n+1})}{d(ABx_{2n+1},Qx_{2n+2})+d(ABx_{2n+1},STx_{2n+2})}\right]$$
$$+\alpha_2[d(STx_{2n+2},Px_{2n+1})+d(STx_{2n+2},Qx_{2n+2})+d(ABx_{2n+1},STx_{2n+2})]$$
$$+\alpha_3[d(ABx_{2n+1},Qx_{2n+2})+d(ABx_{2n+2},Px_{2n+1})+d(STx_{2n+2},ABx_{2n+2})]$$
$$+\alpha_4[d(ABx_{2n+1},Px_{2n+1})+d(ABx_{2n+2},Qx_{2n+2})]$$

$$\le \alpha_1\left[\frac{d(y_{2n+1},y_{2n+2})d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})d(y_{2n},y_{2n+1})}{d(y_{2n},y_{2n+2})+d(y_{2n},y_{2n+1})}\right]$$
$$+\alpha_2[d(y_{2n+1},y_{2n+1})+d(y_{2n+1},y_{2n+2})+d(y_{2n},y_{2n+1})]$$
$$+\alpha_3[d(y_{2n},y_{2n+2})+d(y_{2n+1},y_{2n+1})+d(y_{2n+1},y_{2n+1})]+\alpha_4[d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})].$$

$$d(y_{2n+1},y_{2n+2}) \le \frac{(2\alpha_1+\alpha_2+\alpha_3+\alpha_4)}{(1-\alpha_2-\alpha_3-\alpha_4)}d(y_{2n},y_{2n+1}).$$

Putting $h = \dfrac{(2\alpha_1+\alpha_2+\alpha_3+\alpha_4)}{(1-\alpha_2-\alpha_3-\alpha_4)}$, we find $h<1$, since $2\alpha_1+2\alpha_2+2\alpha_3+\alpha_4<1$.

Hence

$$d(y_{2n+1},y_{2n+2}) \le hd(y_{2n},y_{2n+1}).$$

Similarly we can see that

$$d(y_{2n}, y_{2n+1}) \leq hd(y_{2n-1}, y_{2n}).$$

Proceeding in this way, we have

$$d(y_{2n}, y_{2n+1}) \leq h^{2n-1} d(y_0, y_1).$$

By routine calculations the following inequalities hold for $k > n$

$$d(y_n, y_{n+k}) \leq \sum_{i=1}^{k} (dy_{n+i-1}, y_{n+1})$$

$$\leq \sum_{i=1}^{k} h^{n+i-1} d(y_0, y_1)$$

$$\leq \frac{h^n}{1-h} d(y_0, y_1) \to 0 \text{ as } n \to \infty.$$

Here $h < 1$. Hence $\{y_n\}$ is a Cauchy sequence and by completeness of X we see that $\{y_n\}$ is convergent to a point z in X.

Since Px_{2n}, Qx_{2n+1}, ABx_{2n+1} and STx_{2n+2} are subsequences of $\{y_n\}$, they also converge to a point z, that is as $n \to \infty$,

$$Px_{2n}, Qx_{2n+1}, STx_{2n+1} \to z.$$

Since A and B are continuous and the pair $\{P, AB\}$ is compatible of type α, by proposition 3 we have, as $n \to \infty$,

$$P(AB)x_{2n+1} \to ABz, \ (AB)^2 x_{2n+1} \to ABz.$$

Now putting $x = ABX_{2n+1}$ and $y = x_{2n+2}$ in (3.5), we write

$d\ (P(AB)x_{2n+1}, Qx_{2n+2})$

$$\leq \alpha_1 \left[\frac{d(STx_{2n+2}, Qx_{2n+2}) d((AB)^2 x_{2n+2}, ABx_{2n+2}) + d(ABx_{2n+2}, Qx_{2n+2}) d((AB)^2 x_{2n+1}, P(AB)x_{2n+1})}{d((AB)^2 x_{2n+1}, Qx_{2n+2}) + d((AB)^2 x_{2n+1}, STx_{2n+2})} \right]$$

$$+ \alpha_2 \left[d(STx_{2n+2}, P(AB)x_{2n+1}) + d(STx_{2n+2}, Qx_{2n+2}) + d((AB)^2 x_{2n+1}, STx_{2n+2}) \right]$$

$$+ \alpha_3 \left[d((AB)^2 x_{2n+1}, Qx_{2n+2}) + d(ABx_{2n+2}, P(AB)x_{2n+1}) + d(STx_{2n+2}, ABx_{2n+2}) \right]$$

$$+ \alpha_4 \left[d((AB)^2 x_{2n+1}, P(AB)x_{2n+1}) + d(ABx_{2n+2}, Qx_{2n+2}) \right]$$

which implies that, as $n \to \infty$

$$d(ABz, z) \leq (2\alpha_2 + 2\alpha_3) d(ABz, z),$$

which is a contradiction since $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$.

Therefore, we have $ABz = z$.

By putting $x = Px_{2n}$ and $y = x_{2n+1}$ in (3.5), we write

$d(P(Px_{2n}), Qx_{2n+1})$

$$\leq \alpha_1\left[\frac{d(STx_{2n+1},Qx_{2n+1})d(AB(Px_{2n}),ABx_{2n+1})+d(ABx_{2n+1},Qx_{2n+1})d(AB(Px_{2n}),P(Px_{2n}))}{d(AB(Px_{2n}),Qx_{2n+1})+d(AB(Px_{2n}),STx_{2n+1})}\right]$$

$$+\alpha_2[d(STx_{2n+1},P(Px_{2n}))+d(STx_{2n+1},Qx_{2n+1})+d(AB(Px_{2n}),STx_{2n+1})]$$

$$+\alpha_3[d(AB(Px_{2n}),Qx_{2n+1})+d(ABx_{2n+1},P(Px_{2n}))+d(STx_{2n+1},ABx_{2n+1})]$$

$$+\alpha_4[d(AB(Px_{2n}),P(Px_{2n})+d(ABx_{2n+1},Qx_{2n+1})].$$

Taking the limit $n\to\infty$, we have

$$d(Pz,z)\leq(2\alpha_2+2\alpha_3+\alpha_4)d(Pz,z),$$

which is a contradiction since $2\alpha_1+2\alpha_2+2\alpha_3+\alpha_4<1$.

Therefore, we have $Pz=z=Abz$.

Now, we show that $Bz=z$. By putting $x=Bz$ and $y=x_{2n+1}$ in (3.5), we write $d(p(Bz),Qx_{2n+1})$

$$\leq \alpha_1\left[\frac{d(STx_{2n+1},QX_{2n+1})d(AB(Bz),ABx_{2n+1})+d(ABx_{2n+1},Qx_{2n+1})d(AB(Bz),P(Bz))}{d(AB(Bz),Qx_{2n+1})+d(AB(Bz),STx_{2n+1})}\right]$$

$$+\alpha_2[d(STx_{2n+1},P(Bz))+d(STx_{2n+1},Qx_{2n+1})+d(AB(Bz),STx_{2n+1})]$$

$$+\alpha_3[d(AB(Bz),Qx_{2n+1})+d(ABx_{2n+1},P(Bz))+d(STx_{2n+1},ABx_{2n+1})]$$

$$+\alpha_4[d(AB(Bz),P(Bz))+d(ABx_{2n+1},Qx_{2n+1})].$$

Taking the limit $n\to\infty$, we have

$$d(Bz,z)\leq(2\alpha_2+2\alpha_3)d(Bz,z),$$

which is a contradiction since $2\alpha_1+2\alpha_2+2\alpha_3+\alpha_4<1$ and which implies that $Bz=z$. Since $ABz=z$, $Az=z$.

Now by putting $x=z$ and $y=STx_{2n+2}$ in (3.5), we write $d(Pz,\ Q(STX_{2n+2}))$

$$\leq \alpha_1\left[\frac{d((ST)^2x_{2n+2},Q(ST)x_{2n+2})d(ABz,AB(ST)x_{2n+2})+d(AB(ST)x_{2n+2},Q(ST)x_{2n+2})d(ABz,Pz)}{d(ABz,Q(ST)x_{2n+2})+d(ABz,(ST)^2x_{2n+2})}\right]$$

$$+\alpha_2[d((ST)^2x_{2n+2},Pz)+d((ST)^2x_{2n+2},Q(ST)x_{2n+2})+d(ABz,(ST)^2x_{2n+2})]$$

$$+\alpha_3[d(ABz,Q(ST)x_{2n+2})+d(AB(ST)x_{2n+2},Pz)+d((ST)^2x_{2n+2},AB(ST)x_{2n+2})]$$

$$+\alpha_4[d(ABz,Pz)+d(AB(ST)x_{2n+2},Q(ST)x_{2n+2})].$$

Taking the limit $n\to\infty$, we have

$$d(z,STz)\leq(2\alpha_2+2\alpha_3+\alpha_4)d(z,STz),$$

which is a contradiction since $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$.

Therefore we have $STz=z$.

Now by putting $x=z$ and $y=Qx_{2n+1}$ in (3.5), we write

$d(Pz,Q(Qx_{2n+1}))$

$$\leq \alpha_1\left[\frac{d(ST(Qxx_{2n+1}),Q(Qx_{2n+1}))d(ABz,AB(Qx_{2n+1}))+d(AB(Qx_{2n+1}),Q(Qx_{2n+1}))d(ABz,Pz)}{d(ABz,Q(Qxx_{2n+1}))+d(ABz,ST(Qx_{2n+1}))}\right]$$

$$+\alpha_2[d(ST(Qx_{2n+1}),Pz)+d(ST(Qx_{2n+1}),Q(Qx_{2n+1}))+d(ABz,ST(Qx_{2n+1}))]$$

$$+\alpha_3[d(ABz,Q(Qx_{2n+1}))+d(AB(Qx_{2n+1}),Pz)+d(ST(Qx_{2n+1}),AB(Qx_{2n+1}))]$$

$$+\alpha_4[d(ABz,Pz)+d(AB(Qx_{2n+1}),Q(Qx_{2n+2}))].$$

Taking the limit $n \to \infty$, we have

$$d(z,Qz) \leq (\alpha_2 + \alpha_3 + \alpha_4)d(z,Qz),$$

which is a contradiction since $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$.

Therefore we have $Qz=z$ and hence $STz=z=Qz$.

Finally we show that $Tz=z$. By putting $x=z$ and $y=Tz$ in (3.5), we write

$$d(Pz,Q(Tz)) \leq \alpha_1\left[\frac{d(ST(Tz),Q(Tz))d(ABz,AB(Tz))+d(AB(Tz),Q(Tz))d(ABz,Pz)}{d(ABz,Q(Tz))+d(ABz,ST(Tz))}\right]$$

$$+\alpha_2[d(ST(Tz),Pz)+d(ST(Tz),Q(Tz))+d(ABz,ST(Tz))]$$

$$+\alpha_3[d(ABz,Q(Tz))+d(AB(Tz),Pz)+d(ST(Tz),AB(Tz))]$$

$$+\alpha_4[d(ABz,Pz)+d(AB(Tz),Q(Tz))],$$

which implies that

$$d(z,Tz) \leq (2\alpha_2 + 2\alpha_3)d(z,Tz).$$

Therefore we have $Tz=z$. Since $STz=z$, we have $z=STz=Sz$.

Therefore by combining the above results, we have

$$Az=Bz=Sz=Tz=Pz=Qz=z,$$

that is z is common fixed point of A, B, S, T, $P$and Q.

For uniqueness let w ($w \neq z$) be another fixed point of A, B, S, T, P and Q, then by (3.5), we write

$$d\ (Pz,\ Qw) \leq \alpha_1\left[\frac{d(STw,Qw)d(ABz,ABw)+d(ABw,Qw)d(ABz,Pz)}{d(ABz,Qw)+d(ABz,STw)}\right]$$

$$+\alpha_2[d(STw,Pz)+d(STw,Qw)+d(ABz,STw)]$$

$$+\alpha_3[d(ABz,Qw)+d(ABz,Pz)+d(STw,ABw)]$$
$$+\alpha_4[d(ABz,Pz)+d(ABw,Qw)],$$

which implies that

$$d(z,w)\leq(2\alpha_2+2\alpha_3)d(z,w),$$

which is a conradiction since $2\alpha_1+2\alpha_2+2\alpha_3+\alpha_4<1$.

Therefore we have $z = w$.

This completes the proof of the theorem.

Remark 1. Taking $AB=ST=I$ (I is the identify map on X) and $P=Q$ and $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0$ in the Theorem, we obtain the result due to Fisher [6].

Remark 2. Taking $AB=I$, $P=Q$ and $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0$ in our Theorem, we obtain the result due to Kannan [10].

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REFERENCES

[1] S. Banach, Surles operations dans les ensembles abstracts etleur applications aux equations integrals, *Fund. Math.*, **3** (1922), 133-181.

[2] S.K. Chatterjee, Fixed point theorem, *Comte. Rend. Acad. Bulgare Sc.*, **25** (1962), 727-730.

[3] Y.J. Cho, Fixed point in Fuzzy metric spaces, *J. Fuzzy. Math.*, **5(4)** (1997), 949-962.

[4] M.L. Diviccaro, Some remarks on common fixed points of four mappings, *Jñānābha* **15** (1985), 139-149.

[5] X.P. Ding, Some common fixed point theorems of commuting mappings II, *Math. Seminar Notes*, **11** (1983), 301-305.

[6] B. Fisher, Some results on fixed points, *Publ. Inst. Math.* **25** (1976), 193-198.

[7] G. Jungck, Compatible mappings and common fixed points, Intern. *J. Math. and Math. Sci.*, **9**(1986), 771-779.

[8] G. Jungck, Compatible mappings and common fixed points (2), *Internat. J. Math. and Math Sci.*, **11** (1988), 285-288.

[9] G. Jungck, P.P Murthy and Y.J. Cho, Compatible mapping of type (α) and common fixed points, *Math. Japonica* **38(2)** (1993), 381-390.

[10] R. Kannan, Some results on fixed points, *Bull. Cal. Math. Soc.*, **60** (1968), 71-76.

[11] M.S. Khan and M. Imdad, Some common fixed point theorems, *Glasnik Math.* **18** (38) (1983), 321-326.

[12] S.M. Kang, Y.J. Cho and G. Jungck, Common fixed point of compatible mappings, *Internat. J. Math. and Math. Sci.*, **13** (1990), 61-66.

[13] S. Park and J.S. Bae, Extensions of a common fixed point theorem of Mier and Keeler, *Ark. Math.*, **19** (1981), 223-228.

[14] S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math.*, **32(46)** (1982), 149-153.

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EFFECTS OF MASS TRANSFER AND ROTATION ON FLOW PAST A PLATE WITH THE DEPENDENT SUCTION IN NON-HOMOGENEOUS POROUS MEDIUM

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ABSTRACT

In the present paper effects of mass transfer and rotation on the unsteady free convection flow of an incompressible, homogeneous, electrically conducting, viscous fluid past an infinite porous vertical plate bounded by time dependent porous medium is studied. It is considered that the plate is subjected to time dependent suction velocity and the flow is under the influence of transversely applied uniform magnetic field. The effects of important parameters on primary and secondary velocity and corresponding skin-friction are discussed with the help of graph and tables.

1. Introduction. In nature and in industries, many transport processes exists, where the transfer of heat and mass takes place as a result of combined bouyancy effects of thermal diffusion and diffusion of chemical species. The phenomena of heat and mass transfer also takes place in chemical processing industry such as food processing and polymer production. In addition heat and mass transfer effects in flow of fluids have application in engineering and technology.

A comprehensive study on the theory of rotating fluids and theoroy of heat transfer has been respectively, presented by Greenspan [5] and Morgan [8]. Several authors including Morgan [9], Kapur and Malik [7], Rao and Gupta [12], Thornely [18] Nanda and Mohenti [11], Gupta and Goyal [6], Debnath [2], Acheson [1], Debnath and Mukherjee [3], Reismann [13], Seth and Maiti [14], Devi et al [4], Pillai [10], Singh and Kumar [15] and Singh et al. [16] have studied problems on flow in rotating system under different physical situations and boundary conditions.

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More recently, Singh et al. [17] have studied unsteady oscillatory flow of an incompressible, electrically conducting, viscous liquid, through a porous medium past an infinite vertical porous plate with constant suction and transcerse uniform magnetic field. In this study the suction velocity and the permeability of the medium is assumed constant. In the proposed study the investigation of Singh et al. [17] is extended to observe the heat and mass transfer effects on the unsteady flow of an incompressible, homogeneous, electrically conducting, viscous fluid through a porous medium with time dependent permeability, time dependent suction velocity and thermal diffusion under the influence of uniform magnetic field.

2. Formulation of the Problem. Consider unsteady oscillatory free convection flow of an electrically conducting, homogeneous, incompressible, viscous liquid through a porous medium past an infinite porous vertical plate in the presence of uniform magnetic field. Under cartesian coordinate system (x,y,z), the plate is considered in z-plane and the liquid and the plate both are in a state of rigid body rotation with uniform angular velocity Ω about z-axis. The x-axis and y-axis are assumed in the plane of the vertical porous plate so that z-axis is normal to these axes at the point of intersection and the components of velocity in these directions are u, v, w respectively. In addition the suction velocity at the porous plate is considered $w=-w_0(1+\in e^{int})$ and the porous medium $k(t)= k_0(1+\in e^{int})$ where w_0, n are real and positive. Initially, when $t \leq 0$ the plate and the fluid are assumed to be at the same temperature T_∞ and the foreign mass is assumed to be at low level and uniformly distributed in the flow region such that it is every where C_∞. When, $t>0$, the temperature of the plate is instantaneously raised to T_w and the species concentration is also raised to C_w and thereafter maintained constant. In addition, the analysis is based on the following assumptions.

(i) The uniform magnetic field $\vec{B}\,[=\mu_e H$ so that $\vec{H} = (0,0,H_0)]$ acts in the z-direction i.e. normal to the flow of fluid.

(ii) The magnetic Reynolds number is very small so that induced magnetic field is negligible in comparison to applied magnetic field.

(iii) No external electric field is applied in the flow region so that the effect of polarization of ionized fluid is negligible.

(iv) The Hall effect and viscous dissipation effect have been ignored while electromagnetic body force (Lorentz force) is considered.

(v) An absorption type heat source $Q=Q_0(T-T_\infty)$ and usual Boussinesq's approximation is taken into account.

Under the above-stated assumptions, the governing equations of momentum, energy and diffusion are :

$$\frac{\partial u}{\partial t}-\left(1+\in e^{int}\right)\frac{\partial u}{\partial z}-2Ev=G_rT+G_mC+\frac{\partial^2 u}{\partial z^2}\left[M^2+\frac{1}{k_0\left(1+\in e^{int}\right)}\right]u, \qquad ...(2.1)$$

$$\frac{\partial v}{\partial t}-\left(1+\in e^{int}\right)\frac{\partial u}{\partial z}-2Eu=\frac{\partial^2 v}{\partial z^2}-\left[M^2+\frac{1}{k_0\left(1+\in e^{int}\right)}\right]v, \qquad ...(2.2)$$

$$P_r\frac{\partial T}{\partial t}-P_r\left(1+\in e^{int}\right)\frac{\partial T}{\partial z}=\frac{\partial^2 T}{\partial z^2}-\alpha_0 T, \qquad ...(2.3)$$

$$S_C\frac{\partial C}{\partial t}-S_C\left(1+\in e^{int}\right)\frac{\partial C}{\partial z}=\frac{\partial^2 C}{\partial z^2}+S_C S_0\frac{\partial^2 T}{\partial z^2}. \qquad ...(2.4)$$

The boundary conditions relevant to the problem are :

$$W=1+\in e^{int},\quad T=1+\in e^{int},\quad C=1+\in e^{int},\quad \text{at}\quad z=0 \qquad ...(2.5)$$
$$W\to 0,\quad T\to 0,\quad C\to 0\quad \text{as}\quad z\to\infty,$$

The non-dimensional quantities introduced to obtain the above equations are :

$u^*=u/U_0$, $v^*=v/U_0$, $z^*=w_0 z/\upsilon$, $t^*=w_0^2 t/\upsilon$, $K^*=w_0^2 K/\upsilon^2$, $k_o^*=w_0^2 k_0/\upsilon^2$,

$W^*=\left(u/U_0+iv/U_0\right)$, $n^*=\upsilon n/w_0^2$, $T^*=\left(T-T_\infty\right)/\left(T_w-T_\infty\right)$,

$P_r=\mu C_p/K$ (Prandtl number) $\quad S_c=\upsilon/D$ (Schmidt number) ,

$G_r=\upsilon g\beta^*\left(T_w-T_\infty\right)/w_0^2 U_0$ (Grashof number) , $S_0=\dfrac{D_T\left(T_w-T_\infty\right)}{\vartheta\left(C_w-C_\infty\right)}$ (Soret number),

$G_m=\dfrac{\upsilon g\beta^*\left(C_w-C_\infty\right)}{w_0^2 U_0}$ (modified Grashof number) $E=\upsilon\Omega/w_0^2$ (Rotation parameter),

$M=\mu_e H_0\sqrt{\sigma\upsilon/\rho}/w_0$ (Hartmann number) and $\alpha_0=\upsilon^2 Q_0/Kw_0^2$ (Heat source parameter)

where υ is the kinematic coefficient of viscosity, β^* is the volumetric coefficient of thermal expansion, β is the volumetric coefficient of thermal expansion with concentration, σ is the electrical conductivity of the liquid, ρ is the density of the liquied, μ_e is the magnetic permeability, H_0 is the uniform magnetic field, k_0 is the constant permeability of the porous medium, K is the thermal conductivity, C_p is the specific heat at constant pressure, T is the temperature, T_w is the plate temperature, C_w is the concentration at the plate, C_∞ is the concentration of species far away from the plate, T_∞ is the temperature far away from the plate, D is the thermal diffusivity and other symbols have their usual meanings.

Combining (2.1) and (2.2) and using $u+iv=W$, we obtain

$$\partial W/\partial t-\left(1+\in e^{int}\right)\partial W/\partial z+2iEW+\left(M^2+\left[k_0\left(1+\in e^{int}\right)\right]^{-1}\right)W=\partial^2 W/\partial z^2+G_r T+G_m C$$

3. Solution of the Probelm. We assume the velocity, temperature and conecentration of the liquid in the neighbourhood of the plate as:

$$W(z,t)=W_1(z)+\in W_2(z)e^{int}, \qquad ...(3.1)$$
$$T(z,t)=T_1(z)+\in T_2(z)e^{int}, \qquad ...(3.2)$$
$$C(z,t)=C_1(z)+\in C_2(z)e^{int}. \qquad ...(3.3)$$

Using (3.1), (3.2) and (3.3) in (2.3), (2.4) and (2.6), we obtain

$$T'''_1(z)+P_rT''_1(z)-\alpha_0T_1(z)=0 \qquad ...(3.4)$$

$$T''_2(z)+P_rT''_2(z)-(inP_r+\alpha_0)T_2(z)=-P_rT''_1(z), \qquad ...(3.5)$$

$$C''_1(z)+S_CC'_1(z)=-S_cS_0T'''_1(z), \qquad ...(3.6)$$

$$C''_2(z)+S_CC'_2(z)-inS_CC_C=-S_cC'_1(z)-S_cS_0T'''_2(z), \qquad ...(3.7)$$

$$W''_1(z)+W'_1(z)-(M_1+2iE)W_1(z)=-G_rT_1(z)-G_mC_1(z), \qquad ...(3.8)$$

and

$$W''_2(z)+W'_2(z)-[(M_1+i(2E+n))]W_2(z)=-G_rT_2(z)-G_mC_2(z)-W'_1(z)-\frac{1}{k_0}W_1(z), \qquad ...(3.9)$$

where $M_1=M^2+1/k_0$.

Using (3.1), (3.2) and (3.3) in (2.5), the boundary conditions become

$W_1=1,\ W_2=1,\ T_1=1,\ T_2=1,\ C_1=1,\ C_2=1$ at $z=0$

$W_1=0,\ W_2=0,\ T_1=0,\ T_2=0,\ C_1=0,\ C_2=0$ as $z\to\infty$ ...(3.10)

The solutions of the coupled equations (3.4)–(3.9) under the boundary conditons (3.10) are

$$T_1(z)=e^{-H_1z}, \qquad ...(3.11)$$

$$T_2(z)=(1-F_1)e^{-H_2z}+F_1e^{-H_1z}, \qquad ...(3.12)$$

$$C_1(z)=(1-F_0)e^{-S_cz}+F_0e^{-H_1z}, \qquad ...(3.13)$$

$$C_2(z)=F_2e^{-S_cz}-F_3e^{-H_2z}-F_4e^{-H_1z}+F_5e^{-H_3z}, \qquad ...(3.14)$$

$$W_1(z)=F_8e^{-H_4z}-F_7e^{-S_cz}-F_6e^{-H_1z}, \qquad ...(3.15)$$

$$W_2(z)=F_{14}e^{-H_5z}+F_9e^{-H_1z}+F_{10}e^{-H_2z}-F_{11}e^{-H_3z}+F_{12}e^{-H_4z}-F_{13}e^{-S_cz}. \qquad ...(3.16)$$

From (3.15), the steady parts of the primary velocity $u_1(z)$ and secondary velocity $v_1(z)$ are

$$u_1(z)=e^{-P_3z}(A_9\cos Q_3z+B_9\sin Q_3z)-A_8e^{-S_cz}-A_7e^{-H_1z}, \qquad ...(3.17)$$

$$v_1(z)=e^{-P_3z}(B_9\cos Q_3z-A_9\sin Q_3z)-B_8e^{-S_cz}-B_7e^{-H_1z}. \qquad ...(3.18)$$

From (3.16), the time dependent parts of the primary velocity $u_2(z)$ and the secondary velocity $v_2(z)$ are :

$$\begin{aligned}u_2(z)=&\,e^{-P_4z}(A_{14}\cos Q_4z+B_{14}\sin Q_4z)-e^{-P_3z}(A_{12}\cos Q_3z+B_{12}\sin Q_3z)\\&-e^{-P_2z}(A_{11}\cos Q_3z+B_{11}\sin Q_2z)+e^{-P_1z}(A_{10}\cos Q_1z+B_{10}\sin Q_1z)\\&+A_9e^{-H_1z}-A_{13}e^{-S_cz}, \qquad ...(3.19)\end{aligned}$$

$$\begin{aligned}v_2(z)=&\,e^{-P_4z}(B_{14}\cos Q_4z-A_{14}\sin Q_4z)+e^{-P_3z}(B_{12}\cos Q_3z-A_{12}\sin Q_3z)\\&-e^{-P_2z}(B_{11}\cos Q_3z-A_{11}\sin Q_2z)+e^{-P_1z}(B_{10}\cos Q_1z-A_{10}\sin Q_1z)\\&+B_9e^{-H_1z}-B_{13}e^{-S_cz}. \qquad ...(3.20)\end{aligned}$$

Therefore the primary velocity $u(z,t)$ and secondary velocity $v(z,t)$ are

$$u(z,t) = u_1(z) + \in [u_2(z)\cos nt - v_2(z)\sin nt] \quad ...(3.21)$$

and $$v(z,t) = v_1(z) + \in [v_2(z)\cos nt + u_2(z)\sin nt] \quad ...(3.22)$$

4. Skin-Friction, Rate of Heat and Mass Transfer. The non dimensional skin-friction (τ) at the plate at $z=0$ is

$$\tau = (dW/dz)_{z=0} = (dW_1/dz)_{z=0} + \in e^{int}(dW_2/dz)_{z=0} = \tau_p + i\tau_s \quad ...(4.1)$$

Using (4.1) primary skin-friction (τ_p) due to primary velocity and secondary skin-friction (τ_s) due to secondary velocity are :

$$\tau_p = \tau_{0_p} + \in (A_{15}\cos nt - B_{15}\sin nt) \quad ...(4.2)$$

$$\tau_s = \tau_{0_s} + \in (B_{15}\cos nt + A_{15}\sin nt) \quad ...(4.3)$$

Also, the rate of heat transfer (N_u) and the rate of mass transfer (S_h) at the plate at $z=0$ are :

$$N_u = (dT/dz)_{z=0} = (dT_1/dz)_{z=0} + \in e^{int}(dT_2/dz)_{z=0}, \quad ...(4.4)$$

$$S_h = (dC/dz)_{z=0} = (dC_1/dz)_{z=0} + \in e^{int}(dC_2/dz)_{z=0}. \quad ...(4.5)$$

Hence, considering that the real part only is of significance, we get

$$N_u = -H_1 + \in [A_{16}\cos nt - B_{16}\sin nt], \quad ...(4.6)$$

$$S_h = H_1F_0 - S_c(1+F_0) + \in [A_{17}\cos nt - B_{17}\sin nt]. \quad ...(4.7)$$

In terms of amplitude and phase we write

$$N_u = -H_1 + \in |L_1|\cos(nt+\alpha), \quad ...(4.8)$$

$$S_h = H_1F_0 - S_c(1+F_0) + \in |L_2|\cos(nt+\beta). \quad ...(4.9)$$

Table 1

Skin-friction due to primary and secondary velocity

($P_r=0.71$, $k_0=10.0$ $\alpha_0=1.0$, $n=5.0$, $G_m=10.0$, $t=1.0$ and $\varepsilon=0.002$)

S_c	M	G_r	S_0	E	τ_p	τ_s
0.22	1.0	10.0	1.0	1.0	10.30405	–5.49032
0.78	1.0	10.0	1.0	1.0	9.035954	–3.49691
0.22	1.5	10.0	1.0	1.0	9.607512	–3.23245
0.22	1.0	20.0	1.0	1.0	14.27663	–6.73981
0.22	1.0	10.0	2.0	1.0	10.11975	–5.63872
0.22	1.0	10.0	1.0	2.0	7.311972	–5.03227

Table 2

Amplitude $|K_1|$, phase $tan\alpha$ and rate of heat transfer N_u

($n=5.0$, $t=1.0$ and $\varepsilon=0.002$)

S.No.	P_r	α_0	$\|K_1\|$	$tan\alpha$	N_u
1	0.71	1.0	1.6957	0.4207	0.6879
2	0.025	1.0	0.2138	0.4714	0.0214
3	7.00	1.0	11.5739	0.0286	5.9864
4	11.4	1.0	20.2873	0.0095	9.8197
5	0.71	2.0	1.9871	0.4998	0.6792

Table 3

Amplitude $|K_2|$, phase $tan\beta$ and rate of heat transfer S_h

($n=5.0$, $t=1.0$ and $\varepsilon=0.002$)

S.No.	S_c	S_0	$\|K_2\|$	$tan\beta$	S_h
1	0.22	1.0	0.8271	0.7014	0.1987
2.	0.30	1.0	1.0092	0.6813	0.2632
3	0.60	1.0	1.5482	0.6192	0.5389
4	0.66	1.0	1.6412	0.6138	0.6218
5	0.78	1.0	1.9173	0.6085	0.7485
6	0.22	2.0	0.9915	0.7754	0.1991

5. Discussion and Conclusions. The effects of Schmidt number (S_c), magnetic parameter (M), Grashof number (G_r), Soret number (S_0) and rotation parameter (E) on primary velocity (u) and secondary velocity (v) at $P_r=0.71$, $k_0=10.0$, $n=5.0$, $\alpha_0=1.0$, $t=1.0$, $G_m=10.0$ and $\varepsilon=0.002$ are numerically observed and are shown in Fig.-1 and Fig.-2. The above stated effects on skin-friction (τ_p) due to primary velocity and skin-friction (τ_S) due to secondary velocity at the plate are observed numerically and are represented in Table 1. To be realistic the values of Prandtl number (P_r) are chosen for mercury ($P_r=0.025$), air ($P_r=0.71$), water ($P_r=7.0$) and water at 4^0C ($P_r=11.4$). The values of Schmidt number (S_c) are chosen for Hydrogen ($S_c=0.22$), Helium ($S_c=0.30$), Water-vapour ($S_c=0.60$), Oxygen ($S_c=0.66$) and Ammonia ($S_c=0.78$). The values of the remaining parameters are chosen arbitrarily. The conclusions of the study are as follows:

(i) An increase in G_r or S_0 increases primary velocity while an increase in S_c, M or E decreases the primary velocity.

(ii) An increase in G_r and M both decreases the primary velocity.

(iii) An increase in S_0 and E both decreases the primary velocity.

(iv) The primary velocity increases near the plate and after attaining a maximum value it decreases as z increases.

(v) An increase in $S_c M$ or E increases secondary velocity while an increase in G_r or S_0 decreases the secondary velocity.

(vi) An increase in G_r and M both, the secondary velocity increases.

(vii) An increase in G_m and E both, the secondary velocity increases.

(viii) The secondary velocity decreases near the plate and after attaining a minimum value it increases as z increases.

(ix) An increase in S_c, M, S_0 or E decreases the skin-friction due to primary velocity while an increase in G_r increases the skin-friction due to primary velocity.

(x) An increase in S_c, M or E increases the skin-friction due to secondary velocity while an increase in G_r or S_0 decreases the skin-friction due to secondary velocity.

(xi) The values of $|K_1|$ and N_u are least for mercury and highest for water at 4^0C while reverse effects are noted for $tan\alpha$.

(xii) An increase in α_0 results in an increase in amplitude $|K_1|$ and phase $tan\alpha$ but decreases the rate of heat transfer N_u.

(xiii) The values of $|K_2|$ and S_h increase due to the increase in S_c corresponding to the said gases while reverse effect is noted for the phase $tan\beta$.

(xiv) An increase in S_0 results in an increase in amplitude $|K_2|$ and phase $tan\beta$ but decreases the rate of mass transfer S_h.

REFERENCES

[1] D.J. Acheson, Hydromagnetic wave like instabilities in a rapidly rotating stratified fluid, *J. Fluid. Mech.*, **61** (1973), 609-624.

[2] L.Debnath, On Ekman and Hartman Boundary Layers in a rotating fluid, *Acta. Mech.*, **18** (1973), 333-341.

[3] L. Debnath and S. Mukherjee, Unsteady multiple boundary layers on a porous plane in a rotating system, *Phys. Fluid*, **16** (1973), 1418-1423.

[4] C.D.S. Devi, H.S. Takhar and G. Nath, Unsteady laminar boundary forced flow over a moving wall in magnetic field, *Indian J. Pure Appl. Math.*, **19** (1988), 786-801.

[5] H.P. Greenspan, *The theory of Rotating Fluids*, Cambridge University Press, London, 1968.

[6] M.C. Gupta and M.C. Goyal, Viscous incompressible flow between two coaxial rotating cirular cylinders with small uniform injection at the inner cylinder, *Indian J. Pure Appl. Math.*, **4** (1971), 657-671.

[7] J.N. Kapur and S.C. Malik, Laminar flow in an annulus with rotating porous walls, *Proc. Sixth, Congr. Theoret. Appl. Mech.*, **6** (1960), 125-140.

[8] V.T. Morgan, *Advances in Heat Transfer*, Academic Press, New York, 1975.

[9] G.W. Morgan, A study of motions in rotating liquid, *Proc. Roy. Soc. London Ser. A.*, **206** (1951), 108-130.

[10] J.K. Pillai, S.V.K. Verma and M.S. Babu, Flow of a conducting fluid beetween two coaxial rotating porous cylinders bounded by a permeable bed, *Indian J. Pure Appl. Math.*, **20** (1989), 523-535.

[11] R.S. Nanda and H.K. Mohanti, Hydromagnetic flow in a rotating channel, *Appl. Sci. Res.*, **24** (1971), 65-71.

[12] U.S. Rao and A.S. Gupta, Hydromagnetic flow due to a rotating disk subjected to large suction, *J. Phys. Soc. Japan*, **21** (1966), 2390-2398.

[13] H. Reismann, Two dimensional axisymmetic rotational flow of a viscous liquid, *ZAMM* **55** (1975), 443-445.

[14] G. S. Seth and R. Maiti, (*MHD*) Couette flow and heat transfer in rotating system, *Indian J. Pure Appl. Math.* **13** (1982), 931-945.

[15] B.B. SIngh and A. Kumar, Flow of a second order fluid due to the rotation of an infinite porous disc near a stationary parallel porous disc, *Indian J. Pure Appl. Math.* **20** (1989) 931-943.

[16] N.S. Singh, S.K. Gupta and Atul Kumar Singh, Free convection in *MHD* flow of a rotating viscous liquid in porous medium past a vertical porous plate. *Proc. Nat. Acad. Sci. India. Sect. A* **71** (2001), 149-157.

[17] N.P. Singh, A.K. Singh, M.K. Yadav and Atul Kumar Singh, Hydromagnetic oscillatory flow of a viscous liquid past a vertical porous plate in a rotating system, *Indian. J. Theoret Phys.* **50** (2002), 37-43.

[18] C. Thornely, On Stokes and Rayleigh layers in a rotating system, *Quar. J. Appl. Math.* **21**(1968), 451-466.

APPENDIX

$$H_1 = \left[P_r + \sqrt{p_r^2 + 4\alpha_0}\right]\Big/2, \qquad H_2 = P_1 + iQ_1 = \left[P_r + \sqrt{p_r^2 + 4(inP_r + \alpha_0)}\right]\Big/2,$$

$$H_3 = P_2 + iQ_2 = \left[S_c + \sqrt{S_c^2 + 4inS_c}\right]\Big/2, \; H_4 = P_3 + iQ_3 = \left[1 + \sqrt{(1 + 4M_1 + 8iE)}\right]\Big/2$$

$$H_5 = P_4 + iQ_4 = \left[1 + \sqrt{1 + 4M_1 + 4i(2E + n)}\right]\Big/2, \qquad F_0 = S_c S_0 H_1^2 / [H_1(H_1 - S_c)]$$

$$F_1 = (A_1 + iB_1) = \frac{H_1 P_r}{(H_1^2 - H_1 P_r - \alpha_0) - i(nP_r)}, \qquad F_2 = A_2 + iB_2 = iS_c/n$$

$$F_3 = A_3 + iB_3 = \frac{S_0 S_c H_2^2 (1 - F_1)}{H_2^2 - H_2 S_c - inS_c}, \qquad F_4 = A_4 + iB_4 = \frac{S_0 S_c F_1 H_1^2}{H_1^2 - H_1 S_c - inS_c},$$

$$F_5 = A_5 + iB_5 = 1 - F_2 + F_3 - F_4, \qquad F_6 = A_6 + iB_6 = \frac{G_r + G_m F_0}{H_1^2 - H_1 - M_1 - i2E},$$

$$F_7 = A_7 + iB_7 = \frac{G_m(1 + F_0)}{S_c^2 - S_c - M_1 - i2E}, \qquad F_8 = A_8 + iB_8 = 1 + F_6 + F_7,$$

$$F_9 = A_9 + iB_9 = \frac{F_6(1 + k_0 H_1) + k_0 G_r F_1 - G_m F_4 k_0}{k_0\left[H_1^2 - H_1 - M_1 - i(2E + n)\right]},$$

$$F_{10} = A_{10} + iB_{10} = \frac{G_m F_3 - G_r(1 - F_1)}{\left[(H_2^2 - H_2 - M_1) - i(2E + n)\right]},$$

$$F_{11} = A_{11} + iB_{11} = \frac{G_m F_5}{H_3^2 - H_3 - M_1 - i(2E + n)},$$

$$F_{12} = A_{12} + iB_{12} = \frac{F_9(k_0 H_4 - 1)}{k_0\left[H_4^2 - H_4 - M_1 - i(2E + n)\right]},$$

$$F_{13} = A_{13} + iB_{13} = \frac{k_0 G_m F_2 + F_7(k_0 S_c - 1)}{k_0\left[S_c^2 - S_c - M_1 - i(2E + n)\right]},$$

$$F_{14} = A_{14} + iB_{14} = 1 - F_9 - F_{10} + F_{11} - F_{12} + F_{13},$$

$$P_1 = P_r/2 + \left[\sqrt{(P_r^2 + 4\alpha_0)^2 + 16n^2 P_r^2} + (P_r^2 + 4\alpha_0)\right]^{1/2} \Big/ 2\sqrt{2},$$

$$Q_1=\left[\sqrt{\left(P_r^2+4\alpha_0\right)^2+16n^2P_r^2}+\left(P_r^2+4\alpha_0\right)\right]^{1/2}\Big/\left(2\sqrt{2}\right),$$

$$P_2=S_c/2+\left[S_c\sqrt{S_c^2+16n^2}+S_c^2\right]^{1/2}\Big/2\sqrt{2},\quad Q_2=\left[S_c\sqrt{S_c^2+16n^2}-S_c^2\right]^{1/2}\Big/\left(2\sqrt{2}\right),$$

$$P_3=1/2+\left[\sqrt{(1+4M_1)^2+64E^2}+(1+4M_1)\right]^{1/2}\Big/\left(2\sqrt{2}\right),$$

$$Q_3=\left[\sqrt{(1+4M_1)^2+64E^2}+(1+4M_1)\right]^{1/2}\Big/\left(2\sqrt{2}\right),$$

$$P_4=1/2+\left[\sqrt{(1+4M_1)^2+16(2E+n)^2}+(1+4M_1)\right]^{1/2}\Big/\left(2\sqrt{2}\right),$$

$$Q_4=\left[\sqrt{(1+4M_1)^2+16(2E+n)^2}-(1+4M_1)\right]^{1/2}\Big/\left(2\sqrt{2}\right),$$

$$A_1=\frac{H_1P_ra_1}{a_1^2+b_1^2},\quad B_1=\frac{H_1P_rb_1}{a_1^2+b_1^2},\quad A_2=0,\quad B_2=\frac{S_c}{n},\quad A_3=\frac{a_2a_3+b_2b_3}{a_3^2+b_3^2},\quad B_3=\frac{b_2a_3-a_2b_3}{a_3^2+b_3^2},$$

$$A_4=\frac{S_cS_0H_1^2[A_1a_4-B_1b_4]}{a_4^2+b_4^2},\quad B_4=\frac{S_cS_0H_1^2[A_1b_4-B_1a_4]}{a_4^2+b_4^2},\quad A_5=1-A_2+A_3-A_4,$$

$$B_5=B_3-B_2-B_4,\quad A_6=\frac{(G_r+F_0G_m)a_5}{a_5^2+b_5^2},\quad B_6=\frac{(G_r+F_0G_m)b_5}{a_5^2+b_5^2},\quad A_7=\frac{a_6G_m(1+F_0)}{a_6^2+b_6^2},$$

$$B_7=\frac{b_6G_m(1+F_0)}{a_6^2+b_6^2},\qquad A_8=1+A_6+A_7.\qquad B_8=B_6+B_7.\quad A_9=\frac{a_7a_8-b_7b_8}{a_8^2+b_8^2},$$

$$B_9=\frac{b_7a_8-a_7b_8}{a_8^2+b_8^2},\quad A_{10}=\frac{a_9a_{10}+b_9b_{10}}{a_{10}^2+b_{10}^2},\quad B_{10}=\frac{b_9a_{10}+a_9b_{10}}{a_{10}^2+b_{10}^2},\quad A_{11}=\frac{G_r(A_5a_{11}+B_5b_{11})}{a_{11}^2+b_{11}^2},$$

$$B_{11}=\frac{G_r(B_5a_{11}-A_5b_{11})}{a_{11}^2+b_{11}^2},\quad A_{12}=\frac{a_{12}a_{13}+b_{12}b_{13}}{a_{13}^2+b_{13}^2},\quad B_{12}=\frac{b_{12}a_{13}-a_{12}b_{13}}{a_{13}^2+b_{13}^2},$$

$$A_{13}=\frac{a_{14}a_{15}+b_{14}b_{15}}{a_{15}^2+b_{15}^2},\quad B_{13}=\frac{b_{14}a_{15}-a_{14}b_{15}}{a_{15}^2+b_{15}^2},\quad A_{14}=A_{13}-A_{12}+A_{11}-A_{10}-A_9+1,$$

$$B_{14}=B_{13}-B_{12}+B_{11}-B_{10}-B_9,\qquad a_1=H_1^2-H_1P_r-\alpha_0,$$

$$a_2=S_0S_c\left[(1-A_1)\left(P_1^2-Q_1^2\right)+2P_1Q_1B_1\right],\quad a_3=P_1^2-Q_1^2-P_1S_c\quad a_4=H_1(H_1-S_c),$$

$$a_5=H_1^2-H_1-M_1,\ a_6=S_c^2-S_c-M_1,\ a_7=A_6(1-k_0H_1)-k_0A_1G_r-k_0F_0G_m,\ a_8=k_0a_5,$$

$$a_9=G_mA_3-G_r(1-A_1),\quad a_{10}=P_1^2-Q_1^2-P_1-M_1,\quad a_{11}=P_2^2-Q_2^2-P_2-M_1,$$

$$a_{12}=A_8(k_0P_3-1)-B_8k_0Q_3,\ a_{13}=k_0\left[P_3^2-Q_3^2-P_3-M_1\right],\ a_{14}=A_2k_0G_m+A_7(k_0S_c-1),$$

$$a_{15}=k_0\left[S_c^2-S_c-M_1\right],\ b_1=nP_r,\ b_2=S_cS_0\left[2P_1Q_1(1-A_1)-B_1\left(P_1^2-Q_1^2\right)\right],$$

$$b_3=2P_1Q_1-Q_1S_c-nS_c,\ b_4=nS_c,\ b_5=b_6=2E,\quad b_7=B_6(1-k_0H_1)-B_1k_0G_r,$$

$$b_8=b_{15}=k_0(2E+n),\ b_9=G_mB_3+G_rB_1,\quad b_{10}=2P_1Q_1-Q_1-2E-n,$$

$$b_{11}=2P_2Q_2-Q_2-2E-n,\quad b_{12}=B_8(P_3k_0-1)+A_8Q_3k_0,\quad b_{13}=k_0[2P_3Q_3-Q_3-2E-n],$$

$$b_{14}=B_2k_0G_m+B_7(k_0S_c-1),\quad \tau_{0_p}=A_7H_1+A_8S_c+Q_3B_9-P_3A_9,$$

$\tau_{0_n} = B_7H_1 + B_8S_c - Q_3A_9 - P_3B_9,$

$A_{15}=A_{13}S_c-A_{14}P_4+B_{14}Q_4-A_{12}P_3+B_{12}Q_3-A_{11}P_2+B_{12}Q_2-A_{10}P_1+B_{10}Q_1-A_9H_1$

$B_{15}=B_{13}S_c-A_{14}Q_4-B_{14}P_4-A_{12}Q_3-B_{12}P_3-A_{11}Q_2-B_{12}P_2-A_{10}Q_1-B_{10}P_1-B_9H_1,$

$A_{16}=P_1(A_1-1)-Q_1B_1-H_1A_1, \qquad B_{16}=Q_1(A_1-1)-P_1B_1-H_1B_1,$

$A_{17}=A_3P_1-B_3Q_1-A_5P_2+B_5Q_2-A_2S_c-A_4H_1,$

$B_{17}=A_3Q_1+B_3P_1-A_5Q_2-B_5P_2-B_2S_c-B_4H_1,$

$|L_1|=|A_{16}+iB_{16}|, \quad |L_2|=|A_{17}+iB_{17}|, \quad tan\alpha = B_{16}/A_{16}$ and $tan\beta = B_{17}/A_{17}$

Figure 1. Effects of Various Parameters on Primary Velocity

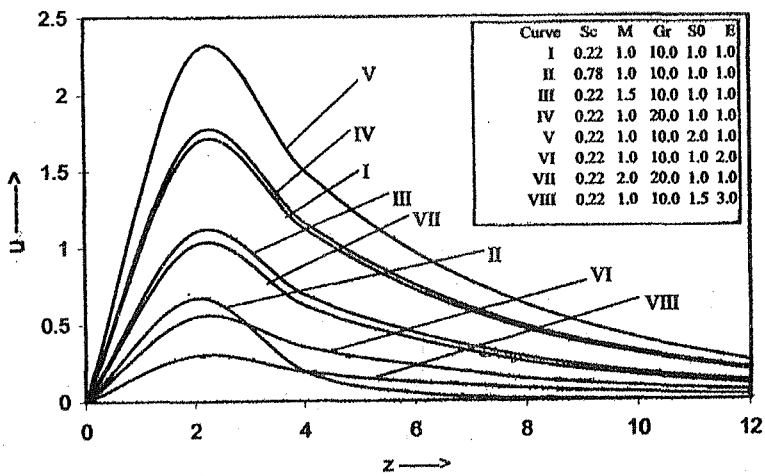
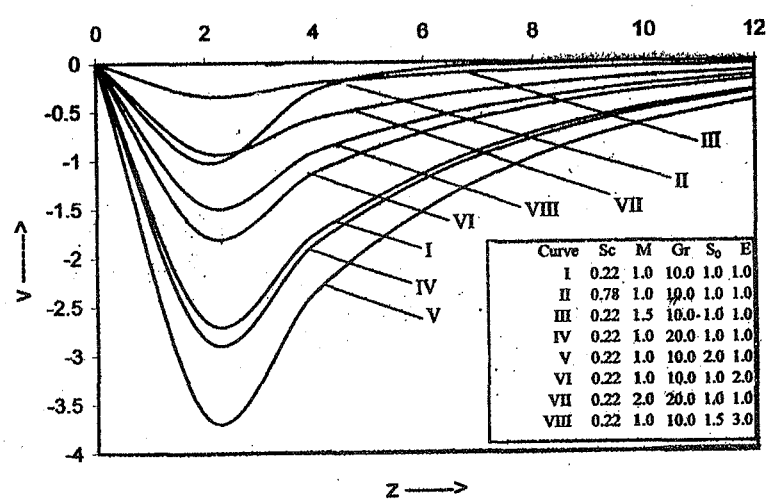


Figure 2. Effects of Various Parameters on Secondary Velocity



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ON A TACHIBANA SPACE WITH PARALLEL BOCHNER CURVATURE TENSOR

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ABSTRACT

In the present paper, we consider the Tachibana space ($n>2$) with parallel Bochner curvature tensor. Several theorems will be investigated.

1. Introduction. An n $(=2m)$ dimensional Kaehlerian space K_n^c is a Riemannian space admitting a structure tensor F_i^h satisfying :

(1.1) $F_i^h F_h^j = -\delta_i^j$,

(1.2) $F_{ij} = -F_{ji}, \left(F_{ij} = F_j^h g_{hi}\right)$

and

(1.3) $\nabla_j F_i^h = 0$,

where ∇_j denotes the operator of covariant differentiation with respect to Christoffel symbol $\left\{ {h \atop ji} \right\}$ formed with g_{ji}, where g_{ji} is Riemannian metric satisfying

(1.4) $F_j^t F_i^s g_{ts} = g_{ji}$.

From (1.2) and (1.3), we obtain

(1.5) $\nabla_k F_{ji} = 0$.

Now, from (1.3) and the Ricci identity

$\nabla_k \nabla_j F_i^h - \nabla_j \nabla_k F_i^h = R_{kjs}^h F_i^s - R_{kji}^s F_s^h$,

we obtain

(1.6) $R_{kji}^s F_s^h = R_{kjs}^h F_i^s$.

The Riemannian curvature tensor, R_{ijk}^h, is given by

(1.7) $R_{ijk}^h = \partial_i \left\{ {h \atop jk} \right\} - \partial_j \left\{ {h \atop ik} \right\} + \left\{ {h \atop il} \right\}\left\{ {l \atop jk} \right\} - \left\{ {h \atop jl} \right\}\left\{ {l \atop ik} \right\}$,

where $\partial_i = \dfrac{\partial}{\partial x^i}$ and $R_{ik} = R_{ijk}^i$ is the Ricci tensor, while $R = R_{jk} g^{jk}$ is a scalar curvature.

S. Tachibana [3] has defined the Bochner curvature tensor (with respect to real local co-ordinates) as

(1.8) $K_{ijk}^h = R_{ijk}^h + \dfrac{1}{n+4}\Big(R_{ik}\delta_j^h - R_{jk}\delta_i^h + g_{ik}R_j^h - g_{jk}R_i^h + S_{ik}F_j^h - S_{jk}F_i^h + F_{ik}S_j^h - F_{jk}S_i^h$

$$+2S_{ij}F_k^h+2F_{ij}S_k^h\Big)-\frac{R}{(n+2)(n+4)}\Big(g_{ik}\delta_j^h-g_{jk}\delta_i^h+F_{ik}F_j^h-F_{jk}F_i^h+2F_{ij}F_k^h\Big),$$

where $S_{jk}=F_j^hR_{hk}$.

An almost Tachibana space is first of all an almost Hermite space. That is, a $2n$-dimensional space with an almost complex structure F satisfying (1.1), (1.2) and finally has the property that the skew symmetric tensor F_{ih} is killing tensor:

(1.9) $\nabla_jF_{hi}+\nabla_iF_{jh}=0$,

from which

(1.10) $\nabla_jF_i^h+\nabla_iF_j^h=0$

and

(1.11) $F_i=-\nabla_jF_i^j$.

Now, the Nijenhuis tensor N_{ji}^h is written in the form:

(1.12) $N_{ji}^h=-4\left(\nabla_jF_i^t\right)F_t^h+2G_{ji}^tF_t^h+F_j^tG_{ti}^h-F_i^tG_{tj}^h$,

and consequently, we have [5]

Theorem 1. In an almost Tachibana space, we obtain

(1.13) $N_{ji}^h=-4\left(\nabla_jF_i^t\right)F_t^h$,

and consequently $\nabla_jF_i^h$ is pure in j and i.

When the Nijenhuis tensor vanishes, the almost Tachibana space is called a Tachibana space. In this case, we have from (1.13) the following [5]

$$\nabla_jF_i^h=0.$$

Hence we have

Theorem 2. A Tachibana space is a Kaehlerian space.

In this paper, we shall consider the Tachibana space ($n>2$) with parallel Bochner curvature tensor, i.e.,

(1.14) $\nabla_jK_{ijk}^h=0.$

We shall call the Tachibana space satisfying (1.14) a TPB_n-space and the space satisfying $K_{ijk}^h=0$ will be called an TVB_n-space

In order to avoid complicated calculations, we put

(1.15) $\pi_{ij}=\dfrac{1}{n+4}\left(R_{ij}-\dfrac{R}{2(n+2)}g_{ij}\right)$,

(1.16) $M_{ij}=F_i^h\pi_{hj}=\dfrac{1}{n+4}\left(S_{ij}-\dfrac{R}{2(n+2)}F_{ij}\right)$,

and

(1.17) $D_{ijk}^h=\pi_{ik}\delta_j^h-\pi_{jk}\delta_i^h+g_{ik}\pi_j^h-g_{jk}\pi_i^h+M_{ik}F_j^h-M_{jk}F_i^h+F_{ik}M_j^h-F_{jk}M_i^h+2M_{ij}F_k^h+2F_{ij}M_k^h.$

Thus, (1.8) can be written as

(1.18) $K^h_{ijk} = R^h_{ijk} + D^h_{ijk}$.

Moreover, π_{ij}, M_{ij} and D^h_{ijk} satisfy the following conditions:

(1.19) $\pi = g^{ab}\pi_{ab} = \dfrac{R}{2(n+2)}$,

(1.20) $M_{ij} = -M_{ji}$

and

(1.21) $D_{ijkh} = D_{khij}$, $(D_{khij} = D^l_{ijk} g_{lh})$.

A Tachibana space is said to be a recurrent space if its curvature tensor R^h_{ijk} satisfies

(1.22) $\nabla_l R^h_{ijk} - \lambda_l R^h_{ijk} = 0$,

for a non-zero recurrence vector λ_l. From (1.3), (1.8), (1.14) and (1.22), we have the following:

Theorem 3. Every recurrent TPB_n-space is a TVB_n-space.

Let a TPB_n-space be an Einstein space, then the Ricci tensor satisfies

(1.23) $R_{ij} = \dfrac{R}{n} g_{ij}, \nabla_l R_{ij} = 0$, from which, we get

(1.24) $S_{ij} = \dfrac{R}{n} F_{ij}, \nabla_l R_{ij} = 0$ and $\nabla_l S_{ij} = 0$.

Substituting (1.23) and (1.24) in (1.14), we get

$$\nabla_l R^h_{ijk} = 0.$$

Thus, we have the following rusult:

Theorem 4. Every Tachibana Einstein TPB_n - space is symmetric in the sense of Cartan.

2. Ricci Recurrent. TPB_n- space : A Tachibana space is said to be a Ricci recurrent space, if the Ricci tensor R_{ij} satisfies

$$\nabla_l R_{ij} - \lambda_l R_{ij} = 0,$$

for a non-zero recurrence vector λ_l.

We, now, have the following result.

Theorem 5. A necessary and sufficient condition for R_{ij} to be a recurrent tensor is that π_{ij} be a recurrent tensor.

Proof. Suppose that R_{ij} is a recurrent tensor, i.e.,

(2.1) $\nabla_l R_{ij} - \lambda_l R_{ij} = 0$.

Multiplying (2.1) with respect to g^{ij}, we have

(2.2) $\nabla_l R - \lambda_l R = 0$,

which by virtue of (1.15) and (2.1) gives

(2.3) $\nabla_l \pi_{ij} - \lambda_l \pi_{ij} = 0.$

Hence π_{ij} is a recurrent tensor.

Conversely, if we have (2.3), then by multiplication with g^{ij} it gives

(2.4) $\nabla_l \pi - \lambda_l \pi = 0.$

Consequently, from (1.19) and (1.15), we have

$$\nabla_l R_{ij} - \lambda_l R_{ij} = 0,$$

i.e., the Ricci tensor is recurrent one.
This completes the proof of theorem 5.

Theorem 6. In a TPB_n-space, if $\nabla_l R_{ij} - \lambda_l R_{ij} = 0$, for a non-zero recurrence vector λ_l and a non-zero tensor R_{ij}, then the tensor D^h_{ijk} is not equal to zero.

Proof. Suppose on the contrary that $D^h_{ijk} = 0$. Since the space is a TPB_n-space, we have from (1.18) that

$$\nabla_l R^h_{ijk} = 0.$$

Contracting the above equation with respect to h and i, we get $\nabla_l R_{ik} = 0$, from which we have $\lambda_l R_{ik} = 0$, which is impossible, since $\lambda_l \neq 0$ and $R_{jk} \neq 0$. Hence, the tensor $D^h_{ijk} \neq 0$.

The following two Lemmas were proved by Walker [4] and Ruse, Walker and Willmore [2]:

Lemma 1. The curvature tensor R_{hijk} satisfies the identity

(2.5) $\nabla_l \nabla_m R_{hijk} - \nabla_m \nabla_l R_{hijk} + \nabla_h \nabla_i R_{jklm} - \nabla_l \nabla_h R_{jklm} + \nabla_j \nabla_k R_{lmhi} - \nabla_k \nabla_j R_{lmhi} = 0.$

Lemma 2. If $a_{\alpha\beta}$, b_α are the quantities satisfying

(2.6) $a_{\alpha\beta} = a_{\beta\alpha}, a_{\alpha\beta} b_\gamma + a_{\beta\gamma} b_\alpha + a_{\gamma\alpha} b_\beta = 0,$

for $\alpha, \beta, \gamma = 1,2,\ldots,N$, then either all the $a_{\alpha\beta}$ are zero or all the b_α are zero.

Using the above Lemmas, we can prove the following result:

Theorem 7. In a TPB_n-space, if $\nabla_l R_{ij} - \lambda_l R_{ij} = 0$ for a non-zero vector λ_l and a non-zero tensor R_{ij}, then the recurrence vector λ_l is gradient.

Proof. If we suppose that $\nabla_l R_{ij} - \lambda_l R_{ij} = 0$, then by Theorem 5, we get $\nabla_l \pi_{ij} - \lambda_l \pi_{ij} = 0$ and consequently from (1.16) and (1.17), we have $\nabla_l M_{ij} - \lambda_l M_{ij} = 0$ and

(2.7) $\nabla_l D^h_{ijk} - \lambda_l D^h_{ijk} = 0.$

From (1.14), (1.18) and (2.7), we get

(2.8) $\nabla_l R^h_{ijk} - \lambda_l D^h_{ijk} = 0$.

Differentiating (2.8) covariantly and using (2.7), we obtain

(2.9) $\nabla_m \nabla_l R^h_{ijk} + \nabla_m \lambda_l D^h_{ijk} + \lambda_l \lambda_m D^h_{ijk} = 0.$

Hence

$$\nabla_l \nabla_m R^h_{ijkh} + \nabla_m \nabla_l D^h_{ijkh} + \lambda_{lm} D_{ijkh} = 0,$$

where $\lambda_{lm} = \nabla_l \lambda_m - \nabla_m \lambda_l$.

The identity (2.5) gives

(2.10) $\lambda_{lm} D_{ijkh} + \lambda_{ij} D_{khlm} + \lambda_{kh} R_{lmij} = 0$

The equation (2.10) is of the form (2.6) because $D_{ijkh}=D_{khij}$. By Theorem 6 the tensor D_{ijkh} is not zero. Hence, by the Lemma 2, $\lambda_{lm} = 0$, i.e., $\nabla_l \lambda_m = \nabla_m \lambda_l$, which is the condition for λ_l to be gradient.

The next theorem is easy to prove.

Theorem 8. A TPB_n-space satisfying $R_{ij}=0$ is symmetric in the sense of Cartan.

The following Lemma has been proved by Matsumoto [1]:

Lemma 3. In a compact Tachibana space with parallel Bochner curvature tensor $g^{ij}\ \nabla_j R$ is a contravariant analytic vector.

Now, suppose that the TPB_n-space is Ricci recurrent space, then we have $\nabla_l R_{ij} - \lambda_l R_{ij} = 0$ and $\nabla_l R - \lambda_l R = 0$. Hence, from the above Lemma, $\lambda^i R$ is a contravariant analytic vector and we get

Theorem 9. If a compact TPB_n-space is Ricci recurrent, then the vector $\lambda^i R$ is a contravarient analytic vector.

3. Parallel Vector Fields in TPB_n–Space. Let us assume that a TPB_n-space admits a parallel vector field v^i, that is,

(3.1) $\nabla_l v^i = 0,$

whence we have

(3.2) $\nabla_l \bar{v}^i = 0,$ where $\bar{v}^i = F^i_j v^j$.

Making use of the Ricci and Bianchi is identities, we get

(3.3) (a) $v^a R_{hajk} = 0, v^a R_{aj} = 0,$

(b) $\bar{v}^a R_{hajk} = 0, \bar{v}^a R_{aj} = 0,$

(3.4) (a) $v^a \nabla_l R_{hajk} = 0, v^a \nabla_l R_{aj} = 0,$

(b) $\bar{v}^a \nabla_l R_{hajk} = 0, \bar{v}^a \nabla_l R_{aj} = 0,$

(3.5) (a) $v^a \nabla_a R_{hijk} = 0, v^a \nabla_a R_{ij} = 0, v^a \nabla_a R = 0,$

(b) $\bar{v}^a \nabla_a R_{hijk} = 0, \bar{v}^a \nabla_a R_{ij} = 0, \bar{v}^a \nabla_a R = 0.$

We know that (Tachibana [3])

(3.6) $\nabla_a K^a_{ijk} = \dfrac{n}{n+4} K_{ijk},$

where

(3.7) $K_{ijk} = \nabla_I R_{jk} - \nabla_j R_{ik} + \dfrac{1}{2(n+2)}\left(g_{ik}\delta^l_j - g_{jk}\delta^l_i + F_{i,k}F^l_j - F_{jk}F^l_i + 2F_{ij}F^l_k\right)\nabla_l R.$

Therefore, for a TPB_n-space, we have

(3.8) $\nabla_i R_{jk} - \nabla_j R_{ik} + \dfrac{1}{2(n+2)}\left(g_{ik}\delta^l_j - g_{jk}\delta^l_i + F_{ik}F^l_j - F_{jk}F^l_i + 2F_{ij}F^l_k\right)\nabla_l R = 0.$

As the differential form $S = \frac{1}{2} S_{ij} dx^i dx^j$ is closed (Yano [5], page 72), it follows that

(3.9) $F^l_k \nabla_i S_{ij} = \nabla_j R_{ik} - \nabla_i R_{jk}.$

Multiplying (3.8) by $F^k_m F^i_n$, we get

(3.10) $\nabla_m R_{nj} = F^k_m F^i_n \left(\nabla_j R_{ik} - \nabla_l R_{jk}\right)$

Multiplying (3.8) by $F^k_m F^i_n$ and taking account of (3.10), we get

(3.11) $\nabla_m R_{nj} = \left(g_{mn}\nabla_j R + g_{mj}\nabla_n R - F_{mn}F^l_j \nabla_l R - F_{mj}F^l_n \nabla_l R + 2g_{nj}\nabla_m R\right)/2(n+2)$

Multiplying (3.11) by $v^n v^j$ and using (3.3), (3.4) and (3.5), we get

$V^2 \Delta_m R = 0,$ (where v is the magnitude of v^i) from which it follows that R is constant.

We know that (Matsumoto [1])

Lemma 4. If a Kaehlerian (Tachibana) space with parallel Bochner curvature tensor has constant scalar curvature, then it is a symmetric space.

Using the above Lemma, we obtain

Theorem 10. If a TPB_n-space admits a parallel vector field v^i, then at least one of the following two cases occurs :

(i) the space is symmetric,

(ii) v^i is a null vector.

REFERENCES

[1] M. Mtsumoto, On Kaehlerian spaces with parallel or vanishing Bochner curvature tensor, *Tensor (N.S.)*, **20** (1969), 25-28.

[2] H.S. Ruse, A.G. Walker and T.J. Willmore, *Harmonic Spaces*, Edizioni Cre-monese, Roma (1961).

[3] S. Tachibana, On the Bochner curvature tensor; *Natur. Sci. Rep.* Ochanomizu Univ. **18** (1967), 15-19.

[4] A.G. Walker, On Ruse's spaces of recurrent curvature, *Proc. London Math.Soc.*, (2), 52, (1950), 36-64.

[5] K. Yano, Differential Geometry on Complex and Almost Complex Spaces, Pergamon Press (1965).

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VARIATION OF δ-COINCIDENCE SETS

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ABSTRACT

David Gauld [2] proved that in many familiar cases the upper semi-finite topology on the set of closed subsets of a space is the largest topology making the coincidence function continuous, when the collection of functions is given the graph topology. For a $\delta>0$ and for continuous maps $f, g : X \to Y$, where X,Y are metric spaces, by taking the δ-coincidence set to consist of points x for which $f(x)$ and $g(x)$ lie at a distance less than or equal to δ, we obtain various results of Gauld in the new setting.

Key Words and Phrases . Graph topology, Upper semifinite topology, Coincidence set, δ-active homotopy, Property W (δ-strong).

1. Introduction. For metric spaces X,Y the collection $\mathscr{F}(X,Y)$ of continuous maps from a metric space X to a metric space Y is equipped with the graph toplogy: The family $\{<W> \mid W$ is an open set of $X\times Y\}$, where

$$<W> = \{f \in \mathscr{F}(X,Y) \mid \text{graph } \Gamma f \text{ of } f \text{ is contained in } W\}$$

forms a basis for this toplogy [4].

Let $h : X\to Y$ be a continuous map from a metric space X to a metric space Y. Then for a continuous map $f: X\to Y$, the conicidence set $\psi_h(f)$ consists of those points of X at which f and h agree. Because $\psi_h(f) \in \zeta(X)$, a map $\psi_h:\mathscr{F}(X,Y)\to\zeta X$ can be defined which sends $f \in \mathscr{F}(X,Y)$ to the coincidence set of f and h. In [2], Gauld studied the variation $\psi_h(f)$ with f. Also the continuity of the coincidence function $\chi : \mathscr{F}(X,Y)\times\mathscr{F}(X,Y)\to\zeta X$ sending (f,h) to their coincidence set is considered.

For a real number $\delta>0$, we let the δ-coincidence set $D(f,h)$ be the set

$$D(f,h)=\{x\in X \mid d(f(x),g(x))\leq\delta\}$$

and for an element $h\in \mathscr{F}(X,Y)$, define a map

$$D_h:\in \mathscr{F}(X,Y)\to\zeta X$$

which sends an element $f\in \mathscr{F}(X,Y)$ to the δ-coincidence set of f and h. The δ-coincidence function D is defined in a similar way. In section 2 of this paper we prove continuity of the map D_h and δ-coincidence function D. Finally we transfer some results of Gauld in this new setting in section 3 of this paper.

For terms and definition not explained here, we refer to [1,2].

2. Variation of coincidence Sets. Throghout the section X,Y are metric spaces. The section begins with the following definitions :

Definition 1. Let $f, h : X \to Y$ be two continuous maps where X,Y are metric spaces. The set $D_{f,h}=\{x\in X \mid d(f(x),g(x))\leq\delta\}$
is called the δ-coincidence set of f and h.
Note that $D_{f,h}$ is a closed set of X.

The δ-coincidence set and the coincidence set of two maps, $f, h : X \to Y$ are different. For $\delta=0$, two sets are the same.

Definition 2. The map D: $\mathcal{C}(X,Y)\times\mathcal{C}(X,Y)\to\zeta X$ defined by

$$D(f,g) = \{x\in X \mid d(f(x),g(x))\leq \delta\}$$

is called the δ-coincidence function.

If h is fixed then the restriction of the map D to $\mathcal{C}(X,Y)\times\{h\}=\mathcal{C}(X,Y)$ is denoted by D_h . For $\delta=0$, D and D_h are easily seen to be the maps χ and ψ_h as described in [2].

Theorem 1. Let $h:X\to Y$ be a continuous map from a metric space X to a metric space Y. If $\mathcal{C}(X,Y)$ is toplogized with the graph topology and ζX by the upper semi-finite topology then

$D_h : \mathcal{C}(X,Y)\to\zeta X$ is continuous.

Proof. Let V be an open set of X. Take $f \in D_h^{-1}[V]$. Then

$D(f,h)=\{x\in X \mid d(f(x), h(x))\leq\delta\}\subseteq V$.

Take $U=V\times Y\cup\{(x,y)\in X\times Y \mid d(y,h(x))>\delta\}$.

Then $f \in\langle U\rangle\subset D_h^{-1}[V]$.

3. Upper Semi-Finite Topology and Continuity of $D(D_h)$

Definition 3. Let X, Y be metric spaces. A homotopy $H:X\times I\to Y$ is called δ-active if $d(H(x,t)), H(x,0))>\delta$ for all $x \in X$ and $t\in(0,1]$ where I denotes the unit closed interval of the real line R with the subspace toplogy.

Let $H:X\times I\to Y$ be a δ-active homotopy. We recall the map $\tau_H:\mathcal{C}(X,I) \to\mathcal{C}(X,Y)$ defined by $\tau_H(\alpha)=H(x, \alpha(x))$; where $\alpha \in \mathcal{C}(X,I)$ and $x\in X$ [2]. Denote by H_0, the restriction of H to the base $X\times\{0\}$. Identify $X\times\{0\}$ with X and note that $D_{H_0}o\tau_H=\sigma$, where $\sigma :\mathcal{C}(X,I) \to\zeta X$ maps α to its zero set. Since for a perfectly normal space X the upper semi-finite toplogy is the largest toplogy on ζX making σ continuous [See 2, Prop. 1.4], we conclude the following:

Let X be a perfectly normal space and $H:X\times I\to Y$ be a δ–active homotopy. Then the upper semi-finite topology is the largest topology on ζX making D_{H_0} continuous.

Call a path α in X, to be δ-non overlapping if $d(\alpha(t),\alpha(0))>\delta$ for any $t\in(0,1]$.

A space X is said to have the property W (δ-strong) if there exists a δ-active deformation of X i.e. there is a δ-active homotopy $H:X\times I \to X$ satisfying that $H_0=I_x$.

We obtain the analogues of corollaries 2.6 and 2.7 of [2] as follows:

Let X be a perfectly normal space. Then the upper semi-finie topology on ζX is the largest topology making D_h continuous provided either of the following holds:

(a) Y has the property W (δ-strong).

(b) $h:X\to Y$ is a constant map and Y has a δ-non-overlapping path beginning from the image point of h.

REFERENCES

[1] J. Dugundji, *Toplogy*, Allyn and Bacon, Boston, 1966.

[2] D.B. Gauld, Variation of fixed point and coincidence sets, *J. Austral. Math. Soc.*, **44** (1988), 214-224.

[3] E. Michael, Topologies on spaces of subsets, *Trans. Amer. Math. Soc.* **71** (1951), 152-182.

[4] S.A. Naimpally, Graph Topology for function spaces, *Trans. Amer. Math. Soc.* **123** (1966), 267-272.

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RELAXATION LABELING BASED GRAPH MATCHING METHODOLOGY TO COMPARE VHR SATELLITE IMAGES WITH GIS DATA

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ABSTRACT

In this paper, we propose a graph matching methodology based on relaxation labeling to compare road junction on *VHR* satellite images with *GIS* road data. Use is made of the spatial layout between points based on the relative angle. The technique finds correspondences between sets of points taking into account error on the spatial location and spurious or missing points. An analysis is given to determine the weights of the algorithm based on the expected graph error for remotely sensed *GIS* data.

Key Words. registration, change detection, graph matching

1. Introduction. A major challenge in the production and use of geographic information is assessment and control of the quality of the database. The rapid growing number of sources of geospatial data, ranging from high-resolution satellite and airborne sensors, *GPS*, and derivative geospatial products, pose severe problems for integrating data. One of the major challenges for content providers to face is the problem of upgrading their current databases to a higher accuracy any ensuring the quality of the information. Current techniques can't support this in a cost-effective way due to the necessary manpower. Automated detection of change and anomalies in the existing databases using very high resolution (*VHR*)

satellite images can form an essential tool to support quality control and maintenance of spatial information. We examine to what extent remote sensing and computer vision can be used to detect deviation of a *GIS* database vs "real world".

We introduce a system that is based on object recognition and searches for the correspondences between the road information in images and existing road vector data. The information that is used consists out of road junctions. Junctions have the advantage that the representation extracted from the image and the vector representations are semantically close: in the simple case, junctions can be represented by a single point in the image and vector data. In contrast, roads are much more difficult in the sense that road detection in the image often leads to fragmented pixel chains with no easy correspondence with the idealized polylines in the vector data. In [1], a simple technique based on ridge detection is introduced for the automatic detection of road junctions.

In this paper, we examine the correspondence probelm between two datasets containing junctions, modeled as a graph matching problem, and how this can be used for registration and change detection. Using the spatial layout of the junctions (e.g. the angle formed by two junctions), we define local constraints that determine which correspondences are allowed. This defines a constraint satisfaction problem comparing two graphs that is solved using continuous relaxation labeling [2]. The matching is error robust, meaning that corresponding road junctions need not be at the exact same spatial position but deviations are allowed. This is important for change detection, allowing the corresponding junction to be found although it has changed position from one dataset to the other. This sensitivity for local changes or deformations make graph matching an ineresting alternative for registration techniques based on a global spatial transformation model [3]. Of course, since the technique is error robust one needs to be able to define which errors are allowable and which are not. The constraint satisfaction problem holds this information implicitly, but we have derived an analytical expression, which allows the user to determine the error thresholds within the system. The expression is general and useful for optimisation techniques other than relaxation labeling like e.g. Markov random fields.

2. Continuous Relaxation Labeling. Error tolerant graph matching techniques lock for graph morphism between two graphs, which allows for distortions. A general distortion model contains the deletion and addition of nodes/ arcs and the replacement of attribute values. Based on this model, a distance function is defined between graphs that penalize the occurring distortions when comparing graphs. The matching problem can be defined as a graph labelling problem. The following are defined:

1. a set of object i, corresponding to image features;
2. a set of lables x_i, corresponding to *GIS* features;
3. a neighbour relationship over the objects;
4. restrictions on possible labels between pairs of neighbouring objects.

Relaxation labeling techniques use an iterative process to determine the probabilities of each object. Different update rules have been proposed. In [2] the relation between different update rules is analytically shown. The problem of finding consistent solutions is shown to be equivalent to solving a variational inequality which is based on the mathematical concept of "consistency". This concept is interesting because it lays bare the foundations of the labeling process and offeres guidance in determining good compatibility coefficients.

To each object i a probability distribution is associated that i has lable $X_k(p_i(x_1),\ldots,p_i(x_m))$,

$$0 \le (p_i(x_k)) \le 1, \quad \sum_{k=1}^{m} p_i(x_k) = 1. \tag{2.1}$$

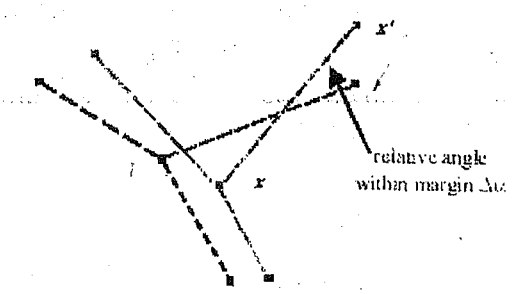
For each pair of neighbouring objects i and j and for each pair of labels X_k and X_l a compatibility coefficient $r_{ij}(X_k,X_l)$ exists. These coefficients express the compatibility of assigning label X_k to obejct i in combination with assigning lable x_l to object j. Negative values express incompatibility, positive values compatibility. Given these quantities, the support of a label X_k for the object i given by the correspondence â is defined by

$$S_i(X_k) = s_i(X_k,\hat{a}) \sum_{j=1}^{n} \sum_{l=1}^{m} r_{ij}(x_k,x_l) p_j(x_l). \tag{2.2}$$

To find a consistent labeling, we optimize the average local consistency, given by

$$A(\hat{a}) = \sum_{j=1}^{n} \sum_{k=1}^{m} p_i(x_k) s_i(X_k,\hat{a}), \tag{2.3}$$

which is a quadratic function which we optimize using a constrained gradient descent mathod, taking into account the restrictions of *Eq.* (2.1)



Figrue 1. Illustration of relative angle constraint.

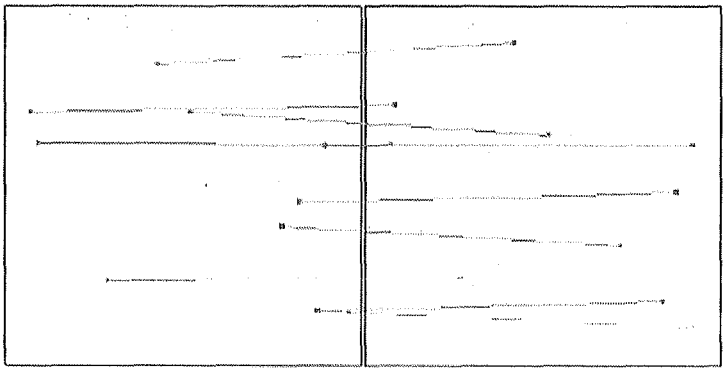


Figure 2. Matching of point sets using relative angle constraint.

3. Matching Sets of Points. In [1] road junctions were proposed as stable registration objects. The problem of comparing images with a *GIS* road database can then be represented as finding the correspondence between two sets of points: one originating from the image and one which can be extracted from the database. This can of course be applied to situations other than image-to-*GIS* registration (e.g. image-to-image or *GIS* -to-*GIS*).

The technique makes use of the spatial relations between points to find correspondences. This makes the correspondence technique less vulnerable for road junctions which are detected in the righ location but whose detected shape does not fully correspond to its counterpart in the database (defects can occur due to image noise). So instead of using features based on the shape of single crossroad to guide matching, we use geometric relations (e.g. angle, distance) between a crossroad and its neighbours to find correspondences matching tree [4]. These are much stabler features given the detection quality which we can realistically expect from line detection. In these experiments we rely only on the relative angle between pairs of points. Fig.1 shows an illustration of the constraint. The black points show object points i and the grey points show the lable points x. In mapping a pair of points i and j on x and x' the relative angle between the lines ij and xx' should lie whithin a margin $\Delta\alpha$ (e.g. $\pm\pi/4$). Although this is a weak constraint, it has been shown that the constraint is adequate to find correspondences between sets of points, disturbed by location noise [1].

4. Null Lable Assignment. To guarantee a good performance of the matching algorithm, the compatibility matrix $r_{ij}(X, X')$ needs to be determined correctly. In most applications, the value of these coefficients are determined using heuristics which basically impose a relative ordering on the restrictions. Stronger restrictions receive a higher absoulte value then weaker restrictions. The specific proportion of the value is usually determined through trial-and-error. For the null-label assignment it is however difficult to determine a correct value for the compatibility coefficient (noted as r_{ij}^{0}). Since each object is a priori a possible null

object, every assignment is consistent with the null assignment, i.e. $r_{ij}^{0}(X_0, X') > 0$ for $\forall j, X'$. The problem is to assess the relative importance of the null-label assignment with respect to the other restrictions. It should be avoided that the null solution is the most consistent solution of the system. On the other hand, false matchings of spurious points should be less consistent than the null assignment.

The definition of consistency can be used to determine the correct values. The definition not only determines the optimal solution of the labeling problem, it also determines what values the compatibility coefficients should take for an "ideal" solution to become the optimal solution of the system. The ideal solution is the matching we wish to find given the noise properties of the detection. For a correct null label assignment, we need to determine when the errors, which occur in the neighbour structure of a node, are that the null label should be assigned. To analyse this, we should look at the support of the different assignments. In the case of the null label assignment, the support can be written as:

$$S_i(X_0, \hat{a}) = \sum_{j=1}^{n} \sum_{x'=1}^{m+1} r_{ij}^{0}(X_0, X') p_i(x')$$

$$= r^0 \sum_{j \in \Omega} \sum_{x'=1}^{m+1} p_i(X') = r^0 d(i) \tag{4.1}$$

with $d(i)$ the degree of node i (i.e. the n number of neighbours). We have simplified $r_{ij}^{0}(X_0, X') = r_i^0$ if $j \in \Omega$ (else $r_{ij}^{0}(X_0, X') = 0$). The constant factor r^0 is reasonable in the absence of prior knowledge of assignments.

The support for a non-full label can be splite up into three classes, namely positive coefficients which express compatibility, negative coefficients which express incompatibility and negative coefficients which control the null assignment. If we consider the first two coefficients constant within the neighbourhood of node i, resp., then the support for c_i can be simplified to

$$S_i(X_i) = \sum_{j=1}^{n+} r_{ij}^{+}(X_i, X_j) + \sum_{j=n_+ +1}^{l} r_{ij}^{-}(X_i, X_j) + \sum_{j=n_l+1}^{n_0} r_i^0 \approx \hat{c}_i^{+} n_+ + \hat{c}_i^{-} n_- + r_i^0 n_0 \,. \tag{4.2}$$

Here n_+ is the number of compatible neighbours, n the number of incompatible neighbours and n_0 the number of nullneighbours, with $n_+ + n_- + n_0 = d(i)$. The number of non-full neighbours is given by $n_1 = n_+ + n_-$ Eq. (4.1) and (4.2) give the following condition which holds in the optimal solution:

$$\hat{c}_i^{+} n_+ + \hat{c}_i^{-} n_- + r_i^0 n_0 > r_i^0 d(i) \tag{4.3}$$

or equivalently

$$\left(1 - f^0\right) r_i^0 < f^+ r_i^+ + f^- r_i^- \tag{4.4}$$

where f^+, f^- and f^0 are the fractions of the number of compatible, incompatible

and null assignments in the neighbourhood of object i. Equation (4.4) can be used to determine the weights for the compatibility matrix given the expected graph error. Based on this expression a distinction can be made between allowable distortions, which should find a correspondent in the other dataset, and significant changes, which should be assigned the null label.

5. Applications

In [1], reports were given on artificially generated sets of points. We have investigated the problem of automatic registration of satellite the images with a remotely sensed *GIS* database based on feature points like road junctions. The road junctions can be manually selected or through automatic detection based on the results of [1] as shown in Fig. 3. In this example we work with manually selected junctions. 82 points were selected in the image. The equivalent region in the road database consists of 200 junctions. A rough initial tie point is givne between image map [5] and *GIS* data after which a search radius of 150 meters is defined around each point in which the corresponding point is sought. We have manually registered the 82 image points to the corresponding points in the dataset. The *RMS* on the position is 7 meters and the maximum error is 50 meters. Of the 82 points, 71 were correctly matched, 11 were assigned the null label and no false matching was performed. Based on the matching results, a rubber sheeting transformation is performed which compensates the local distortions between image and *GIS* data. Fig 4. shows the result of this registration.

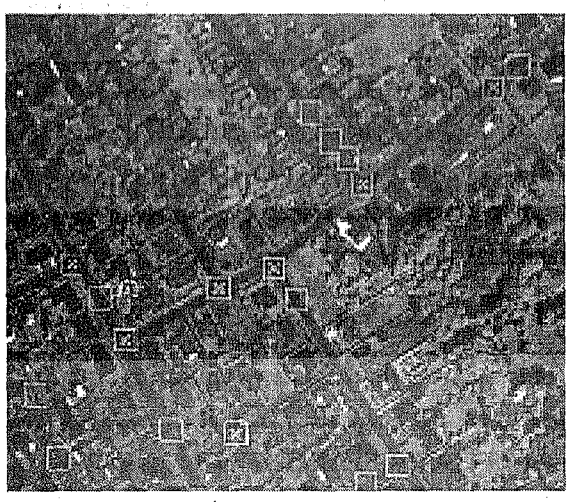


Figure 3. Example of automatic detection of road junctions in a panchromatic ikonos image (green crosses show detected junctions)

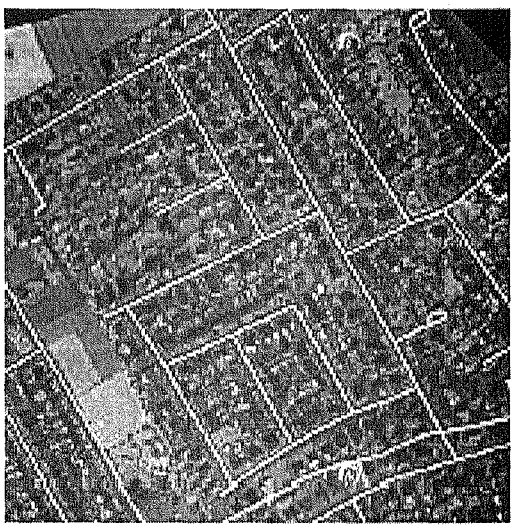


Figure 4. Results after rubbersheet transformation based on matching results.

6. Conclustions

We have proposed a methodology for comparing *VHR* images with remotely sensed *GIS* data based on the corresponding road junctions using continuous relxation labeling. A thorough analysis is given of the null label assignment, which leads to an expression on the optimal weights based on the expected graph error. The method allows for local distortions between two remotelly sensed datasets to be compensated and to distinguish these distortions from significant changes.

REFERENCES

[1] S. Gautama, A. Borghgraef and I. Bruyland, Automatic registration of satellite images with *GIS* database. *Proceedings Advanced Concepts for Intelligent Vision Systems ACIVS 2002, Gent. Belgie* **7** (2002).

[2] R. Hummel, S. Zucker, On the foundations of relaxation labelling processes. *IEEE Transactions Pattern Analysis Machine Intelligence.,* **5** No. 3 (1983), 742-776.

[3] D. Mount, N. Netanyahu, J. Letmoigne, Efficient Algorithms for Robust Point Pattern Matching and Applications to Image registration, *Pattern Recognition,* **32** (1999), 17-38.

[4] J. R. Quinlan, Induction of decision trees, *Machinel Learning* **1** (1986), 81-106.

[5] T. Kohonen, The Self-organizing map. *Procedings of the IEEE,* **78** (1990), 1464-1480.

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SOME RANDOM FIXED POINTS OF RANDOM MULTIVALUED OPERATORS ON PROBABILISTIC POLISH SPACES

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ABSTRACT

The main purpose is to obtain a random fixed point theorem for random multivalued operators generalized contraction in probabilistic polish spaces.

Key Words. Random fixed point; random multivalued operators; measurable mapping; probabilistic Polish space.

2000AMS Subject Classification. Primary 54H25; Secondary 47H10.

1. Introduction. Dedeic and Sarapa [4] proved some theorems on common fixed points for two commuting mappings on complete Menger space. They extended the Ciric's results, which proved the fixed point theorem for generalized contractions. As another consequence of his result, they proved several fixed point theorems for two commuting mappings on complete metric spaces. Dedeic and Sarapa [5] proved some theorems on common fixed point for a sequence of mappings on complete menger space. They proved several fixed point theorems for a sequence of continuous mappings. Dedeic and Sarapa [6] also proved and generalized several known results of Sehgal and Bharucha Reid [8] and their own result and established a theorem on common fixed point for three mappings on complete Menger space. Before establishing the theorem, the assumed that (X,F,t) is a complete Menger space with continuous T-norm t, where $t(a,b)=min(a,b)$ for $a,b \in [0,1]$.

In 1992, Beg and Azam [1] gave a stochastic version as: Let (X,d) be a polish space that is a separable complete metric space and $(\Omega,\mathfrak{A})$ be a measurable space. Let 2^X be the family of all subsets of X, $K(X)$ be the family of all non-empty closed subsets of X. Let $T:X \to CB(X)$ be a mapping and $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is said to be asymptotically T-regular with respect of F, if $d(Fx_n,Tx_n)\to 0$.

Beg and Azam [2] proved that a random coincidence points of compatible random coincidence points of compatible random multivalued operators. Mappings $T:X\to CB(X)$ and $F:X\to X$ are compatible if whenever there is a sequence in X

satisfying $lim\ Fx_n \in lim\ Tx_n$ (provided $lim\ Fx_n$ exist and $lim\ Tx_n$ exist in $CB(X)$), then $lim\ H(FTx_n, TFx_n)=0$ where H is Hausdorff metric space on $CB(X)$ induced by the metric space on $CB(X)$ induced by the metric d. Beg and Azam [3] generalized Jungck [9] gives random operators $f: \Omega\times X\to X$ and $T:\Omega\times X\to CB(X)$ are compatible if $f(\omega,.)$ and $T(\omega,.)$ are compatible for each $\omega\in\Omega$.

2. Random Fixed Point of Random Multivalued Generalized Contractions. Random fixed point theorems are stochastic generalization of classical fixed point theorem required for the theory of random equations. This section is in continuation of generalization of Beg and Shahzad [3], the existence of a common random fixed point of two random multivalued genralized contraction by using functional inequality.

3. Preliminaries. Let (X,d) be a polish space that is a separable complete metric space and $(\Omega,\mathfrak{A})$ be measurable space. The polish space (X,F) is said to be "Probabilistic Polish Space" if the following conditions hold : (i) $F_{(x,y)}(0)=0$; (ii) $F_{(x,y)}(t)=1$ iff $x=y$; (iii) $F_{(x,y)}(t)=F_{(y,x)}(t)$; (iv) $F_{(x,y)}(t_1)=1$ and $F_{(y,z)}(t_2)=1$, then $F_{(x,z)}(t_1+t_2)=1$. Condition (ii) is equivalent to the statement $x=y$ iff $F_{x,y}=H$. We remark that any metric space may be regarded as PM-space indeed, if (X,d) is a metric space, then we define for each x and y in X, the distribution function $F_{x,y}$ as follows : $F_{x,y}(t)=H(t-d(x,y))$. Let 2^X be a family of all subsets of X and $CB(X)$ denote the family of all non-empty bounded closed subests of X. A mapping $T:\Omega\to 2^X$, is called measurable if for any open subset C of X, $T^{-1}(C)=\{\omega\in\Omega: T(\omega)\cap C\neq\phi\}\in\mathfrak{A}$. A mapping $\xi:\Omega\to X$ is said to be measurable selector of a measurable mapping $T:\Omega\to 2^X$, if ξ is measurable and for any $\omega\in\Omega$, $\xi(\omega)\in T(\omega)$. A mapping $f:\Omega\times X\to X$ is called random operator if for any $x\in X$, $f(.,X)$ is measurable. A mapping $T:\Omega\times X\to CB(X)$ is a random multivalued operator if for every $x\in X$, $T(.,X)$ is measurable. A measurable mapping $\xi:\Omega\to X$ is called random fixed point of a random multivalued operator $T:\Omega\times X\to CB(X)$ $(f:\Omega\times X\to X)$ if for every $\omega\in\Omega$, $\xi(\omega)\in T(\omega,\xi(\omega)(f(\omega,\xi(\omega)))=\xi(\omega)$. Let $T:\Omega\times X\to CB(X)$ be a random operator and $\{\xi_n\}$ a sequence of meaurable mapping $\xi_n:\Omega\to X$. The sequence $\{\xi_n\}$ is said to be asymtotically T-regular if $d(\xi_n(\omega),T(\omega,\xi_n(\omega))\to 0$.

In 1992, Beg and Azam [1] proved a fixed point on asymptotically regular multivalued mappings. Recentrly Beg and Shahzad [3] further extended random fixed points of multivalued mapping satisfying more general contractive type conditions.

In the present paper we give generalized version of a result of Beg and Shahzad [3] in probabilistic polish space in the form of the following theorem :

4. Theorem. *Let (X,Ω) be a Probabilistic Polish Space. Let $S:\Omega\times X\to CB(X)$ be a continuous random multivalued operator, if there exists measurable mapping* $\alpha:\Omega\to(0,1)$:

$$F_{S(\omega,x),S(\omega,y)}(t) \geq F_{x,S(\omega,x)}(t/\alpha(\omega)) + F_{y,S(\omega,y)}(t/\alpha(\omega))$$

for all $x,y \in X$, $\omega \in \Omega$ and $\alpha(\omega)$ is non-negative $\alpha(\omega)<2$ with then there exists a random fixed point of S.

Proof. Let $\xi_0:\Omega \to X$ be an arbitrary measuable mapping and choose a measurable mapping $\xi:\Omega \to X$ such that $\xi_1(\omega) \in S(\omega,\xi_0(\omega))$ for each $\omega \in \Omega$ then for $\omega \in \Omega$

$$F_{S(\omega,\xi_0(\omega)),S(\omega,\xi(\omega))}(t) \geq F_{\xi_0(\omega),S(\omega,\xi_0(\omega))}(t/\alpha(\omega)) + F_{\xi_1(\omega),S(\omega,\xi_1(\omega))}(t/\alpha(\omega)).$$

It further implies that there exists a measurable mapping $\xi_2:\Omega \to X$ for any $\omega \in \Omega$, $\xi_2(\omega) \in S(\omega,\xi(\omega))$;

$$F_{\xi_1(\omega),\xi_2(\omega)}(t) = F_{S(\omega,\xi_0(\omega)),S(\omega,\xi(\omega))}(t),$$

$$F_{\xi_1(\omega),\xi_2(\omega)}(t) \geq F_{\xi_0(\omega),\xi_1(\omega)}(t/\alpha(\omega)) + F_{\xi_1(\omega),\xi_2(\omega)}(t/\alpha(\omega)),$$

$$F_{\xi_1(\omega),\xi_2(\omega)}(1-t/\alpha(\omega))t \geq F_{\xi_0(\omega),\xi_1(\omega)}(t/\alpha(\omega))$$

$$\Rightarrow F_{\xi_1(\omega),\xi_2(\omega)}(t) \geq F_{\xi_0(\omega),\xi_1(\omega)}(t/(\alpha(\omega)-1))$$

$$\Rightarrow F_{\xi_1(\omega),\xi_2(\omega)}(t) \geq F_{\xi_0(\omega),\xi_1(\omega)}(t/k), \qquad \text{(I)}$$

where $k=\alpha(\omega)-1<1$.

There exists a measurable mapping $\xi_3:\Omega \to X$ such that for any $\omega \in \Omega$, $\xi_3(\omega) \in S(\omega,\xi_2(\omega))$ and

$$\begin{aligned} F_{\xi_2(\omega),\xi_3(\omega)}(t) &= F_{S(\omega,\xi_1(\omega)),S(\omega,\xi_2(\omega))}(t) \\ &\geq F_{\xi_1(\omega),S(\omega,\xi_1(\omega))}(t/\alpha(\omega)) + F_{\xi_2(\omega),S(\omega,\xi_2(\omega))}(t/\alpha(\omega)) \\ &\geq F_{\xi_1(\omega),\xi_2(\omega)}(t/\alpha(\omega)) + F_{\xi_2(\omega),\xi_3(\omega)}(t/\alpha(\omega)) \end{aligned}$$

$$\Rightarrow F_{\xi_2(\omega),\xi_3(\omega)}(1-1/\alpha(\omega)) \geq F_{\xi_1(\omega),\xi_2(\omega)}(t/\alpha(\omega))$$

$$\Rightarrow F_{\xi_2(\omega),\xi_3(\omega)}(t) \geq F_{\xi_1(\omega),\xi_2(\omega)}(t/(\alpha(\omega)-1)).$$

Then by (I)

$$\begin{aligned} F_{\xi_1(\omega),\xi_2(\omega)}(t/(\alpha(\omega)-1)) &\geq F_{\xi_0(\omega),\xi_1(\omega)}(t/(\alpha(\omega)-1)^2) \\ &\geq F_{\xi_0(\omega),\xi_1(\omega)}(t/k^2). \end{aligned}$$

Then we have

$$F_{\xi_2(\omega),\xi_3(\omega)}(t) \geq F_{\xi_1(\omega),\xi_2(\omega)}(t/(\alpha(\omega)-1)) \geq F_{\xi_0(\omega),\xi_1(\omega)}(t/(\alpha(\omega)-1)^2)$$

$$\Rightarrow F_{\xi_2(\omega),\xi_3(\omega)}(t) \geq F_{\xi_1(\omega),\xi_2(\omega)}(t/k) \geq F_{\xi_0(\omega),\xi_1(\omega)}(t/k^2).$$

Similarly proceeding in the same way induction we produce a sequence of measurable mapping $\xi_n:\Omega \to X$ and

$$F_{\xi_n(\omega),\xi_{n+1}(\omega)}(t) \geq F_{\xi_0(\omega),\xi_1(\omega)}(t/k^n)$$

Further more for $m>n$

$$F_{\xi_n(\omega),\xi_m(\omega)}(t) \geq F_{\xi_n(\omega),\xi_{n+1}(\omega)}(t) + F_{\xi_{n+1}(\omega),\xi_{n+2}(\omega)}(t) + \ldots + F_{\xi_{m-1}(\omega),\xi_m(\omega)}(t)$$
$$\geq F_{\xi_0(\omega),\xi_1(\omega)}\left(1/k^n + 1/k^{n+1} + \ldots + 1/k^{m-1}\right)(t)$$
$$\geq F_{\xi_0(\omega),\xi_1(\omega)}\left\{\left(1 + 1/k + 1/k^2 + \ldots + 1/k^{m-n-1}\right)/k^n\right\}(t)$$
$$\geq F_{\xi_0(\omega),\xi_1(\omega)}\left((1-k)/k^n\right)(t) \to 1-\lambda$$

i.e. $F_{\xi_n(\omega),\xi_m(\omega)}(t) \to 1-\lambda$.

It follows that $\{\xi_n(\omega)\}$ is a Cauchy sequence for $\gamma > 0$ and any $\omega \in \Omega$, $\xi_{2\gamma+1}(\omega) \in S(\omega, \xi_{2\gamma}(\omega))$ then there exists a measurable mapping $\xi:\Omega \to X$ such that $\xi_n(\omega) \to \xi(\omega)$ for each $\omega \in \Omega$. It further implies that $\xi_{2\gamma+1}(\omega) \to \xi(\omega)$

$$F_{\xi(\omega),S(\omega,\xi(\omega))}(t) \geq \left[F_{\xi(\omega),\xi_{2\gamma+1}(\omega)}(t), F_{\xi_{2\gamma+1}(\omega),(\omega,\xi(\omega))}(t)\right]$$
$$\geq \left[F_{\xi(\omega),\xi(\omega)}(t), F_{\xi(\omega),(S(\omega),\xi(\omega))}\right](t) \to 0$$

i.e. $\{\xi_{2\gamma+1}\}$ is an asymptotically S-regular.

Consider

$$F_{\xi(\omega),S(\omega,\xi(\omega))}(t) \geq F_{\xi(\omega),\xi(\omega)}(t/\alpha(\omega)) + F_{\xi(\omega),S(\omega,\xi(\omega))}(t/\alpha(\omega))$$
$$\Rightarrow F_{\xi(\omega),S(\omega,\xi(\omega))}(1 - 1/\alpha(\omega))t \geq 1$$
$$\Rightarrow F_{\xi(\omega),S(\omega,\xi(\omega))}(t) \geq \alpha(\omega)/(\alpha(\omega) - 1)$$
$$\Rightarrow \xi(\omega) \in S(\omega, \xi(\omega))$$

i.e. $\xi(\omega)$ is a fixed point of S.

REFERENCES

[1] I. Beg and A. Azam, Fixed point of asymptotically regular multivalued mappings, *J. Austral. Math. Soc. (Ser A) Vol.* **53** (1992), 313-326.

[2] I. Beg and A. Azam, Common fixed points for commuting and compatible maps, *Discussion Mathematical Differential Inclusion,* **16** (1996), 121-135.

[3] I. Beg and N. Shahzad, Random fixed point theorems on product spaces, *J. Appl. Math. and Stochastic Analysis,* **6** (1993), 95-106.

[4] R. Dedeic and N. Sarapa, On common fixed point theorems for commuting mappings on Menger spaces, *Radovi Mathematicki,* **4** (1998), 269-278.

[5] R. Dedeic and N. Sarapa, Fixed point theorems for a sequence of mappings on Menger spaces, *Math. Japonica,* **34** (1989), 525-539.

[6] R. Dedeic and N. Sarapa, A common fixed point theorem for three mappings on Menger space, *Math. Japonica,* **34** (1989), 919-923.

[7] G. Jungck, Common fixed points for commuting and compatible maps on compacta, *Proc. Amer. Math. Soc.,* **103** (3) (1988), 977-983.

[8] V.M. Sehgal and A.T. Bharucha-Reid, Fixed points of contraction mappings on Probabilistic metric spaces, *Mathematicae, System Theory,* **6** (1972), 97-102.

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UNSTEADY STATE RADIAL HEAT FLOW IN THE OUTER LAYERS OF SPHERICAL REGIONS OF HUMAN OR ANIMAL BODY

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ABSTRACT

Exact solution has been obtained using Laplace transform for one dimensional unsteady state heat migration problem in annular layers in spherical regions of a human or animal body under different atmospheric and physiological conditions assuming that skin surface is exposed to the environmental and loss of heat from the surface is due to radiation, convection and evaporation. The mathematical model incorporates the variations of blood mass flow rate and metabolic heat generating in different layers taking metabolic heat generation tissue temperature dependent.

Key Words. Stratum Corneum, Stratum Germinativum, Blood Mass Flow Rate, Metabolic Heat Generation, Laplace Transform.

1. Introduction. The body core temperature remains almost constant under normal conditions. The skin and underlying tissue layers undergoes temperature variation due to change in physiological parameters and environmental conditions in order to maintain a uniform body core temperature. The skin mainly consists of two layers, epidermis and dermis. The epidermis is composed of a living stratum germinativum which rests upon the dermis, and a dead, horny, superficial stratum corneum. The dermis is composed of matted masses of connective tissues and elastic fibres through which pass numerous bleed vessels, lymphatics and nerves. There are no blood vessels in the epidermis (Fig. 1). The population density of blood vessels in the dermis is very thin near the interface of epidermis and dermis, but increases gradually becoming almost uniform in the sub-dermal part. This help to understand the variation of quantities like rate of metabolic heat generation, rate of blood mass flow and thermal conductivity of tissue in this region with respect to its position.

In the present study, it is assumed that outer skin surface is exposed to atmosphere and the loss of heat is due to convection, radiation and evaporation of sweat. We consider only those part of human or animal body which are almost

spherical in s irection only. We divide the re

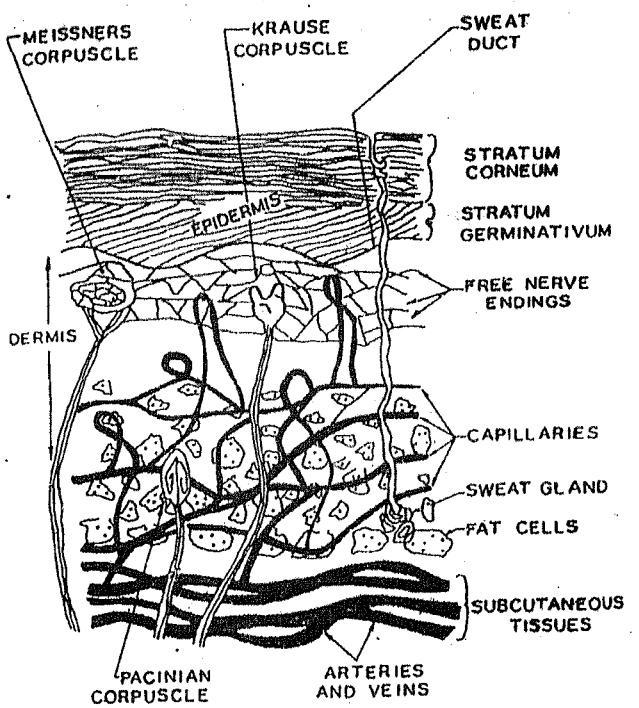

Fig 1. *A section through hairless skin showing epidermis, dermis and part of subcutaneous tissues*

its structure. It is assumed that rate of blood mass flow and rate of metabolic heat generation are position dependent which are negligible (zero) in epidermis vary linearly in dermis and constant in the sbdermal part. Further the rate of metabolic heat generation depends on the tissue temperature and its variation is in a manner that makes it self controlled. The thermal conductivity is taken as constant but different in three layers.

Mathematical Formualtion. The general mathematical model for the heat flow in skin and underlying tissues is given by

$$div[K\ gard\ T] + M(T_b - T) + S = \rho c(\partial T/\partial t) \qquad ...(1)$$

and $$-K(\partial T/\partial n) = h(T - T_a) + LE \qquad ...(2)$$

at the skin surface, here $M = m_b c_b$.

The inner boundary is taken at the interface of subdermal tissues and body core, where

$$T = T_b = 37.2^0\text{C}. \qquad ...(3)$$

Here K is the thermal conductivity of the tissue, m_b the rate of blood mass flow, c_b the specific heat of the blood. T_b the body core temperature, S the rate of metabolic heat generation ρ the tissue density, c the specific heat of the tissue, h is the heat transfer coefficient, L the latent heat of evaporation, E the rate of evaporation, T_a the atmospheric temperature and $\partial T/\partial n$ the partical derivative of T along normal to the boundary. We apply equation (1) and (2) to one dimensional unsteady state case taking skin surface insulated at time $t=0$.

Pearl [1] and Cooper and Trezek [2] have studied solution of some simple problems of infinite tissue media assuming all the parameters as constant throughout the region. Saxena [3] and Saxena and Arya [4] applied analytical and numerical methods to find solutions of certain unsteady state and steady state problems of temperature distribution in skin and subcutaneous tissues. Saxena [5] has studied the heat transport in peripheral layers of human body under normal and abnormal conditions. Some related work has been done by Pardasani and Saxena [6], Saxena, Juneja and Yadav [8] and Saxena, Pardasani and Saxena [7].

Solution of the Problem. The equations (1) and (2) are transformed into spherical polar coordinates for a one-dimensional unsteady state case to obtain

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(Kr^2(\partial T/\partial r)\right)+M(T_b-T)+S=\rho c(\partial T/\partial t) \qquad ...(4)$$

and $$-K(\partial T/\partial r)=h(T-T_a)+LE \qquad ...(5)$$

at the skin surface

In equation (4) the parameter $M(=m_b c_b)$ and S are assumed to be position dependent. The following assumptions have been made for each layer when the total thickness of the region under consideration is $r_0 \le r \le r_3$:

(i) For Epidermis

$K=K_1,\ M_1=0,\ S_1=0$

The thickness of epidermis is $r_2 \le r \le r_3$

(ii) For Dermis

$K=K_2,\ M_2=m[(r_2-r)/(r_2-r_1)]$

$S_2=S\ [(r_2-r)/(r_2-r_1)][1+q_d(T_b-T_2)]$,

where $q_d=2/(T_a+T_b)$. The thickness of dermis is $r_1 \le r \le r_2$.

At interface of epidermis and dermis the temperature is same i.e. $T_1=T_2$ where T_1 is temperature at r_2.

(iii) For Subdermal Tissues

$K=K_3,\ M_3=m,\ S_3=S[1+q_s(T_b-T_3)]$,

where $q_s=1/T_b$. The thickness of subdermal layer is $r_0 \le r \le r_1$ and the temperature

at interface of subdermal dermal layers is same i.e. $T_2=T_3$ if T_2 is temperature at r_1.

Now taking Laplace transformation with respect to t on both sides of equation (4), we get

$$d/(r^2dr)(Kr^2(d\overline{T}/dr))+M(T_b/p-\overline{T})+S/p=\rho cp\overline{T}-\rho cpT(0,r). \quad ...(9)$$

The initial condition is given by

$T(0,r)=T_b$.

Here $\overline{T}(r,p)$ is Laplace Transform of $T(r,t)$.

Now applying the initial condition, equation (9) becomes

$$d^2\overline{T}/dr^2+(2/r)(d\overline{T}/dr)-(M+\rho cp)\overline{T}/K=-(MT_b+\rho cpT_b+S)/(pK). \quad ...(10)$$

The solution of equation (10) is

$$\overline{T}(r,p)=\frac{1}{r}\left(Ae^{\alpha r}+Be^{-\alpha r}\right)+\frac{MT_b+\rho cpT_b+S}{p(M+\rho cp)}, \quad ...(11)$$

where $\alpha=\sqrt{(M+\rho cp)/K}$.

Now taking Laplace transformation with respect to t on both sides of equation (3) and equation (5) we get

$$\overline{T}(r_3,p)=T_a/p-LE/(hp) \quad ...(12)$$

at skin surface,

and

$$\overline{T}(r_0,p)=T_a/p \quad ...(13)$$

at inner boundary.

Applying boundary conditions (12) and (13) to equation (11) and substituting the value of α we get

$$\overline{T}(r,p)=\frac{r_3}{r}\left[\left(T_a-T_b-\frac{LE}{h}-\frac{S}{M}\right)\frac{1}{p}\left\{\frac{sinh(p+M/\rho c)^{1/2}(\rho c/K)^{1/2}(r-r_0)}{sinh(p+M/\rho c)^{1/2}(\rho c/K)^{1/2}(r_3-r_0)}\right\}+\frac{S}{M(p+M/\rho c)}\right.$$

$$\left.\left\{\frac{sinh(p+M/\rho c)^{1/2}(\rho c/K)^{1/2}(r-r_0)}{sinh(p+M/\rho c)^{1/2}(\rho c/K)^{1/2}(r_3-r_0)}\right\}\right]+\frac{Sr_0}{Mr}\left(\left\{\frac{sinh(p+M/\rho c)^{1/2}(\rho c/K)^{1/2}(r_3-r)}{p\,sinh(p+M/\rho c)^{1/2}(\rho c/K)^{1/2}(r_3-r_0)}\right\}\right)$$

$$-\frac{1}{(p+M/\rho c)}\left\{\frac{sinh(p+M/\rho c)^{1/2}(\rho c/K)^{1/2}(r_3-r)}{sinh(p+M/\rho c)^{1/2}(\rho c/K)^{1/2}(r_3-r_0)}\right\}\Bigg]+\frac{(M+\rho cp)T_b+S}{p(M+\rho cp)} \quad ...(14)$$

Now taking inverse Laplace transformation on both side of equation (14) we obtain

$$T(r,t)=\frac{4(K\rho c)^{1/2}}{r}\left[r_3\left\{\left(T_a-T_b-\frac{LE}{h}\right)\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{M/\rho c+2\pi^2Kn^2}\right)\times\right.\right.$$

$$\left.\left(e^{-(M/\rho c+2\pi^2Kn^2)t}-1\right)n\sin(2n\rho c(r-r_0)(r_3-r_0))\right\}+(S/M)r_3\times\left\{\sum_{n=1}^{\infty}(-1)^n\right.$$

$$\left.\left(\frac{Me^{-(M/\rho c+2\pi^2Kn^2)t}}{2\pi^2Kn^2\rho c(M/\rho c+2\pi^2Kn^2)}-\frac{e^{-(M/\rho c)t}}{2\pi^2Kn^2}+\frac{1}{M/\rho c+2\pi^2Kn^2}\right)\times n\sin(2n\rho c(r-r_0)(r_3-r_0))\right\}$$

$$+\frac{S}{M}\left\{\sum_{n=1}^{\infty}(-1)^n\left(\frac{Me^{-(M/\rho c+2\pi^2Kn^2)t}}{2\pi^2Kn^2\rho c(M/\rho c+2\pi^2Kn^2)}-\frac{e^{-(M/\rho c)t}}{2\pi^2Kn^2}+\frac{1}{M/\rho c+2\pi^2Kn^2}\right)\right.$$

$$\left.\left.n\sin(2n\rho c(r_3-r_0)(r_3-r_0))\right\}\right]+T_b+S/M-(S/M)e^{-(M/\rho c)t}. \qquad ...(15)$$

The equation (15) gives the value of temperature distribution in different layers of skin and subcutaneous tissue (*SST*) region.

Numerical Results and Discussion. The numerical results have been obtained using the following values (Saxena and Bindra [9])
$K_1=0.030$ *cal cm*$^{-1}$ *min*$^{-1}$ $^0C^{-1}$, $K_2=0.045$ *cal cm*$^{-1}$min$^{-1}C^{-1}$, $K_3=0.060$ *cal cm*$^{-1}$ *min*$^{-1}$ $^0C^{-1}$, $L=579$ *cal/g*, $h=0.009$ *cal cm*$^{-2}$ *min*$^{-1}$ $^0C^{-1}$, $\rho=1.05$ *g/cm*^{2}, $c=0.83$ *cal/g*.
The numerical calculations have been made for three cases of atmospheric temperatures and the values of m, S and E have been taken accordingly as follows:

(i) $T_a=15^0C$, $m=m_bc_b=0.003$ *cal cm*$^{-3}$ *min*$^{-1}$ $^0C^{-1}$,
$S=0.0357$ *cal cm*$^{-3}$ *min*$^{-1}$ and $E=0$ *g cm*$^{-2}$ *min*$^{-1}$

(ii) $T_a=23^0C$, $m=m_bc_b=0.018$ *cal cm*$^{-3}$ *min*$^{-1}$ $^0C^{-1}$,
$S=0.018$ *cal cm*$^{-3}$ *min*$^{-1}$ and $E=0$, 0.24×10^{-3}, 0.48×10^{-3} *g cm*$^{-2}$ *min*$^{-1}$

(iii) $T_a=33^0C$, $m=m_bc_b=0.0315$ *cal cm*$^{-3}$ *min*$^{-1}$ $^0C^{-1}$,
$S=0.018$ *cal cm*$^{-3}$ *min*$^{-1}$ and
$E=0.24\times10^{-3}$, 0.48×10^{-3}, 0.72×10^{-3} *g cm*$^{-2}$ *min*$^{-1}$.

Depending upon the structure of skin and underlying tissues the constant $r_i(i=0,1,2,3)$ are assigned following values

$r_0=8cm$, $r_1=8.5cm$, $r_2=8.9cm$, $r_3=9.1cm$.

We have taken total thickness of skin 1.1*cm*.
The physical parameter rate of metabolic heat generation has been taken as self controlled, described by factor $q_d(T_b-T)$ where q_d depends on the atmospheric temperature and body core temperature. As the temperature changes, the thermoreceptors present in the skin send information to the hypothalamus centres

which in turn controls the rate of blood mass flow and rate of metabolic heat generation. At low temperature for e.g. at 15^0C, the blood vessels constrict, causing low blood mass flow rate and q_d and (T_b-T) increases, causing increase in the rate of metabolic heat generation. Therefore the structure of skin and underlying tissues decreases the heat carried by blood to the surface and increases the rate of metabolic heat generation to regulate the body core temperature at low atmospheric temperature.

At higher atmospheric tgemperature for e.g. at 33^0C these process are reversed that is more heat is carried by the blood to the surface due to increased blood flow rate. The metabolic heat generation also decreases in order to regulate the body core temperature. In spite of these processes, the rate of sweat evaporation at the skin surface also play an important role in heat regulation in *SST* region.

The time factor also play an imporatnat role in order to regulate the body temperature. Initially when blood enters in the body tissue (i.e. at time zero) the temperature of the *SST* region is same as the blood temperature (i.e. 37.2^0C) because no heat loss takes place at time zero but after some time (say t) the skin temperature varies in different layers i.e. in epidermis, dermis and subcutaneous part of skin. The body core temperature remains constant i.e. 37.2^0C. The temperature of region changes up to 10 min and very small change is observed upto 20 and 30 minute but after that it will remain nearly constant. The values of temperatures in *SST* region at time 0.5 minute for different values of T_a, E are r_i $(i=0,1,2,3)$ are given in table-I

Table -I

Temperature of SST region at time 0.5 min. at different values of atmospheric temperature and for different values of E

Atmospheric temperature →		$T_a=15^0C$	$T_a=23^0C$ E in $gcm^{-2}min^{-1}$			$T_a=33^0C$, E in $gcm^{-2}min^{-1}$		
	Radial Distance	$E=0$	0	0.24×10^{-3}	0.48×10^{-3}	0.24×10^{-3}	0.48×10^{-3}	0.72×10^{-3}
Temperature of SST region for r_0, r_1, r_2 and r_3 →	r_3=9.1cm	27.289750	30.861011	23.968477	17.075942	28.432554	21.540019	14.647485
	r_2=8.9cm	33.050795	34.546008	31.660251	28.774499	33.539262	30.787608	27.757756
	r_1=8.5cm.	35.839102	36.337438	35.375649	34.413860	35.998570	35.036781	34.074992
	r_0=8.0cm.	37.20	37.20	37.20	37.20	37.20	37.20	37.20

Fig 2 is drawn for different values of temperature and radial distance given in Table -I

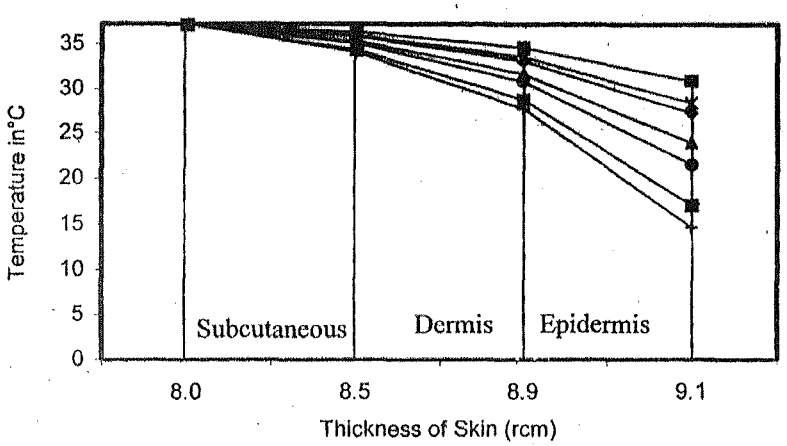

Fig 2.

In the above figure (Fig. 2) the first, fourth and sixth curves from top to bottom are for $T_a=23^0C$ and $E=0$, 0.24×10^{-3} and 0.4×10^{-3} $gcm^{-2}m^{-1}$, second, fifth and seventh curves from top to bottom are for $T_a=33^0C$ and $E=0.24\times10^{-3}$, 0.48×10^{-3} and 0.72×10^{-3} gcm^{-2} min^{-1} and third curve from top to bottom is for $T_a=15^0C$ and $E=0$.

From Fig. 2 we make the following observation :

We see in Fig.2 that the gaps between the curves for different values of E at each T_a shows that the rate of evaporation has significant effect on temperature distribution and this effect decreases as we move towards the body core. The skin surface temperature for $T_a=15^0C$ and $E=0$ is less than that when $T_a=23^0C$ and $E=0$ because the rate of blood mass flow is much less than at $T_a=15^0C$ then that at $T_a=23^0C$. Lastly the skin surface temperature for some evaporation rate is lower at $T_a=23^0C$ than that at $T_a=33^0C$. This may be because of the effect of lower atmospheric temperature and lower rate of blood mass flow at $T_a=23^0C$.

REFERENCES

[1] W. Perl, Heat and matter distribution in body tissues and determination of tissue blood flow by local clearance methods. *J. Theor. Biol.* **2** (1962), 201-235.

[2] T.E. Cooper and G.J. Tezek, A prose technique for determining the thermal conductivity of tissue *J. Heat Trans, ASME* **94** (1972), 133-140.

[3] V.P. Saxena, Application of similarity transformation to unsteady state heat migration problems in human skin and subcutaneous tissues. *Proc. VI Int. Heat Tr. Conf.* (Canada) (1978) 65-68.

[4] V.P. Saxena and D. Arya, Variational finite element approach to heat distribution problems in human skin and subdermal tissues. *Proc. I. Int. Conf. Numerical Methods in Thermal Problems* (ed.K. Morgan (U.K: Pineridge Press) (1981), 1067-1076.

[5] V.P. Saxena, Mathematical study of heat transport in peripheral layers of human body under normal and abnormal conditions. Proceedings of 8th *Int. Congress of Biomathematics,* Panama 1997, 122.

[6]. K.R. Pardasani and V.P. Saxena, Exact solutions to temperature distribution problems in annular skin layers, *Bull. Cal. Math. Soc.* **81** (1989), 1-8.

[7] V.P. Saxena, K.R. Pardasani and P.M. Saxena, Exact solution to temperature distribution problems in spherical tissue layers, *Ganita Sandesh* **5** (1991), 272-278.

[8] V.P. Saxena, A. Juneja and A.S. Yadav, Heat and water transport in thermally damaged skin, *Proc. Nat. Acad. Sci. India*, **74** (A), II, 2004.

[9] V.P. Saxena and J.S. Bindra, Steady state temperature distribution in dermal regions of human body with variable blood flow, perspiration and self-controlled metabolic heat generation, *Indian J. Pure Appl. Math.*, **15** (1984), 31-42.

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SOLUTION OF THE DETERMINISTIC EPIDEMIOLOGICAL MODEL IN TERMS OF THE HYPERGEOMETRIC FUNCTIONS ${}_0F_1$

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ABSTRACT

An Susceptible Infective model (*SI*) for a communicable disease is considered here and solved analytically. The bsic feature of the model is that it employes more realistic assumption regarding infective contact rate which is assumed to be variable and population dependent as distinguished from the classical models in which it was assumed to be constant. The model is solved in terms of the function ${}_0F_1$ which is a special case of the hypergeometric function ${}_2F_1$.

Key words . Confluent hypergeometric function, hypergeometric functions, Epidemiology, Susceptible Infective.

1. Introduction. Epidemiology as a scientific discipline has been fron *t*-runner in the field of studies generally termed as population dynamics. An epidemic usually described as occurrence of a disease in excess to normal expectancy. Since contagiousness is one of the main causes of spread of such epidemics, the term epidemiology has been applied to general study and devising measures of controlling all communicable disease. Preoccupation with prevention and cure of infectious diseases has been one of the basic features of researches in medical sciences, since the middle ages. This has resulted in invention of various vaccines and drugs and most of killer diseases such as Plague, Cholera, Smallpox etc. are more or less eradicated from almost entire globe. Still situation in certain under-developed nations is as grim as it was in the middle ages. Moreover the whole world is in the grip of fear of a new epidemic in the form of *AIDS* (Acquired Immune Deficiency Syndrome). Hence Mathematical modeling assumes greater significance in epidemiological studies. As in the case with most of the mathematical modeling such a study should begin with identifying all the significant parameters of the problem.

The spread of an infectious diesease in a host population depends on a number of disease related factors e.g. infectious agent mode of transmission, susceptibility and resistance of the host, incubation peirod, the infectious period

etc. There are several extraneous considerations such as social, economic, geographic and demographic fators which also play significant role in this respect. A meaningful communicable disease model can be constructed only if these factor are incorporated in it. Depending on population size, the epidemiological models can be categorised in to two categories: Deterministic Models and Stochastic models.

The deterministic models involving differential equations were described and several threshold theorems were obtained by Kermack and Mckendriek [3,4]. A landmark was achieved in the field of mathematical theory of epidemiology with publication of the monumental work in the form of a book N.T.J. Bailey [1] dealing with both deterministic and stochastic models. A survey of results in epidemiology obtained till 1967 has been detailed by K. Dietz [2]. Recent decades have witnessed enormous growth in mathematical modeling for epidemics.

2. Mathematical Formulation of Model. The general model for a communicable disease (fatal or otherwise) in which an infected person does not recover is known as *SI* model. In this model at a time 't', total population '$N(t)$' is divided into two disjoint classes namely the infective class '$I(t)$' consisting of totality of infected individuals who can transmit disease and the suceptible class '$S(t)$' of individuals who can incur disease by contact the infected individuals.

The population size '$N(t)$' is so large that it can be considered as a continuous variable of time. The population in changing on account of immigration, births, emigration and deaths (due to disease in question or other causes). Let 'β' be the rate at which population is receiving new individuals due to immigration and birth and 'μ' be the rate at which individuals are being removed on account of emigration and death. Hence all the new entrants are assumed to be susceptible.

The population is assumed to be uniform or homogeneous. The daily contact rate '$C(N)$' is assumed to vary linearly with population and is taken as 'δN'. The rate at which total number of susceptible are infected by class I of infective can be taken as

$$(I\delta N)S/N = \delta IS$$

The initial value problem for the *SI* model can be put as follows:

$$ds/dt = \beta N - \mu S - \delta IS,$$

$$dS/dt = \delta IS - \mu I,$$

$$N = I + S,$$

$$I(0) = I_0 > 0;\ S(0) > 0;\ \text{and}\ N(0) > N_0 > 0. \tag{2.1}$$

The rate of total number of susceptible transferred to infective class is 'δIS'. Total population removed per unit time on account of emigration and death is 'μN'. It is divided into two classes namely suscepible class and infective class in proportion to number of individuals in those classes. All newborns and immigrants

are susceptible.

The fact account for the first term 'βN' in expression for dS/dt.

3. Solution of the Model Equation. The solution of the model is given by system of equations (2.1) can be effected as follows. On adding first two equations of the system (2.1) and employing the relation $N=I+S$, we obtain

$$N' = (\beta - \mu)N. \tag{3.1}$$

Here prime denotes differentiation with respect to t.

This equation has solution

$$N = N_0 e^{(\beta-\mu)t}. \tag{3.2}$$

Employing this value of N in the first equation in (2.1), we get

$$S' = \beta N - \mu S - \delta S(N - S)$$

or

$$S' = \delta S^2 - \left(\mu + \delta N_0 e^{(\beta-\mu)t}\right) + \beta N_0 e^{(\beta-\mu)t}. \tag{3.3}$$

Let us choose 'u' such that

$$\delta S u + u' = 0.$$

Making this substitution in equation (3.3) and simplifying we obtain

$$u'' + \left(\mu + \delta N_0 e^{(\beta-\mu)t}\right)\mu' + \beta N_0 e^{(\beta-\mu)t}\mu = 0 \tag{3.4}$$

Using relations (3.1) and (3.2) equation (3.4) takes the form

$$(\beta-\mu)^2 N^2 \frac{d^2u}{dN^2} + (\mu + \delta N)N(\beta-\mu)\frac{du}{dN} + \delta\beta N u = 0. \tag{3.5}$$

Putting $M = \{\delta/(\beta-\mu)\}\, N$, equation (3.5) becomes

$$M\frac{d^2u}{dM^2} + \left\{\frac{\mu}{(\beta-\mu)} - M\right\}\frac{du}{dM} - \delta u = 0. \tag{3.6}$$

Differential equation (3.6) is the well known Confluent Hypergeometric equation [6], whose general solution is:

$$u = C_1\, {}_1F_1\left(\delta; \frac{\mu}{(\beta-\mu)}; M\right) + u + C_2 M^{1-\mu/(\beta-\mu)}\, {}_1F_1\left(1+\delta-\frac{\mu}{(\beta-\mu)}; 2-\frac{2}{(\beta-\mu)}; M\right) \tag{3.7}$$

where C_1 and C_2 are arbitrary constants and $\mu/(\beta-\mu)$ is non-integral.

In (3.6) put $\mu/(\beta-\mu) = 2\delta$, $M=2Z$ and $u=e^z w$, the result is [6],

$$zw'' + 2aw' - zw = 0 \tag{3.8}$$

of which one solution must be $w = e^{-z}\, {}_1F_1(\delta; 2\delta; 2z)$.

In equation (3.8) change the independent variable to $\sigma = z^2/4$ and thus arrive at the equation

$$\sigma^2 \frac{d^2w}{d\sigma^2} + (\delta + 1/2)\sigma^2 \frac{dw}{d\sigma} - \sigma w = 0 \tag{3.9}$$

or

$$[\theta(\theta+\delta+1/2-1)-\sigma]w=0, \theta=\sigma d/d\sigma \tag{3.10}$$

Equation (3.10) is a differential equation for the ${}_0F_1$ function with denominator parameter $(\delta+1/2)$ and argument $\sigma=z^2/4$. Hence if $\delta+1/2$ is non integral (That is if 2δ is not an odd integer, the general solution of equation (3.10) is

$$w=A\,{}_0F_1\left(-;\delta+1/2;z^2/4\right)+\beta\left(z^2\right)^{1/2-\delta}{}_0F_1\left(-;3/2-\delta;z^2/4\right) \quad ...(3.11)$$

Hence the solution of equation (3.6) can be obtained in the form

$$u=e^{M/2}\left\{A\,{}_0F_1\left(-,\frac{\mu}{2(\beta-\mu)}+\frac{1}{2};\frac{M^2}{16}\right)+B\left(M^2/4\right)^{\left(\frac{1}{2}-\frac{\mu}{2(\beta-\mu)}\right)}{}_0F_1\left(-,\frac{3}{2}-\frac{\mu}{2(\beta-\mu)};\frac{M^2}{16}\right)\right\} \tag{3.12}$$

where A and B are arbitrary constants.

4. Conclusion. The exact solution obtained the hyper form of hypergeometric functions can be employed to predict the number of infective in a population in the grip of a contagious disease in which infected population does not recover, if the numerical values of the parameters are known. Hence if the infective class is to be reduced to negligible proportions after a given time span, corresponding value of contact rate can be computed and if need be strong measures such as quarantine be imposed to bring the contact rate to the requisite level.

REFERENCES

[1] N.T. J. Baily, *The Mathematical Theory of Epidemics*, Ist Edition Griffin, London (1957).

[2] K. Dietz, Epidemics and Tumours : a survey, *J. Roy. Stat. Soc. A.*, **130** (1967) 505-538.

[3] W.O. Kermack, and A.G. Mc-Kendrik, Introduction to the Mathematical Theory of Epidemics, Part I, *Proc. Roy. Soc. London. Sect. A.*, **115** (1927), 700-721.

[4] W.O. Kermack and A.G. Mc-Kendrik, Introduction to the Mathematical Theory of Epidemics, Part I, *Proc. Roy. Soc. London, Sect. A.*, (1927), 138, 55-63.

[5] A.M. Mathai and R.K. Saxena, *H-Function with Applications in Statistics and Other Disciplines.* Wiley Eastern Ltd. (1978).

[6] E.D. Rainville, *Special Functions*, MacMillan, Chelsea Pub. Co. Bronx, New York, (1971).

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ON DECOMPOSITION CURVATURE TENSOR FIELDS IN A TACHIBANA RECURRENT SPACE

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ABSTRACT

In the present paper, we prove some theorems on decomposition curvature tensor fields in a Tachibana recurrent space.

1. Introduction. A Tachibana space T_n is a Riemannian space, which admits a structure tensor field F_i^h satisying the conditions [4]

$$F_j^h F_h^i = \delta_j^i, \qquad ...(1.1)$$

$$F_{ij} = -F_{ji}\left(F_{ij} = F_i^a g_{aj}\right) \qquad ...(1.2)$$

and

$$F_{ij}^n = 0, \qquad ...(1.3)$$

where the comma (,) followed by an index denotes the operation of covariant differentiation with respect to the Riemannian metric tensor g_{ij}.

The Riemannian curvature tensor field R_{ijk}^h is defined by

$$R_{ijk}^h = \partial_i \begin{pmatrix} h \\ jk \end{pmatrix} - \partial_j \begin{pmatrix} h \\ ik \end{pmatrix} \begin{pmatrix} h \\ il \end{pmatrix} \begin{pmatrix} l \\ jk \end{pmatrix} - \begin{pmatrix} h \\ jl \end{pmatrix} \begin{pmatrix} l \\ jk \end{pmatrix}, \qquad ...(1.4)$$

where $\partial_i \equiv \partial/\partial x^i, \{x^i\}$ denotes real local coordinates.

The Ricci tensor and scalar curvature are given by $R_{ij} = R_{aij}^a$ and $R = R_{ij} g^{ij}$ respectively.

It is well known that these tensors satisfy the following identities [3] :

$$R_{ijk,a}^h = R_{jk,i} - R_{ikj}, \qquad ...(1.5)$$

$$R_i = 2R_{i,a}^a, \qquad ...(1.6)$$

$$F_i^a R_{aj} = -R_{ia} F_j^a, \qquad ...(1.7)$$

and

$$F_i^a R_a^j = R_i^a F_a^j. \qquad ...(1.8)$$

The Bianchi identities in T_n are given by

$$R_{ijk}^h + R_{jki}^h + R_{kij}^h = 0, \qquad ...(1.9)$$

and

$$R^h_{ijk,a} + R^h_{ika,j} + R^h_{iaj,k} = 0 \qquad ...(1.10)$$

A Tachibana space is said to be Tachibana recurrent if its curvature tensor field satisfies the condition [1]

$$R^h_{ijk,a} = \lambda_a R^h_{ijk}, \qquad ...(1.11)$$

where λ_a is a non-zero recurrence vector field.

The following relations follow immediately from equation (1.11):

$$R_{ik,a} = \lambda_a R_{ik} \qquad ...(1.12)$$

and

$$R_{,a} = \lambda_a R. \qquad ...(1.13)$$

2. Decomposition of Curvature Tensor Field R^h_{ijk}. Consider the docomposition of recurrent curvature tensor field R^h_{ijk} in the following form:

$$R^h_{ijk} = A_{jk} v^h_{,i}, \qquad ...(2.1)$$

where A_{jk} and $v^h_{,i}$ are two non-zero tensor fields such that

$$\lambda_h v^h_{,i} = p_i, \qquad ...(2.2)$$

$$\lambda_h v^h = 1 \qquad ...(2.3)$$

and

$$A_{jk} = \lambda_{j,k} - \lambda_{k,j} \neq 0. \qquad ...(2.4)$$

The vector field p_i is also a non-zero vector field and is called decomposed vector field.

We shall prove the following:

Theorem 1. Under the docomposition (2.1), the Bianchi identities take the forms:

$$P_i A_{jk} + p_j A_{ki} + p_k A_{ij} = 0 \qquad ...(2.5)$$

and

$$\lambda_a A_{jk} + \lambda_j A_{ka} + \lambda_k A_{aj} = 0 \qquad ...(2.6)$$

Proof. In view of equation (1.9) and (2.1), we have

$$A_{jh}, v^h_{,i} + A_{ki} v^h_{,j} + A_{ij} v^h_{,k} = 0. \qquad ...(2.7)$$

Multiplying the above equation by λ_h and using (2.2), we obtain the required result (2.5). Making use of equation (1.10), (1.11) and (2.1), we get

$$v^h_{,i}\left(\lambda_a A_{jk} + \lambda_j A_{ka} + \lambda_k A_{aj}\right) = 0 \qquad ...(2.8)$$

Multiplying the above equation by λ_h and using (2.2), we have

$$P_i\left(\lambda_a A_{jk} + \lambda_j A_{ka} + \lambda_k A_{aj}\right) = 0 \qquad ...(2.9)$$

Since p_i is a non zero recurrence vector field, we obtain

$$\lambda_a A_{jk} + \lambda_j A_{ka} + \lambda_k A_{aj} = 0.$$

This completes the proof of the theorem.

Theorem 2. Under the decomposition (2.1), the tensor field R^h_{ijk}, R_{ij} and A_{jk} satisfy the relations

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} = p_i A_{jk} \qquad ...(2.10)$$

Proof. With the help of equations (1.5), (1.11) and (1.12), we obtain

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} \qquad ...(2.11)$$

Multiplying equations (2.1) by λ_h and using the relation (2.2), we have

$$\lambda_a R^a_{ijk} = p_i A_{jk}. \qquad ...(2.12)$$

In view of equations (2.11) and (2.12), we get the required result (2.10).

Theorem 3. Under the decomposition (2.1), the quantities λ_a and $v^h_{,i}$ behave like recurrent vector and tensor fields respectively. The recurrent forms of these quantities are given by

$$\lambda_{a,m} = \mu_m \lambda_a, \qquad ...(2.13)$$

and

$$v^h_{,i,m} \underset{def}{=} v_m v^h_{,i}, \qquad ...(2.14)$$

where $v^h_{,i,m} = v^h_{,im}$ and μ_m, v_m are non-zero recurrence vector fields.

Proof. Defferentiating (2.11), convariently with respect to x^m and using equations (2.1) and (2.10), we have

$$\lambda_{a,m} A_{jk} v^a_{,i} = \lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}. \qquad ...(2.15)$$

Multiplying the above equation by λ_a and using (2.11), we get

$$\lambda_{a,m}(\lambda_i R_{jk} - \lambda_j R_{ik}) = \lambda_a(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}). \qquad ...(2.16)$$

Now multipling equation (2.16) by λ_h, we get

$$\lambda_{a,m}(\lambda_i R_{jk} - \lambda_j R_{ik})\lambda_h = \lambda_a \lambda_h(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}). \qquad ...(2.17)$$

Since the expression on the right hand side of the above equation is a symmetric in a and h, therefore we get

$$\lambda_{a,m}\lambda_h = \lambda_{h,m}\lambda_a, \qquad ...(2.18)$$

provided that $\lambda_i R_{jk} - \lambda_j R_{ik} \neq 0$.

The vector λ_a being non-zero, we can obtain a proportional vector field μ_m such that $\lambda_{a,m} = \mu_m \lambda_a$.

Further, differentiating (2.2) covariently with respect to x^m and using (2.13), we get

$$\lambda_h v^h_{,im} = p_{i,m} - \mu_m p_i \qquad ...(2.19)$$

From the above equation it is obvious that

$$\lambda_h v^h_{,im} = \lambda_a v^a_{,im} \qquad ...(2.20)$$

Since λ_a is a non-zero recurrence vector field, we can get a proportional vector field v_m such that $V^h_{,im} = v_m v^h_{,i}$.

This completes the proof of the theorem.

Theorem 4. Under the decomposition (2.1), the vector field p_i and tensor field A_{jk} behave like recurrent vector and recurrent tensor fields and their recurrent forms are given respectively by

$$P_{i,m} = (\mu_m + v_m)p_i, \qquad ...(2.21)$$

and

$$A_{jk,m} = (\lambda_m - v_m)A_{jk}. \qquad ...(2.22)$$

Proof. Differentiating (2.2) covariantly with respect to x^m and using equations (2.2), (2.13) and (2.14), we obtain the required result (2.21).

Further, differentiating equation (2.1) covariently with respect to x^m and using equations (1.11), (2.1) and (2.14), we get the required recurrent form (2.22).

Theorem 5. Under decomposition (2.1), the curvature tensor and holomorphically projective curvature tensor are equal iff

$$A_{km}\left\{\left(p_i S_j^h - p_j S_i^h\right) + p_a\left(F_j^h F_i^a - F_i^h F_j^a\right)\right\} + 2p_a A_{ji} F_k^h F_i^a = 0. \qquad ...(2.23)$$

Proof. The holomorphically projective curvature tensor field p_{ijk}^h in T_n is defined by

$$p_{ijk}^h = R_{ijk}^h + \left(R_{ik}\delta_j^h - R_{jk}\delta_i^h + S_{ik}F_j^h - S_{jk}F_i^h + 2S_{ij}F_k^h\right)/(n+2). \qquad ...(2.24)$$

which may be expressed briefly as

$$p_{ijk}^h = R_{ijk}^h + D_{ijk}^h, \qquad ...(2.25)$$

where

$$nD_{ijk}^h = \left(R_{ik}\delta_j^h - R_{jk}\delta_i^h + S_{ik}F_j^h - S_{jk}F_i^h + 2S_{ij}F_k^h\right)/(n+2). \qquad ...(2.26)$$

Contracting indices h and k in (2.1), we get

$$R_{ij} = A_{jl}v_{,i}^l. \qquad ...(2.27)$$

In view of the above equation, we have

$$S_{ij} = F_i^a R_{aj} = F_i^a A_{jl} v_{,a}^l \qquad ...(2.28)$$

Making use of (2.27) and (2.28) in (2.26), we have

$$D_{ijk}^h = \left[A_{kl}\left\{\left(S_j^h v_{,i}^l - S_i^h v_{,j}^l\right) + v_{,a}^D\left(F_j^h F_i^a - F_i^h F_j^a\right)\right\} + 2A_{jl}F_i^a F_k^h v_{,a}^l\right]/(n+2). \qquad ...(2.29)$$

From (2.25) it is clear that

$P_{ijk}^h = R_{ijk}^h$ iff $D_{ijk}^h = 0$, which in view of equation (2.29), gives

$$A_{ki}\left\{\left(\delta_j^h v_{,i}^l - \delta_i^h v_{,j}^l\right) + v_{,a}^l\left(F_j^h F_i^a - F_i^h F_j^a\right)\right\} + 2A_{jl}F_i^a F_k^g v_{,a}^l = 0. \qquad ...(2.30)$$

Multiplying the above equation by λ_a and using (2.2), we obtain the required condition (2.23).

REFERENCES

[1] K.B. Lal and S.S. Singh, On Kaehlerian spaces with recurrent Bochner curvature, *Acc. Naz. Dei. Lincei, Rend., Series VIII,* **Vol. LI, No. 3-4** (1970), 91-96.

[2] K. Takano, Decomposition of curvature tensor in a recurrent space, *Tensor (N.S.),* **18 No. 3,** (1967), 343-347.

[3] S. Tachibana, On the Bochner curvature tensor, *Nat. Sci. Rep., Ochanonizu University* **18** No. 1 (1967), 15-19.

[4] K. Yano, *Differential Geometry on Complex and Almost Complex Spaces, Pergamon Press* (1965)

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HALL EFFECTS ON COUPLE-STRESS FLUID HEATED AND SOLUTED FROM BELOW IN POROUS MEDIUM

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ABSTRACT

A layer of couple-stress fluid heated and soluted from below in porous medium is considered in the presence of uniform horizontal magnetic field to include the effect of Hall currents. For the stationary convection case, the stable solute gradient and magnetic field postpones the onset of convection while the Hall current hastens the onset of convection. The medium permeability and couple-stress both postpone and hasten the onset of convection depending on the Hall parameter M. Graphs have been plotted by giving numerical values to the parameters, to depict the stability characteristics. The stable solute gradient and the magnetic field (and corresponding Hall currents) are found to introduce oscillatory modes in the sytem, which were non-existent in their absence. The fufficient conditions for the non-existence of overstability are also obtained.

Key Words Couple-stress fluid, heated and soluted from below, Porous medium, Magnetic field, Hall currents.

1. Introduction. The formation and derivation of the basic equations of a layer of fluid heated from below in porous medium, using Boussinesq approximation, has been given in a treatise by Joseph [4]. When a fluid permeates an isotropic and homogeneous porous medium, the gross effect is represented by the Darcy's law. The study of a layer of fluid heated from below in porous media is motivated both theoretically and by its practical applications in engineering. Among the applications in engineering disciplines one can find the food process industry, chemical process indurtry, solidification and centrifugal casting of metals. The development of geothermal power resources has increased general interest in the properties of convection in porous medium. An extensive and updated account

of convection in porous media has been given by Nield and Bejan [7]. The forced convection in fluid saturated porous medium channel has been studied by Nield et al. [8]. The effect of a magnetic field on the stability of such a flow is of interest in geophysics, particularly in the study of the earth's core, where the earth's mantle, which consists of conducting fluid, behaves like a porous medium that can become convectively unstable as a result of differential diffusion. Another application of the results of flow through a porous medium in the presence of a magnetic field is in the study of the stability of convective geothermal flow. A good account of the effect of magnetic field on the layer of the fluid heated from below has been given in a treatise by Chandrasekhar [2]. *MHD* finds vital applications in *MHD* generators, *MHD* flow-meters and pumps for pumping liquid metallurgy, geophysics *MHD* couplers and bearing and physiological processes such as magnetic therapy.

Double-diffusive convection concens flow that can arise when a layer of fluid with a dissolved solute (such as salt) is heated from below. Veronis [19] has investigated the problem of thermohaline convection in a layer of fluid heated from below and subjected to a stable salinity gradient. The physics is quite similar in the stellar case in that helium acts like salt in raising the density and in diffusing more slowly than heat. The conditions under which convective motions are important in stellar atmospheres are usually far removed from consideration of single component fluid and rigid boundaries and therefore it is desirable to consider a fluid acted on by solute gradient and free boundaries. The problem of the onset of thermal instability in the presence of a solute gradient is of great importance because of its applications to atmospheric physics and astrophysics, especially in the case of the ionosphere and the outer layer of the atmosphere. The thermosolutal convection problems also arise in oceanography, limnology and engineering. Thermosolutal convection problem arise in oceanography, limnology, and engineering. Examples of particular interest are provided by ponds built to trap solar heat (Tabor and Matz [17] and some Antarctic lakes (Shirtcliffe [16]). Sherman and Sutton [9] have considered the effect of current on the efficiency of a magncto-fluid-dynamic generator.

With the growing importance of non-Newtonian fluids in modern technology and industries, the investigations on such fluids are desirable. Stokes [10] formulated the theory of couple-stress fluid. One of the applications of couple-stress fluid is its use to the study of the mechanisms of lubrication of synovial joints, which has become the object of scientific research. A human joint is a dynamically loaded bearing which has articular cartilage as the bearing and synovial fluid as the lubricant. When a fluid film is generated, squeeze-film action is capable of providing considerable protection to the cartilage surface.

A number of theories of the micro-continuum have been postulated and

applied (Stokes [10], Lai et al. [5], Walicka [20]). The theory due to Stokes [10] allows for polar effects such as the presence of couple stresses and body couples. Stokes [10] theory has been applied to the study of some simple lubrication problems (see e.g. Sinha et al. [11], Bujurke and Jayaraman [1], Lin [6]). According to the theory of Stokes [10], couple-stresses are found to appear in noticeable magnitudes in fluids with very large molecules. Since the long chain hyaluronic acid molecules are found as additives in synovial fluid, Walicki and Walicka [21] modeled synovial fluid as couple-stress fluid in human joints. Sharma and Thakur [12] and Sharma et al. [13] have studied the problems of couple-stress fluid heated from below in porous medium in hydromagnetics and rotation, separately. The Hall effect is likely to be important in many geophysical situations as well as in flow of laboratory plasma. There is growing importance of non-Newtonian fluids in chemical technology, industry and geophysical fluid dynamics. The Hall currents have relevance and importance in geophysics, *MHD* generator and industry.

Keeping in mind the important of non-Newtonian fluids, convection in fluid layer heated and soluted from below, porous medium, magnetic field and Hall currents; the present paper attempts to study the couple-stress fluid heated and soluted from below in porous medium in the presence of uniform horizontal magnetic field to include the effect of Hall currents. The study is motivated by a model of synovial fluid. The synovial fluid is the natural lubricant of points of the vertebrates. The detailed description of the joint lubrication has very important practical implications-practically all diseases of joints are caused by or connected with a malfunction of the lubrication. The extremal efficiency of the physiological joint lubrication is caused by more mechanisms. The synovial fluid is due to the content of the hyaluronic acid a fluid of high viscosity, near to a gel. A layer of such fluid heated and soluted from below in porous medium under the action of magnetic field may find applications in physiological processes e.g. *MHD* finds applications in physiological processes such as magnetic therapy; heating may find application in physio-therapy.

2. Formulation of the Problem and Perturbation Equations. Here we consider an infinite, horizontal, incompressible, electrically conducting couple-stress fluid layer of thickness d, heated from below so that, the temperatures and densities at the bottom surface $z=0$ are T_0 and ρ_0 and at the upper surface $z=d$ and T_d and ρ_d respectively, and that a uniform temperature gradient $\beta(=|dT/dz|)$ and a uniform solute gradient $\beta'(=|dC/dz|)$ are maintained. The gravity field $\vec{g}=(0,0,-g)$ and a uniform horizontal magnetic field $\vec{H}=(H,0,0)$ pervade the system. This fluid layer is flowing through an isotropic and homogeneous porous medium of porostiy ε and medium permeability k_l.

Let $p, \rho, T, C, \alpha, \alpha', g, \eta, \mu_e, N, e$ and $\vec{q} = (u,v,w)$ denote, respectively, the fluid pressure, density, temperature, solute concentration, thermal coefficient of expansion, an analogous solvent coefficient of expansion, gravitational acceleration, resistivity, magnetic permeability, electron number density, charge of an electron and fluid velocity. The equations expressing the conservation of momentum, mass, temperature solute concentration and equation of state of couple-stress fluid through porous medium (Stokes [10], Joseph [4]) are

$$\frac{1}{\varepsilon}\left[\frac{\partial \vec{q}}{\partial t} + \frac{1}{\varepsilon}(\vec{q}.\nabla)\vec{q}\right] = -\frac{1}{\rho_0}\nabla p + \vec{g}\left(1 + \frac{\delta\rho}{\rho_0}\right) - \frac{1}{k_1}\left(\nu - \frac{\mu'}{\rho_0}\nabla^2\right)\vec{q} + \frac{\mu_e}{4\pi\rho_0}\left(\nabla \times \vec{H}\right)\times \vec{H}, \tag{1}$$

$$\nabla.\vec{q} = 0, \tag{2}$$

$$E\,\partial T/\partial t + (\vec{q}.\nabla)T = \kappa\nabla^2 T, \tag{3}$$

$$E'\,\partial C/\partial t + (\vec{q}.\nabla)C = \kappa'\nabla^2 C, \tag{4}$$

$$\rho = \rho_0[1 - \alpha(T - T_0) + \alpha'(C - C_0)], \tag{5}$$

where the suffix zero refers to values at the reference level $z=0$ and in writing Eq. (1), use has been made of the Boussinesq approximation which states that the density variations are ignored in all terms in the equation of motion except the external force term. The magnetic permeability μ_e, the kinematic viscosity, ν, the kinematic viscoelasticity ν', the thermal diffusivity κ and the solute diffusivity κ' are all assumed to be constants.

The Maxwell's equations yield

$$\varepsilon\frac{d\vec{H}}{dt} = \left(\vec{H}.\nabla\right)\vec{q} + \varepsilon\eta\nabla^2\vec{H} - \frac{c\varepsilon}{4\pi Ne}\nabla\times\left[\left(\nabla\times\vec{H}\right)\times\vec{H}\right], \tag{6}$$

$$\nabla.\vec{H} = 0, \tag{7}$$

where $d/dt = \partial/\partial t + \varepsilon^{-1}\vec{q}.\nabla$ stands for the convection derivative.

Here $E = \varepsilon + (1-\varepsilon)((\rho_s c_s)/(\rho_0 c_i))$ is a constant and E' is a constant analogous to E but corresponding to solute rather than heat. ρ_s, c_s and ρ_0, c_i stand for density and heat capacity of solid (porous matrix) material and fluid, respectively. The steady state solution is

$$\vec{q} = (0,0,0), \qquad T = -\beta z + T_0,$$
$$C = -\beta' z + C_0, \qquad \rho = \rho_0(1 + \alpha\beta z - \alpha'\beta' z). \tag{8}$$

Here we use linearized stability theory and normal mode analysis method. Assume small perturbations around the basic solution, and let $\delta\rho$, δp, θ, γ, $\vec{q}$ (u,v,w) and $\vec{h} = (h_x, h_y, h_z)$ denote respectively the perturbations in fluid density ρ, pressure p, temprature T, solute concentration C, velocity $(0,0,0)$ and magnetic field $\vec{H} = (H,0,0)$. The change in density $\delta\rho$, caused mainly by the perturbations θ and γ

in temperature and concentration, is given by

$$\delta\rho = -\rho_0(\alpha\theta - \alpha'\gamma). \tag{9}$$

Then the linearized perturbation equations of the couple-stress fluid become

$$\frac{1}{\varepsilon}\frac{\partial\vec{q}}{\partial t} = -\frac{1}{\rho_0}(\nabla\delta p) - \vec{g}(\alpha\theta - \alpha'\gamma) - \frac{1}{k_1}\left(\nu - \frac{\mu'}{\rho_0}\nabla^2\right)\vec{q} + \frac{\mu_e}{4\pi\rho_0}\left(\nabla\times\vec{H}\right)\times\vec{H}, \tag{10}$$

$$\nabla.\vec{q} = 0, \tag{11}$$

$$E\frac{\partial\theta}{\partial t} = \beta w + \kappa\nabla^2\theta, \tag{12}$$

$$E'\frac{\partial\gamma}{\partial t} = \beta' w + \kappa'\nabla^2\gamma, \tag{13}$$

$$\varepsilon\frac{d\vec{h}}{dt} = \left(\vec{H}.\nabla\right)\vec{q} + \varepsilon\eta\nabla^2\vec{h} - \frac{c\varepsilon}{4\pi Ne}\nabla\times\left[\left(\nabla\times\vec{H}\right)\times\vec{H}\right], \tag{14}$$

$$\nabla.\vec{h} = 0. \tag{15}$$

3. The Dispersion Relation. Analyzing the disturbances into normal modes, we assume that the perturbation quantities are of the form

$$[w, h_z, \theta, \gamma, \zeta, \xi] = [W(z), K(z), \Theta(z), \Gamma(z), Z(z), Z(z)]exp(ik_x x + ik_y y + nt), \tag{16}$$

where k_x, k_y are the wave number along the x-and y-directions respectively, $k = \sqrt{(k_x^2 + k_y^2)}$ is the resultant wave number and n is the growth rate, which is, in general, a complex constant. Here $\zeta = \partial v/\partial x - \partial u/\partial y$ and $\xi = \partial h_y/\partial x - \partial h_x/\partial y$ stand for the z-components of vorticity and current density, respectively.

Expressing the coordinates x,y,z in the new unit of length d and letting $a = kd$, $\sigma = nd^2/\nu$, $p_1 = \nu/\kappa$, $p_2 = \nu/\eta$, $q = \nu/\kappa'$, $P_l = k_1/d^2$, $F = \mu'/(\rho_0 d^2\nu)$ and $D = d/dz$, Eqs. (10)-(15), using (16), yield

$$\left(D^2 - a^2\right)\left[\frac{\sigma}{\varepsilon} + \frac{1}{P_\ell} - \frac{F}{P_\ell}\left(D^2 - a^2\right)\right]W - \frac{ik_x\mu_e Hd^2}{4\pi\rho_0\nu}\left(D^2 - a^2\right)K + \frac{ga^2d^2}{\nu}(\alpha\Theta - \alpha'\Gamma) = 0, \tag{17}$$

$$\left[\sigma/\varepsilon + 1/P_\ell - F/P_\ell\left(D^2 - a^2\right)\right]Z = ik_x\mu_e Hd^2 X/(4\pi\rho_0\nu), \tag{18}$$

$$\left(D^2 - a^2 - p_2\sigma\right)K = -\left(ik_x Hd^2/(\eta\varepsilon)\right)W + cik_x Hd^2 X/(4\pi Ne\eta), \tag{19}$$

$$\left(D^2 - a^2 - p_2\sigma\right)X = -\left(ik_x Hd^2/(\eta\varepsilon)\right)Z - cik_x H/(4\pi Ne\eta)\left(D^2 - a^2\right)K, \tag{20}$$

$$\left(D^2 - a^2 - Ep_1\sigma\right)\Theta = -\left(\beta d^2/\kappa\right)W, \tag{21}$$

$$\left(D^2 - a^2 - E'q\sigma\right)\Gamma = -\left(\beta' d^2/\kappa'\right)W. \tag{22}$$

Consider the case where both boundaries are free as well as perfect conductors of both heat and solute concentration, while the adjoining medium is perfectly conducting. The case of two free boundaries is a little artificial but it enables us to find analytical solutions and to make some qualitative conclusions. The appropriate boundary conditions, with respect to which Eqs. (17)-(22) must

be solved are (Chandrasekhar [2])

$$W = D^2W = X = DZ = 0, \Theta = 0, \Gamma = 0, \text{ at } z = 0 \text{ and } 1,$$

$DX=0$, $K=0$ on a perfectly conducting boundary

and $X=0$, h_x, h_y, h_z are continuous with an external vacuum field

on a non-conducting boundary. (23)

The case of two free boundaries, though little artificial, is the most appropriate for steller atmospheres (Spiegel [14]). Using the above boundary conditions, it can be shown that all the even order derivatives of W must vanish for $z=0$ and 1 and hence the proper solution of W characterizing the lowest mode is

$$W=W_0 \sin \pi z. \tag{24}$$

where W_0 is a constant.

Eliminating Θ, Γ, K, Z and X between Eqs. (17)-(22) and substituting the proper solution $W=W_0 \sin \pi z$, in the resultant equation, we obtain the dispersion relation

$$R_1 = \left(\frac{1+x}{x}\right)\left[\frac{i\sigma_1}{\varepsilon}+\frac{1}{P}+\frac{\pi^2F(1+x)}{P}\right][1+x+iEp_1\sigma_1]+S_1\frac{(1+x+iEp_1\sigma_1)}{(1+x+iE'q\sigma_1)}$$

$$+Q_1\cos^2\theta\frac{(1+x)[1+x+iEp_1\sigma_1]\left\{\left(\frac{i\sigma_1}{\varepsilon}+\frac{1}{P}+\frac{\pi^2F(1+x)}{P}\right)[1+x+ip_2\sigma_1]+Q_1x\cos^2\theta\right\}}{\left\{\begin{array}{l}\left[\frac{i\sigma_1}{\varepsilon}+\frac{1}{P}+\frac{\pi^2F(1+x)}{P}\right][1+x+ip_2\sigma_1]^2+Q_1x\cos^2\theta[1+x+ip_2\sigma_1]+\\ Mx\cos^2\theta(1+x)\left[\frac{i\sigma_1}{\varepsilon}+\frac{1}{P}+\frac{\pi^2F(1+x)}{P}\right]\end{array}\right\}}, \tag{25}$$

where

$$R_1=\frac{g\alpha\beta d^4}{\nu\kappa\pi^4}, S_1=\frac{g\alpha'\beta' d^4}{\nu\kappa'\pi^4}, Q_1=\frac{\mu_e H^2 d^2}{4\pi\rho_0\nu\eta\varepsilon\pi^2}, M=\left(\frac{cH}{4\pi Ne\eta}\right)^2, x=\frac{a^2}{\pi^2}, i\sigma_1=\frac{\sigma}{\pi^2},$$

$k_x=k\cos\theta$ and $P=\pi P_\ell$.

Equation (25) is the required dispersion relation including the effects of magnetic field, Hall current, stable solute gradient and medium permeability on a layer of couple-stress fluid heated and soluted from below in porous medium in the presence of a uniform horizontal magnetic field and Hall currents.

4. The Stationary Convection. When the instability sets in as stationary convection, the marginal state will be characterized by $\sigma=0$. Putting $\sigma=0$, the dispersion relation (25) reduces to

$$R_1=\left(\frac{1+x}{x}\right)\frac{\left(\frac{(1+x)\{1+\pi^2F(1+x)\}}{P}+Q_1x\cos^2\theta\right)^2+\frac{Mx\cos^2\theta(1+x)\{1+\pi^2F(1+x)\}^2}{P^2}}{\left\{\frac{(1+x+Mx\cos^2\theta)\{1+\pi^2F(1+x)\}}{P}+Q_1x\cos^2\theta\right\}}+S_1, \quad (26)$$

which expresses the modified Rayleigh number R_1 as a function of the dimensionless wave number x and the parameters S_1, Q_1, M, F and P.

To study the effects of stable solute gradient, horizontal magnetic field, Hall currents, medium permeability and couple-stress parameter, we examine the natures of $dR_1/dS_1, dR_1/dQ_1, dR_1/dM, dR_1/dP$ and dR_1/dF analytically. Equation (26) yields

$$dR_1/dS_1=+1, \quad (27)$$

$$\frac{dR_1}{dQ}=\left(\frac{1+x}{x}\right)x\cos^2\theta\frac{\left(\frac{(1+x)\{1+\pi^2F(1+x)\}}{P}+Q_1x\cos^2\theta+\frac{Mx\cos^2\theta\{1+\pi^2F(1+x)\}\,Q_1x\cos^2\theta}{P\left(\frac{1+x+Mx\cos^2\theta\{1+\pi^2F(1+x)\}}{P}+Q_1x\cos^2\theta\right)}\right)}{\left\{\frac{(1+x+Mx\cos^2\theta)\{1+\pi^2F(1+x)\}}{P}+Q_1x\cos^2\theta\right\}}, \quad (28)$$

and

$$\frac{dR_1}{dM}=-Q_1x\cos^4\theta(1+x)\frac{\left(\frac{(1+x)\{1+\pi^2F(1+x)\}}{P}+Q_1x\cos^2\theta\right)\left(\frac{\{1+\pi^2F(1+x)\}}{P}\right)}{\left\{\frac{(1+x+Mx\cos^2\theta)\{1+\pi^2F(1+x)\}}{P}+Q_1x\cos^2\theta\right\}^2}. \quad (29)$$

It is clear from (27), (28) and (29) that, for stationary convection, the magnetic field and stable solute gradient postpones the onset of convection whereas, the Hall currents hastens the onset of convection on the thermosolutal instability of couple-stress fluid in porous medium in hydromagnetics in the presence of Hall currents. Equation (26) also yields

$$\frac{dR_1}{dP}=-\frac{(1+x)\{1+\pi^2F(1+x)\}}{xP^2}\frac{\left(\begin{array}{l}\frac{(1+x)}{P^2}(1+x+Mx\cos^2\theta)^2\{1+\pi^2F(1+x)\}^2\\+\frac{2Q_1x\cos^4\theta(1+x)(1+x+Mx\cos^2\theta)\{1+\pi^2F(1+x)\}}{P}\\+Q_1^2x^2\cos^4\theta(1+x-Mx\cos^2\theta)\end{array}\right)}{\left[\frac{(1+x+Mx\cos^2\theta)\{1+\pi^2F(1+x)\}}{P}+Q_1x\cos^2\theta\right]^2}. \quad (30)$$

$$\frac{dR_1}{dF}=\left(\frac{(1+x)^2\pi^2}{xP}\right).\frac{\left(\begin{array}{l}\left(\frac{(1+x)}{P^2}\right)\left(1+x+Mx\cos^2\theta\right)^2\left\{1+\pi^2F(1+x)\right\}^2\\ +\frac{2Q_1x\cos^2\theta\,(1+x)\left(1+x+Mx\cos^2\theta\right)\left\{1+\pi^2F(1+x)\right\}}{P}\\ +Q_1^2x^2\cos^4\theta\left(1+x-Mx\cos^2\theta\right)\end{array}\right)}{\left[\frac{\left(1+x+Mx\cos^2\theta\right)\left\{1+\pi^2F(1+x)\right\}}{P}+Q_1x\cos^2\theta\right]^2}. \tag{31}$$

Hence, it is clear from (30) and (31) that, for stationary convection, the meidum permeability hastens the onset of convection whereas, the couple-stress postpones the onset of convection on the thermal instability of couple-stress fluid in porous medium in hydromagnetics in the presence of Hall currents for all wave numbers

$$(1+x)>Mx\cos^2\theta,$$

which is normally satisfied as the Hall currents parameter M is very small compared to unity.

The dispersion relation (26) is analysed numerically. In Figure 1,R_1 is plotted against x for $S_1=10, 20, 30;\ P=50,\ \theta=45^0,\ F=2,\ Q_1=10$ and $M=10$. It is clear that the stable solute gradient postpones the onset of convection in a couple-stress fluid heated and soluted from below in a porous medium in the presence of Hall currents as the Rayleigh number increases with the increase in stable solute gradient parameter. In Figure 2, R_1 is plotted against x for $Q_1=10,20,30;\ P=50,\ \theta=45^0,\ F=2,\ S_1=10$ and $M=10$. It is clear that the magnetic field postpones the onset of convection in a couple-stress fluid heated from below in a porous medium in the presence of Hall currents as the Rayleigh number increases with the increase in magnetic field parameter. In Figure 3, R_1 is plotted against x for $M=10,20, 30;\ P=50,\ \theta=45^0,\ F=2,\ S_1=10$ and $Q_1=10$. Here we find that the Hall currents hastens the onset of convection for all wave numbers as the Rayleigh number decreases with an increase in the Hall currents parameter.

In Figure 4, R_1 is plotted against x for $P=10, 20, 30;\ M=0.1,\ \theta=45^0,\ F=2,\ S_1=10$ and $Q_1=100$. Here we find that when $M<1$, the medium permeability always hastens the onset of convection for all wave numbers as the Rayleigh number decreases with an increase in medium permeability parameter. In Figure 5, R_1 is plotted against x for $P=10, 20,30;\ M=100,\ \theta=45^0,\ F=2,\ S_1=10$ and $Q_1=100$. Here we find that when $M>1$, the medium permeability postpones the onset of convection for small wave numbers only as the Rayleigh number increases with an increase in medium permeability parameter and hastens the onset of convection for higher wave numbers as the Rayleigh numbers decreases with an increase in medium

permeability parameter.

In Figure 6, R_1 is plotted against x for $F=1,2,3,4$; $Q_1=100$, $\theta=45^0$, $P=10$, $S_1=10$ and $M=0.1$. Here we find that when $M<1$, the couple-stress postpones the onset of convection for all wave numbers as the Rayleigh number increases with the increase in couple-stress parameter. In Figure 7, R_1 is plotted against x for $F=1,2,3$; $Q_1=100$, $\theta=45^0$, $P=10$, $S_1=10$ and $M=100$. Here we find that when $M>1$, the couple-stress hastens the onset of convection for small wave numbers as the Rayleigh number decreases with the increase in couple-stress parameter and postpones the onset of convection in the for higher wave number as the Rayleigh number increases with the increase in couple-stress parameter.

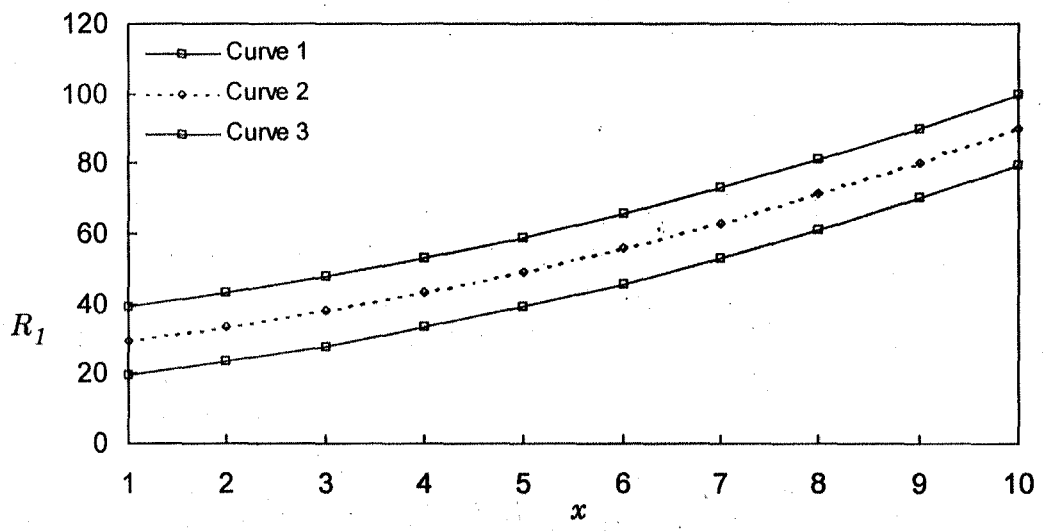

Fig.1: The variation of Rayleigh number (R_1) with wave number (x) for $P=50$, $F=2$, $M=10$, $Q_1=10$; $S_1=10$ for curve 1, $S_1=20$ for curve 2 and $S_1=30$ for curve 3.

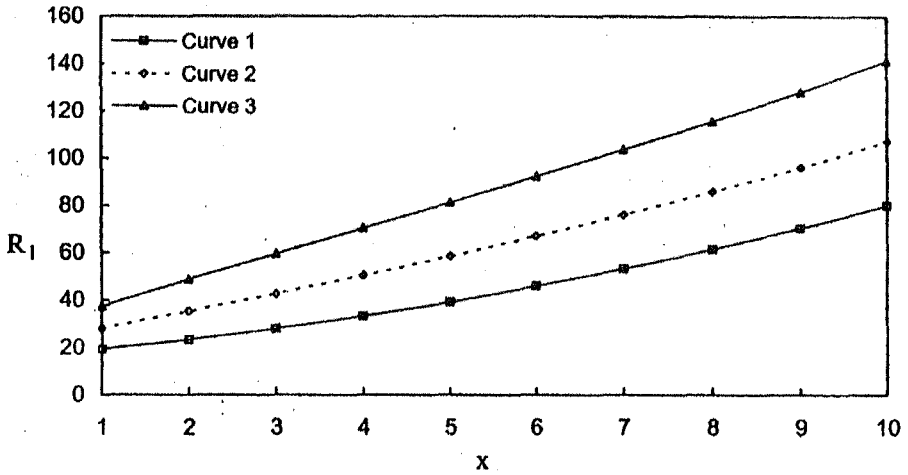

Fig.2: The variation of Rayleigh number (R_1) with wave number (x) for P = 50, F = 2 $\theta = 45^0$, M = 10, S_1 =10; Q_1= 10 for curve 1, Q_1= 20 for curve 2 and Q_1= 30 for curve 3.

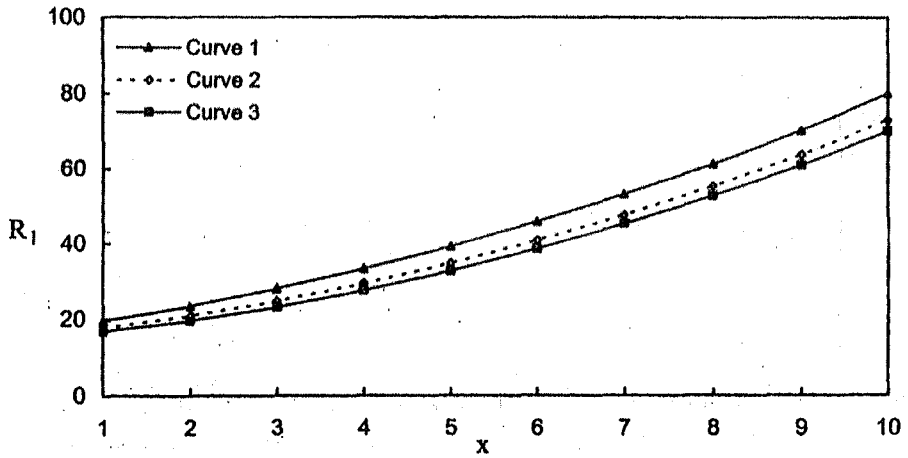

Fig.3: The variation of Rayleigh number (R_1) with wave number (x) for P = 50, F = 2 $\theta = 45^0$, Q_1 = 10, S_1 =10; M= 10 for curve 1, M= 20 for curve 2 and M= 30 for curve 3.

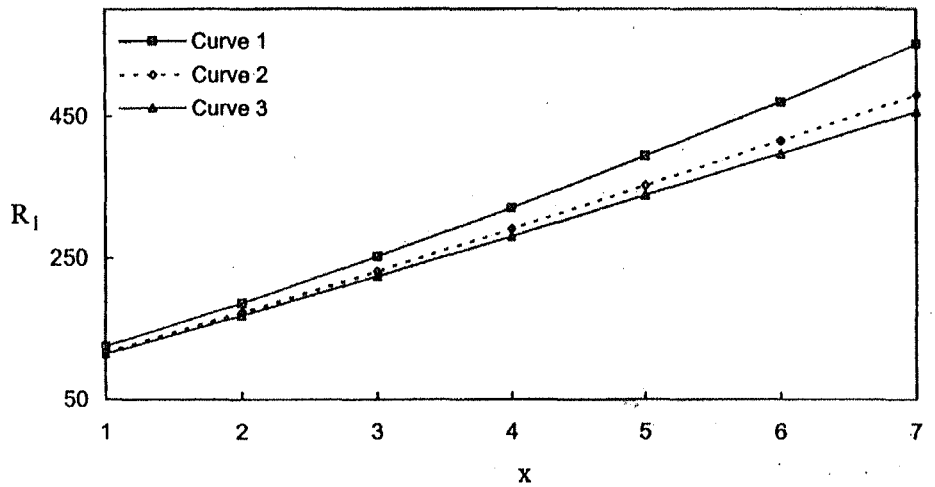

Fig.4: The variation of Rayleigh number (R_1) with wave number (x) for F = 2, Q_1 = 100 $\theta = 45^0$, M = 0.1, S_1 =10; P = 10 for curve 1, P = 20 for curve 2 and P = 30 for curve 3

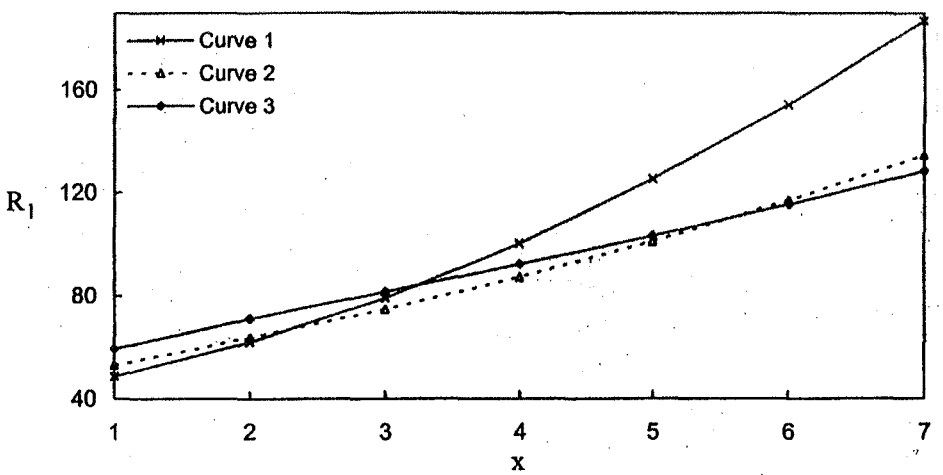

Fig.5: The variation of Rayleigh number (R_1) with wave number (x) for Q_1 = 100, F = 2 $\theta = 45^0$, M = 100, S_1 =10; P = 10 for curve 1, P = 20 for curve 2 and P = 30 for curve 3

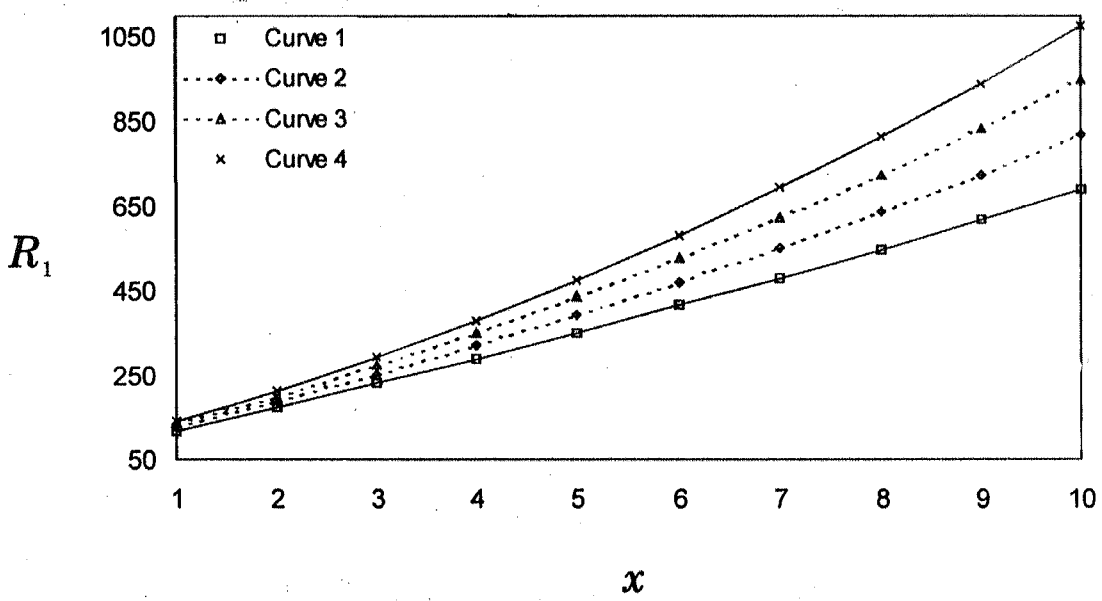

Fig.6: The variation of Rayleigh number (R_1) with wave number (x) for $P = 10$, $\theta = 45^0$, $M = 0.1$, $Q_1 = 100$; S_1 =i0; F=1 for curve 1, $F=2$ for curve 2, F=3 for cur 3 and F= 4 for curve 4.

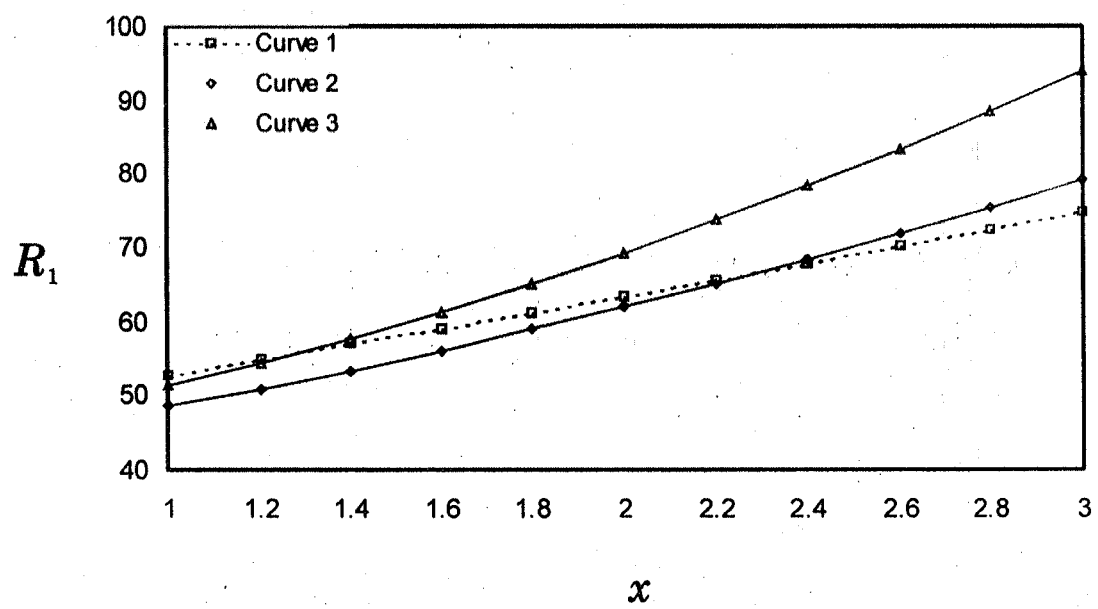

Fig.7: The variation of Rayleigh number (R_1) with wave number (x) for $P = 10$, $\theta = 45^0$, $M = 100$, $Q_1 = 100$; S_1 =10; F=1 for curve 1, $F=2$ for curve 2, F=3 for curve 3.

5. Stability of the System and Oscillatory Modes. Here we examine the possibility of oscillatory modes, on a stability problem due to the presence of the magnetic field and Hall currents. Multplying (17) by W^*, which is the complex conjugate of W, and using (18)-(22) together with the boundary conditions (23), we obtain

$$FI_1 + (1 + P_\ell \sigma/\varepsilon)I_2 - \left(g\alpha\kappa a^2 P_\ell/\nu\beta\right)\left[I_3 + Ep_1\sigma^* I_4\right] + \left(g\alpha'\kappa' a^2 P_\ell/\nu\beta'\right)\left[I_5 + E'q\sigma^* I_6\right]$$
$$+ \frac{\mu_e \varepsilon\eta}{4\pi\rho_0\nu} P_\ell\left(I_7 + p_2\sigma^* I_8\right) + \frac{\mu_e\varepsilon\eta d^2}{4\pi\rho_0\nu} P_\ell\left(I_{11} + p_2\sigma I_{12}\right) + d^2\left[\left(1 + P_\ell\sigma^*/\varepsilon\right)I_{10} + FI_9\right] = 0, \quad (32)$$

where

$$I_1 = \int_0^1\left(\left|D^2W\right|^2 + 2a^2|DW|^2 + a^4|W|^2\right)dz, \qquad I_2 = \int_0^1\left(|DW|^2 + a^2|W|^2\right)dz,$$
$$I_3 = \int_0^1\left(|D\Theta|^2 + a^2|\Theta|^2\right)dz, \qquad I_4 = \int_0^1\left(|\Theta|^2\right)dz,$$
$$I_5 = \int_0^1\left(|D\Gamma|^2 + a^2|\Gamma|^2\right)dz, \qquad I_6 = \int_0^1\left(|\Gamma|^2\right)dz,$$
$$I_7 = \int_0^1\left(\left|D^2K\right|^2 + 2a^2|DK|^2 + a^4|K|^2\right)dz, \qquad I_8 = \int_0^1\left(|DK|^2 + a^2|K|^2\right)dz,$$
$$I_9 = \int_0^1\left(|DZ|^2 + a^2|Z|^2\right)dz, \qquad I_{10} = \int_0^1\left(|Z|^2\right)dz,$$
$$I_{11} = \int_0^1\left(|DX|^2 + a^2|X|^2\right)dz, \qquad I_{12} = \int_0^1\left(|X|^2\right)dz \quad (33)$$

The integrals $I_1,\ldots,I_{12}$ are all positive-definite. Putting $\sigma = \sigma_r + i\sigma_i$ where σ_r, σ_i are real and equating the real and imaginary parts of Eq. (32), we obtain

$$\sigma_r\left[\frac{I_2}{\varepsilon} - \frac{g\alpha\kappa a^2}{\nu\beta}Ep_1 I_4 + \frac{g\alpha'\kappa' a^2}{\nu\beta'}E'qI_6 + \frac{\mu_e\varepsilon\eta}{4\pi\rho_0\nu}p_2\left(I_8 + d^2 I_{12}\right) + \frac{d^2}{\varepsilon}I_{10}\right]$$
$$= -\left[\frac{F}{P_\ell}I_1 + \frac{1}{P_\ell}I_2 - \frac{g\alpha\kappa a^2}{\nu\beta}I_3 + \frac{g\alpha'\kappa' a^2}{\nu\beta'}I_5 + \frac{\mu_e\varepsilon\eta}{4\pi\rho_0\nu}\left(I_7 + d^2 I_{11}\right) + \frac{d^2}{P_\ell}\left(I_{10} + FI_9\right)\right], \quad (34)$$

$$i\sigma_i\left[\frac{I_2}{\varepsilon} + \frac{g\alpha\kappa a^2}{\nu\beta}Ep_1 I_4 - \frac{g\alpha'\kappa' a^2}{\nu\beta'}E'qI_6 - \frac{\mu_e\varepsilon\eta}{4\pi\rho_0\nu}p_2\left(I_8 - d^2 I_{12}\right) - \frac{d^2}{\varepsilon}I_{10}\right] = 0. \quad (35)$$

It is evident from Eq. (34) that σ_r is either positive or negative. The system is, therfore, either stable or unstable. It is clear from Eq. (35) that σ_i may be either zero or non-zero, meaning that the modes may be either non-oscillatory or oscillatory. In the absence of stable solute gradient and magnetic field, equation (35) reduces to

$$\left[I_2/\varepsilon + g\alpha\kappa a^2 Ep_1 I_4/\nu\beta\right]\sigma_i = 0. \quad (36)$$

and the terms in brackets are positive definite. Thus $\sigma_i = 0$, which means that oscillatory modes are not allowed and the principle of exchange of stabilitties is

satisfied for a porous medium, in the absence of stable solute gradient and magnetic field. This result is true for the porous as well as non-porous medium as studied in Chandrasekhar [2]. The oscillatory modes are introduced due to the presence of the stable solute gradient and the magnetic field (and corresponding Hall currents), which were non-existent in their absence.

6. The Case of Overstability. Here we discuss the possibility of whether instability may occur as overstability. Since we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it suffices to find conditions for which (25) will admit of solutions with σ_1 real. Equating real and imaginary parts of Eq. (25) and eliminating R_1 between them, we obtain

$$A_4 c_1^4 + A_3 c_1^3 + A_2 c_1^2 + A_1 c_1 + A_0 = 0, \tag{37}$$

where we have put $c_1 = \sigma_1^2, b = 1 + x$ and

$$A_4 = E'^2 q^2 p_2^4 / \varepsilon^2 \left[\left(1/\varepsilon + E p_1 \pi^2 F / P \right) b + E p_1 / P \right], \tag{38}$$

$$\begin{aligned} A_3 = {} & \left[E'^2 q^2 \left(\frac{1}{\varepsilon} + \frac{E p_1 \pi^2 F}{P} \right) \left(\frac{p_2^2 \pi^4 F^2}{P^2} + \frac{2}{\varepsilon^2} \right) p_2^2 + \frac{p_2^4}{\varepsilon^2} \left(\frac{1}{\varepsilon} + \frac{E p_1 \pi^2 F}{P} \right) \right] b^3 \\ & + \left[2E'^2 q^2 \left(\frac{1}{\varepsilon} + \frac{E p_1 \pi^2 F}{P} \right) \left(\frac{p_2^2 \pi^4 F^2}{P^2} - \frac{M x \cos^2 \theta}{\varepsilon^2} \right) p_2^2 + \frac{p_2^4}{\varepsilon^2} \frac{E p_1}{P} + \frac{E'^2 q^2 E p_1}{P} \left(\frac{p_2^2 \pi^2 F^2}{P^2} + \frac{2}{\varepsilon} \right) p_2^2 \right] b^2 \\ & + E'^2 q^2 \left[\left(\frac{1}{\varepsilon} + \frac{E p_1 \pi^2 F}{P} \right) \left(\frac{p_2}{P^2} - \frac{2 Q_1 x \cos^2 \theta}{\varepsilon} \right) p_2^3 + \frac{2 E p_1 p_2^2}{P} \left(\frac{p_2^2 \pi^2 F^2}{P^2} - \frac{M x \cos^2 \theta}{\varepsilon^2} \right) + \left(\frac{p_2^2}{\varepsilon^2} (E p_1 - p_2) \right) \right] b \\ & + \left[\frac{E'^2 q^2 E p_1}{P} \left(\frac{p_2}{P^2} - \frac{2 Q_1 x \cos^2 \theta}{\varepsilon} \right) p_2^3 + + \left(\frac{p_2^4}{\varepsilon^2} (E p_1 - p_2)(b-1) \right) \right], \end{aligned} \tag{39}$$

and the coefficients A_0, A_1 and A_2, being quite lengthy and not needed in the discussion of overstability, have not been written here.

Since σ_1 is real for overstability, the four values of $c_1 \left(= \sigma_1^2 \right)$ are positive. The sum of roots of (37) is $-A_3/A_4$, and if this is to be negative, then $A_3 > 0$ $A_4 > 0$. It is clear from (38) and (39) that A_3 and A_4 are always positive if

$$E p_1 > p_2, E p_1 > E' q, p_2^2 \pi^2 F / P^2 > M x \left(\cos^2 \theta \right) / \varepsilon^2 \text{ and } p_2 / P^2 > 2 Q_1 x \left(\cos^2 \theta \right) / \varepsilon \tag{40}$$

which imply that

$$\kappa < E\eta, \kappa < \frac{E\kappa'}{E'}, \nu > \frac{\rho_0 k_x^2}{\mu'} \left(\frac{c H k_1}{4 \varepsilon N e} \right)^2, \nu > \left(\frac{\mu_e}{2\pi\rho_0} \right)^{1/2} \left(\frac{k_1 H k_x}{\varepsilon} \right) \tag{41}$$

i.e. $$\kappa < min\left(E\eta, \frac{E\kappa'}{E'} \right), \nu > max\left[\frac{\rho_0 k_x^2}{\mu'} \left(\frac{c H k_1}{4 \varepsilon N e} \right)^2, \left(\frac{\mu_e}{2\pi\rho_0} \right)^{1/2} \left(\frac{k_1 H k_x}{\varepsilon} \right) \right] \tag{42}$$

thus $\kappa < min\left(E\eta, \frac{E\kappa'}{E'}\right), \nu > max\left[\frac{\rho_0 k_x^2}{\mu'}\left(\frac{cHk_1}{4\varepsilon Ne}\right)^2, \left(\frac{\mu_e}{2\pi\rho_0}\right)^{1/2}\left(\frac{k_1 Hk_x}{\varepsilon}\right)\right]$, therefore, are sufficient conditions for the non-existence of overstability, the violation of which does not necessarily imply occurrence of overstability.

7. Discussion. The inclusion of Hall currents gives rise to a cross flow i.e., a flow at right angles to the primary flow in a channel in the presence of a transverse magnetic field, has been shown by Sato [15] whereas Tani [18]. Tani [18] has found that Hall effect produces a cross-flow of double-swirl pattern in incompressible flow through a straight channel with arbitrary cross-section. This breakdown of the primary flow and formation of a secondary flow may be attributed to the inherent instability of the primary flow in the presence of Hall current. Sato [15] has pointed out that even if the distribution of the primary flow velocity be stable to external distrubances. the whole layer may become turbulent if the distribution of the cross-flow velocity is unstable. A similar situation occurs on the three-dimensional boundary layer along a swept-back wing. Gupta [3] has found that the presence of Hall current induces a vertical component of vorticity and this may well the reason for the destabilizing influence.

Therefore, for stationary convection, the magnetic field postpones the onset of convection whereas, the Hall currents hastens the onset of convection on the thermal instability of couple-stress fluid in porous medium in hydromagnetics in the presence of Hall currents. The medium permeability hastens the onset of convection whereas, the couple-stress postpones the onset of convection on the thermal instability of couple-stress fluid in porous medium in hydromagnetics in the presence of Hall currents for all wave numbers $(1+x) > Mx\cos^2\theta$, which is normally satisfied as the Hall currents parameter M is very small compared to unity. Graphs have been plotted by giving some numerical values to the parameters, to depict the stability characteristics. The stable solute gradient and the magnetic field (and corresponding Hall currents) introduce oscillatory modes in the system, which were non-existent in their absence. The sufficient condition for the non-existence of overstability for thermal instability in couple-stress fluid in presence of magnetic field in porous medium are

$\kappa < min\left(E\eta, \frac{E\kappa'}{E'}\right), \nu > max\left[\frac{\rho_0 k_x^2}{\mu'}\left(\frac{cHk_1}{4\varepsilon Ne}\right)^2, \left(\frac{\mu_e}{2\pi\rho_0}\right)^{1/2}\left(\frac{k_1 Hk_x}{\varepsilon}\right)\right]$. Moreover, in the absence of couple-stress vicosity ($\mu'=0$) and solute gradient, sufficient condition for non-existence of overstability for thermal instability in viscous, Newtonian fluid (Gupta [3]), as expected, reduces to $E\eta > \kappa$. Porosity 'ε' factor played an important role in developing the sufficient conditions for the non-existence of overstability. The result of the satisfaction of principal of exchange of stabilities

[Sec. 5] is true in porous as well as non-porous medium.

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REFERENCES

[1] N.M. Bujurke and G. Jayaraman, The influence of couple-stresses in squeeze films, *Int. J. Mech. Sci.*, **24** (1982), 369-376.

[2] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability,* Dover Publication, New York, 1981.

[3] A.S. Gupta, Hall effects on thermal instability, *Roumaine Math Pures Appl.,* **12** (1967), 665-677.

[4] D.D. Joseph, *Stability of Fluid Motions.* Vol. II., Springer-Verlag, Berlin, 1976.

[5] W.M. Lai, S.C. Kuei and V.C. Mow, Rheological equations for synovial fluids, *J. Biomechanical Engng.,* **100** (1978), 169-186.

[6] J.R. Lin, Couple-stress effect on the squeeze film characteristics of hemispherical bearing with reference to synovial joints, *Appl. Mech. Engng.,* **1** (1996) 317-332.

[7] D.A. Nield and A. Bejan, *Convection in Porous Medium,* Springer-Verlag, New York, 1999.

[8] D.A. Nield, S.L.M. Junqueira and J.L. Lage, Forced convection in a fluid saturated porous medium channel with isothermal or isoflux boundaries. *J. Fluid Mech.,* **322** (1996), 201-214.

[9] A. Sherman and G.W. Sutton, *Magnetohydrodynamics,* North-Western University Press, Evanston, IL, 1962.

[10] V.K. Stokes, Couple-stresses in fluids, *Phys. Fluids,* **9** (1966), 1709-1715.

[11] P. Sinha, C. Singh and K.R. Prasad, Couple-stresses in journal bearing lubricants and the effect of convection, *Wear,* **67** (1981), 15-24.

[12] R.C. Sharma and K.D. Thakur, On couple-stress fluid heated from below in porous medium in hydromagnetics, *Czech. J. Phys.,* **50** (2000), 753-758.

[13] R.C. Sharma, Sunil and M. Pal, On couple-stress fluid heated from below in porous medium in presence of rotation, *Appl. Mech. Engg.,* **5(4)** (2000), 883-896.

[14] E.A. Spiegel, Convective instability in a compressible atmosphere, *Astrophys. J.,* **141** (1965), 1068-1070.

[15] H. Sato, The Hall effect in the viscous flow of ionized gas between parallel plates under transverse magnetic field, *J. Phys. Soc. Japan,* **16** (1961), 1427-1433.

[16] T.G.L. Shirtcliffe, Lake Bonney Antarctica: Cause of the Elevated Temperatures, *J. Geophys. Res.,* **69** (1964), 5257-5268.

[17] H. Tabor and R. Matz, Solar Pond Project, *Solar Energy,* **9** (1965), 177-182.

[18] I. Tani, Steady flow of conducting fluids in channels under transverse magnetic fields, with consideration of Hall effect, *J. Aerospace Sci.,* **29** (1962), 297-305.

[19] G. Veronis, On finite amplitude instability in thermohaline convection, *J. Marine Res.,* **23** (1965), 1-17.

[20] A. Walicka, *Micropolar flow in a slot between rotating surfaces of revolution,* Zielona Góra, TU Press, 1994.

[21] E. Walicki and A. Walicka, Inertia effects in the squeeze film of a couple stress fluid in biological bearing, *Appl. Mech. Engng.,* **4** (1999), 363-373.

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DIFFERENTIAL RECURRENCE RELATIONS FOR BIORTHOGONAL TYPE POLYNOMIALS AND THEIR APPLICATIONS

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ABSTRACT

In the present paper, we define biorthogonal type polynomials and establish their differential recurrence relations through special generating functions and then, making an appeal to these relations, we obtain some more generating functions through Lie group theory.

1. Introduction. The biorthogonal polynomials play an important role in the theory of Approximation, Queueing, Coding and other branches of Applied Mathematics and Physics. They are used as weight function. Konhauser [6] has defined rather first the biorthogonal polynomials of degree n such that

$$Z_n^{\alpha}(x,k)=\frac{\Gamma(kn+\alpha+1)}{n!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{x^{kr}}{\Gamma(kr+\alpha+1)}.$$

Again, Carlitz [3], Prabhakar [7,8], Srivastava [10], Srivastava and Singhal [12] and Thakre [13] studied various biorthogonal polynomials along with generating functions, recurrence relations, orthogonal property and other relevant properties.

Motivating by above work in the present paper we define biorthogonal type polynomials such that

$$Y_n^{\alpha}(x,k)=\frac{\Gamma(1+\alpha+nk)}{n!}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(1+r)}{\Gamma(1+kr)}\frac{\Gamma(1+n-r)}{\Gamma(1+\alpha+kr)}\frac{(-x^k)^r}{\Gamma(1+nk-kr)}. \tag{1.1}$$

It is evident that for $k=1$

$$Z_n^{\alpha}(x,1)=Y_n^{\alpha}(x,1)=L_n^{\alpha}(x). \tag{1.2}$$

We also take the generating function concept due to Bateman [1,2] and Rainville [9] to establish the differential recurrence relations for the biorthogonal polynomials and make their applications to find further generating functions of these polynomials with the help of Lie group theory (see Srivastava and Monocha [11]).

2. Theorems. Before evaluation of recurrence relations for the

biorthogonal polynomials defined by (1.1), we prove two theorems here, in this section

Theorem 1. From $e^{t}\psi(x,t)=\sum_{n=0}^{\infty}\sigma_n(x,k)t^{nk}$,

it follows that $\sigma'_0(x,k)=0$ and for $n\geq 1, k>0;$

$$x\sigma'_n(x,k)=nk\sigma_n(x,k)-\sigma_{n-1/k}(x,k). \tag{2.1}$$

Proof. Consider the generating function

$$e^{t}\psi(x,t)=\sum_{n=0}^{\infty}\sigma_n(x,k)t^{nk}=F \tag{2.2}$$

Then,

$$\partial F/\partial x=te^{t}\psi'(x,t) \tag{2.3}$$

and

$$\partial F/\partial t=e^{t}\psi(x,t)+xe^{t}\psi'(x,t). \tag{2.4}$$

On eliminating $\psi(x,t),\psi'(x,t)$ due to (2.2), (2.3) and (2.4), we get

$$x(\partial F/\partial x)-t(\partial F/\partial t)+tF=0 \tag{2.5}$$

Alternatively, (2.5) may be written as

$$\sum_{n=0}^{\infty}x\sigma'_n(x,k)t^{nk}-\sum_{n=0}^{\infty}nk\sigma_n(x,k)t^{nk}+\sum_{n=0}^{\infty}\sigma_n(x,k)t^{nk+1}=0. \tag{2.6}$$

Further (2.6) may be put in the form

$$\sum_{n=0}^{\infty}x\sigma'_n(x,k)t^{nk}-\sum_{n=0}^{\infty}nk\sigma_n(x,k)t^{nk}+\sum_{n=1/k}^{\infty}\sigma_{n-1/k}(x,k)t^{nk}=0. \tag{2.7}$$

Finally, from the relation (2.7), we get the required result

$$x\sigma'_n(x,k)=nk\sigma_n(x,k)-\sigma_{n-1/k}(x,k).$$

Theorem 2. From $A(t)exp(-xt/(1-t))=\sum_{n=0}^{\infty}\rho_n(x,k)t^{nk}$,

it follows that $\rho'_0(x,k)=0, n\geq 1$ and $k>0$,

$$\rho'_n(x,k)=\rho'_{n-1/k}(x,k)-\rho_{n-1/k}(x,k) \tag{2.8}$$

and

$$\rho'_n(x,k) = -\sum_{m=0}^{n-1/k} \rho_m(x,k). \tag{2.9}$$

Proof. Consider the generating function

$$A(t)exp(-xt/(1-t)) = \sum_{n=0}^{\infty} \rho_n(x,k)t^{nk} = F, \tag{2.10}$$

we have

$$(1-t)(\partial F/\partial x) = -tF. \tag{2.11}$$

Hence, we get

$$\sum_{n=0}^{\infty} \rho'_n(x,k)t^{nk} - \sum_{n=0}^{\infty} \rho'_n(x,k)t^{nk+1} = -\sum_{n=0}^{\infty} \rho_n(x,k)t^{nk+1}, \tag{2.12}$$

which yields $\rho'_n(x,k) = 0$ and for $n \geq 1$, $k>0$,

$$\rho'_n(x,k) = \rho'_{n-1/k}(x,k) - \rho_{n-1/k}(x,k). \tag{2.13}$$

Again, rearranging (2.11) in the form

$$\partial F/\partial x = -(t/(1-t))F, \tag{2.14}$$

so that we obtain

$$\sum_{n=0}^{\infty} \rho'_n(x,k)t^{nk} = -\left(\sum_{n=0}^{\infty} t^{n+1}\right)\left(\sum_{n=0}^{\infty} \rho_n(x,k)t^{nk}\right) = -\sum_{n=0}^{\infty}\sum_{m=0}^{n} \rho_m(x,k)t^{nk+1}$$

$$= -\sum_{n=0}^{\infty}\sum_{m=0}^{n-1/k} \rho_m(x,k)t^{nk}$$

Hence, for $n \geq 1$, $k>0$, we get

$$\rho'_n(x,k) = -\sum_{m=0}^{n-1/k} \rho_m(x,k). \tag{2.15}$$

3. Applications of the theorems to obtain the differential recurrence relations. Consider the generating relation

$$e^t {}_0F_1(-;1+\alpha;-xt) = \sum_{n=0}^{\infty} \frac{Y_n^{\alpha}(x,k)t^{nk}}{(1+\alpha)_{nk}} \tag{3.1}$$

and theorem 1, we get the relation

$$\frac{x}{(1+\alpha)_{nk}} \frac{d}{dx} Y_n^{\alpha}(x,k) = \frac{nk}{(1+\alpha)_{nk}} Y_n^{\alpha}(x,k) - \frac{nk}{(1+\alpha)_{nk-1}} Y_{n-1/k}^{\alpha}(x,k), \tag{3.2}$$

which readily yields the relation

$$x(d/dx)Y_n^{\alpha}(x,k) = nkY_n^{\alpha}(x,k) - (\alpha + nk)Y_{n-1/k}^{\alpha}(x,k). \tag{3.3}$$

On replacing n by $n+1/k$, we find

$$x(d/dx)Y_{n+1/k}^{\alpha}(x,k) = (nk+1)Y_{n+1/k}^{\alpha}(x,k) - (1+\alpha+nk)Y_n^{\alpha}(x,k). \tag{3.4}$$

Further, consider the generating relation

$$(1-t)^{-1-\alpha} exp(-xt/(1-t)) = \sum_{n=0}^{\infty} Y_n^{\alpha}(x,k)t^{nk}, \tag{3.5}$$

and theorem 2, we find

$$\frac{d}{dx}Y_n^{\alpha}(x,k) = \frac{d}{dx}Y_{n-1/k}^{\alpha}(x,k) - Y_{n-1/k}^{\alpha}(x,k). \tag{3.6}$$

Now, replacing n by $n+1/k$, we get

$$\frac{d}{dx}Y_{n+1/k}^{\alpha}(x,k) = \frac{d}{dx}Y_n^{\alpha}(x,k) - Y_n^{\alpha}(x,k) \tag{3.7}$$

Again, making an appeal to (3.4) and (3.7), we obtain the relation

$$x\frac{d}{dx}Y_n^{\alpha}(x,k) = (nk+1)Y_{n+1/k}^{\alpha}(x,k) - (1+\alpha+nk-x)Y_n^{\alpha}(x,k). \tag{3.8}$$

4. Applications of the differential recurrence relations to obtain further generating functions by Lie thoery. To find the further generating functions due to Lie theory, we choose following relations for the eigen function $Y_n^{\alpha}(x,k)y^{kn}$ as

$$A\left[Y_n^{\alpha}(x,k)y^{kn}\right] = a_{n,k}Y_n^{\alpha}(x,k)y^{kn}, \tag{4.1}$$

$$B\left[Y_n^{\alpha}(x,k)y^{kn}\right] = b_{n,k}Y_{n-1/k}^{\alpha}(x,k)y^{kn-1}, \tag{4.2}$$

$$C\left[Y_n^{\alpha}(x,k)y^{kn}\right] = C_{n,k}Y_{n+1/k}^{\alpha}(x,k)y^{kn+1}, \tag{4.3}$$

provided that A,B,C are first order differential operators and $a_{n,k}$, $b_{n,k}$ and $c_{n,k}$ are the expressions in n and k, and are independent of x and y but not necessarily of α.

Now, making an appeal to the differential recurrence relations (3.3) and (3.8) and Weisner's technique [14-16], we get the operators

$$A = y(\partial/\partial y) + (\alpha+1)/2, \tag{4.4}$$

$$B = xy^{-1}(\partial/\partial x) - \partial/\partial y, \tag{4.5}$$

and $$C = xy(\partial/\partial x) + y^2(\partial/\partial y) + (1+\alpha-x)y. \tag{4.6}$$

The commutation relations due to above operators follow

$$[A,B] = AB-BA = -B, \tag{4.7}$$

$$[A,C] = AC-CA = C, \tag{4.8}$$

and $$[C,B] = CB-BC=2A. \tag{4.9}$$

Again, we consider, here, an operator

$$D=CB+A^2-A$$

$$= x\left[x\left(\partial^2/\partial x^2\right)+(1+\alpha-x)(\partial/\partial x)+y(\partial/\partial y)\right]+\left(\alpha^2-1\right)/4 \tag{4.10}$$

which commutes with A,B and C.

Again, suppose $f(x,y)$ is a common eigen function of the operators D and B and that satisfy the equations

$$Df(x,y)=(\alpha^2-1)f(x,y)/4, \tag{4.11}$$

and

$$Bf(x,y)=-f(x,y), \tag{4.12}$$

which gives us the partial differential equations

$$\left[x\frac{\partial^2}{\partial x^2}+(1+\alpha-x)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right]f(x,y)=0 \tag{4.13}$$

and

$$\left(xy^{-1}\frac{\partial}{\partial x}-\frac{\partial}{\partial y}-1\right)f(x,y)=0 \tag{4.14}$$

Setting $f(x,y)=e^yR(x.y)$ in (4.13) and with the use of (4.14), we get (4.15)

$$\left[u\frac{d^2}{du^2}+(1+\alpha)\frac{d}{du}+1\right]R(u)=0, \text{ where } u=xy. \tag{4.16}$$

Due to Srivastava and Manocha [§ 1.4 (11)], (4.16) gives the solution

$$R(u)=\Gamma(\alpha+1)u^{-\alpha/2}J_\alpha\left(2\sqrt{u}\right)={}_0F_1\left[\begin{matrix}- \ ; \\ 1+\alpha;\end{matrix}-u\right]. \tag{4.17}$$

Therefore, from (4.15) and (4.17), we have

$$f(x,y)=e^y{}_0F_1\left[\begin{matrix}- \ ; \\ 1+\alpha;\end{matrix}-xy\right] \tag{4.18}$$

Now if $\mathbb{G}$ and $\mathbb{G}'$ are the Lie algebras of local Lie groups G and G' respectively and then G and G' are isomorphic (see Srivastava and Manocha [11] thorem 5 p. 319).

Let T denotes the local isomorphism of G onto a neighbourhood of the identity element in G'. Then T is a multiplier representation of G on the representation space $A_{x\beta}$. Particularly, if $L_\beta \varepsilon G'$ is an isomorphic image of $\beta \varepsilon G$, we have

$$[T(e^{\beta t})f](x) = exp(tL_\beta)f(x) = \sum_{n=1}^{\infty} \frac{t^n}{n!}[L_\beta^n f](x), \tag{4.19}$$

where t lies in a sufficiently small neighbourhood of $0 \ \varepsilon \ \mathbb{C}$.
Now, to determine the multiplier representation

$[T(g)f](x,y)$ with the aid (4.19), where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon SL_2$. (4.20)

corresponding to A,B and C operators (See Srivastava and Manocha [§6.4 (18)]). we have

$$[T(g)f](x,y) = (by+d)^{-\alpha-1} exp\left(\frac{bxy}{by+d}\right).f\left(\frac{xy}{(ay+c)(by+d)}, \frac{ay+c}{by+d}\right)$$

where $|by/d| < 1, |c/ay| < 1$ (4.21)

and g given by (4.20) with $ad-bc=1$, lies in a sufficiently small neighbourhood

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \varepsilon SL_2 . \tag{4.22}$$

Now, making an appeal to (4.18), (4.19), (4.20) and (4.21) with the condition (4.22), we find

$$[T(g)f](x,y) = (by+d)^{-\alpha-1} exp\left(\frac{ay+bxy+c}{by+d}\right) {}_0F_1\left[\begin{matrix} - \ ; \\ 1+\alpha ; \end{matrix} \frac{-xy}{(by+d)^2}\right] \tag{4.23}$$

which satisfies

$$D[T(g)f](x,y) = (1/4)(\alpha^2 - 1)[T(g)f](x,y) \tag{4.24}$$

Since $[T(g)f](x,y)$ is analytic at $y=0$, hence we have

$$[T(g)f](x,y) = \sum_{n=0}^{\infty} p_{n,k}(g) L_n^{\alpha}(x,k) y^{nk} \tag{4.25}$$

To compute the coefficient $p_{n,k}(g)$, we put $x=0$ in (4.25) and with the aid of (4.23) we have

$$(by+d)^{-\alpha-1}\, exp\left(\frac{ay+c}{by+d}\right)=\sum_{n=0}^{\infty}\frac{(1+\alpha)_{nk}}{(nk)!}p_{n,k}(g)y^{nk}. \tag{4.26}$$

from which, on making an appeal to (3.5), we derive

$$(by+d)^{-\alpha-1}\, exp\left(\frac{ay+c}{by+d}\right)=e^{c/d}d^{-1-\alpha}(-b/d)^{nk}Y_n^{\alpha}(1/(bd),k)y^{nk}. \tag{4.27}$$

Now from (4.26) and (4.27), we obtain

$$p_{n,k}(g)=\frac{\Gamma(1+\alpha)\Gamma(1+nk)}{\Gamma(1+\alpha+nk)}e^{c/d}d^{-1-\alpha}(-b/d)^{nk}.Y_n^{\alpha}(1/(bd),k). \tag{4.28}$$

Thus, making an appeal to (4.23), (4.25) and (4.28), we get

$$\left(1+\frac{by}{d}\right)^{-\alpha-1} exp\left[\frac{y(1+bdx)}{d(by+d)}\right]\, {}_0F_1\left[\begin{matrix}-\ ;\\ 1+\alpha;\end{matrix}\frac{-xy}{(by+d)^2}\right]$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma(1+nk)\Gamma(1+\alpha)}{\Gamma(1+\alpha+nk)}Y_n^{\alpha}(x,k)Y_n^{\alpha}\left(\frac{1}{bd},k\right)\left(\frac{-by}{d}\right)^{nk};\left|\frac{by}{d}\right|<1. \tag{4.29}$$

Again, setting $b=i/\sqrt{w}$ and $d=-i/\sqrt{w}$, $i=\sqrt{(-1)}$ in (4.29), we find

$$(1-y)^{-\alpha-1}\, exp\left[\frac{-(x+w)y}{(1-y)}\right]{}_0F_1\left[\begin{matrix}-\ ;\\ 1+\alpha;\end{matrix}\frac{wxy}{(1-y)^2}\right]$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma(1+\alpha)\Gamma(1+nk)}{\Gamma(1+\alpha+nk)}Y_n^{\alpha}(x,k)Y_n^{\alpha}(w,k)y^{nk};|y|<1. \tag{4.30}$$

5. Special Case. For $k=1$, (4.30) with the appeal to (1.1) and (1.2), gives the formula due to Hardy [4] and Hille [5]

$$(1-y)^{-\alpha-1}\, exp\left[\frac{-(x+w)y}{(1-y)}\right]{}_0F_1\left[\begin{matrix}-\ ;\\ 1+\alpha;\end{matrix}\frac{wxy}{(1-y)^2}\right]=\sum_{n=0}^{\infty}\frac{n!}{(1+\alpha)_n}L_n^{\alpha}(x)L_n^{\alpha}(w)y^n;|y|<1 \tag{5.1}$$

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REFERENCES

[1] H. Bateman, A generalization of the Legendre polynomials, *Proc. London Math. Soc. Series* **2,3** (1905), 111-123.

[2] H. Bateman, Some properties of a certian set of polynomials, *Tohoku Math. Journal,* **37** (1933), 23-38.

[3] L. Carlitz, A note on certain bioorthogonal polynomials, *Paci. J. Math.* **24** (1968), 425-430.

[4] G.H. Hardy, Summation of a series of polynomials of Laguerre, *J. London Math. Soc.* **7** (1932), 138-139; Addendum Ibid **7** (1932) 192.

[5] E. Hille, On Laguerre's series I, II and III *Proc. Nat. Acad. Sci. U.S.A.*, **12** (1926), 261-269 and 348-352.

[6] J.D.E. Konhauser, Biorthogonal polynomials suggested by Laguerre polynomials, *Paci. J. Math.* **21** (1967), 303-314.

[7] T.R. Prabhakar, On a set of polynomials suggested by Laguerre polynomials. *Paci. J. Math.* **35**, (1970), 213-219.

[8] T.R. Prabhakar, On the other set of biorthogonal polynomials suggested by Laguerre polynomials, *Paci. J. Math.* **37** (1971), 801-804.

[9] E.D. Rainville, *Special Funcitons,* Mac-Millan, New York (1960); Chelsea Pub. Co. Bronx, New York (1971).

[10] H.M. Srivastava, A note on the Konhauser sets of biorothogonal polynomials suggested by Laguerre polynomials, *Paci. J. Math.*, **90** (1980), 197-200.

[11] H.M. Srivastava and H.L. Manocha, *A Treatise on generating Functions,* John Willey and Sons, (1984).

[12] H.M. Srivastava and J.P. Singhal, A class of polynomials defined by generalized Rodrigues' formula, *Ann. Math. Pure. Appl.* (4) **90** (1971), 75-80.

[13] N.K. Thakare, Some open problems in the theory of biorthogonal polynomials, *Proc. S.S.F.A.*, (2002), 13-39.

[14] L. Weisner, Group theoretic origin of certain generating functions, *Pacific J. Math.* **5** (1955), 1033.

[15] L. Weisner, Generating functions for Hermite polynomials. *Canad. J. Math.* **14** (1959), 141.

[16] L. Weisner, Generating functions for Bessel functions, *Canad. J. Math.*, **11** (1959), 148.

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A THEOREM ON LOCALIZATION OF $|N,p_n|$ SUMMABILITY OF A FACTORED FOURIER SERIES

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ABSTRACT

In the present paper, our object is to investigate a suitable type of factor so that the summability $|N,p_n|$ of the factored Fourier series becomes a local property. It may be observed that for $p_n=1/(n+1)$, our theorem reduces to the results of a well known theorem due to Lal (1963) on the absolute harmonic summability of the factored Fourier series.

1. Introduction. Let Σu_n be an infinite series with the sequence $\{S_n\}$ of its partial sums. Let $\{p_n\}$ be sequence of constants with $p_n>0$ and

$$P_n = \sum_{m=0}^{n} p_m.$$

The sequence-to sequence transformation given by

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}S_k \tag{1.1}$$

defines the nth Nörlund mean or the nth (N,p_n) mean of the sequence $\{S_n\}$ of partial sums of the series Σu_n.

If t_n tends to a fixed and finite sum S as $n\to\infty$, then the series Σu_n or the sequence $\{S_n\}$ of its partial sums is said to be summable (N,p_n) to the sum S [Hardy [2]]. The conditions for the regularity of the summability (N,p_n) defind by (1.1) are

$$\lim_{n\to\infty}(p_n/P_n) \text{ and } \sum_{v=0}^{n}|p_v| = o(p_n).$$

For $p_n = 1/(n+1), P_n \sim \log n,$

the Nörlund mean (N,p_n) reduces to the harmonic mean $(N+1/(n+1))$ [Riesz [5]].

The series Σu_n is said to be absolutely summable (N,p_n) or simply summable $|N,p_n|$ if the sequence $\{t_n\}$ is of bounded variation. Mears [4].

Let $f(t)$ be a 2π-periodic and Lebesgue integrable function of t in the inerval $(-\pi,\pi)$. Without any loss of generality, we may assume that the constant term in

the Fourier series of $f(t)$ is zero, i.e $\int_{-\pi}^{\pi} f(t)dt = 0$,

and then

$$f(t) \sim \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t).$$

We use the following notations :

$$\phi(t) = [f(x+t) + f(x-t) - 2f(x)]/2,$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma\alpha}\int_0^t (t-u)^{\alpha-1}\phi(u)du, \qquad (\alpha > 0)$$

$$\Phi_0(t) = \phi(t),$$

$$\phi_n(t) = \Gamma(\alpha+1)t^{-\alpha}\Phi_\alpha(t), \qquad (0 \le \alpha \le 1)$$

$$\Delta p_n = p_n - p_{n+1}.$$

Lal [3] has discussed the localization problem for $|N,1/(n+1)|$ summability of the factored Fourier series by proving the following

Theorem A. If $\{\lambda_n\}$ is a convex sequence such that the serise $\sum n^{-1}\lambda_n$ is convergent, then the summability $|N,1/(n+1)|$ of the series $\sum(A_n(t)log(n+1)\lambda_n/n)$ at a point can be ensured by a local property.

Applying the absolute Nörlund summability method, which is more genral than $|N,1/(n+1)|$ summability, the object of this paper is to investigate a suitable type of factor so that the summability $|N,p_n|$ of the factored Fourier series becomes a local property.

In what follows we establish the following main

Theorem. If $\{p_n\}$ and $\{\Delta p_n\}$ are both non-negative, monotonic, non-increasing and $\{\lambda_n\}$ is a convex sequence such that is $\sum p_n\lambda_n$ is convergent, then $|N,p_n|$ summability of $\Sigma A_n(t)\lambda_n P_n p_n$ depends only on the behaviour of the generating function $f(t)$ in the immediate neighbourhood of the points $t=x$, provided

$$\sum_{k=0}^{m-2}\left|p_{k+1}^2\lambda_{k_{+1}}\right| = o(1) \tag{1.2}$$

and

$$\sum_{k=0}^{m-2} P_{k+1}p_{k+1}\Delta\lambda_{k_{+1}} = o(1). \tag{1.3}$$

2. **Proof of the Main Theorem.** To prove our theorem we require the following

Lemma (Basanquet and Kastelman [1] Theorem 1).

Suppose that $f_n(x)$ is measurable in (a,b) where $b\text{-}a\leq\infty$ for $n=1,2,\ldots$. Then a necessary and sufficient condition that, for every function $\lambda(x)$ integrable (L) over (a,b) the functions $f_n(x)\lambda(x)$ should be integrable (L) in (a,b) and

$$\sum_{n=0}^{\infty}\left|\int_a^b\lambda(x)f_n(x)dx\right|\leq K,$$

is that

$$\sum_{n=1}^{\infty}|f_n(x)|\leq K,$$

where K is an absolute constant, for almost every x in (a,b).

Since

$$t_n=\frac{1}{P_n}\sum_{\upsilon=0}^{n}P_{n-\upsilon}u_\upsilon=\frac{1}{P_n}\sum_{\upsilon=0}^{n}P_\upsilon u_{n-\upsilon}, \tag{2.1}$$

where

$$u_n=A_n(t)p_nP_n\lambda_n,$$

we have

$$\begin{aligned}t_n-t_{n-1}&=\sum_{\upsilon=0}^{n-1}(P_\upsilon/P_n-P_{\upsilon-1}/P_{n-1})u_{n-\upsilon}\\&=\frac{1}{P_nP_{n-1}}\sum_{\upsilon=0}^{n-1}(P_np_\upsilon-P_\upsilon p_n)u_{n-\upsilon}\\&=\frac{1}{P_nP_{n-1}}\sum_{\upsilon=0}^{n-1}(P_np_{n-\upsilon-1}-P_{n-\upsilon-1}p_n)u_{\upsilon+1}.\end{aligned} \tag{2.2}$$

For the Fourier series of $f(t)$ at $t=x$

$$A_n(t)=\frac{2}{\pi}\int_0^\pi\phi(t)\cos nt\,dt,$$

so that

$$\begin{aligned}t_n-t_{n-1}&=\frac{2}{\pi}\int_0^\pi\phi(t)\left\{\frac{1}{P_nP_{n-1}}\sum_{k=0}^{n-1}(P_np_{n-k-1}-P_{n-k-1}p_n)P_{k+1}p_{k+1}\lambda_{k+1}\cos(k+1)t\right\}dt\\&=\frac{2}{\pi}\int_0^\pi\phi(t)K(n,t)dt,\text{say}\end{aligned} \tag{2.3}$$

Hence

$$\sum_{n=2}^{\infty}|t_n-t_{n-1}|\leq\sum_{n=2}^{\infty}\left|\frac{2}{\pi}\int_0^\delta\phi(t)K(n,t)dt\right|+\sum_{n=2}^{\infty}\left|\frac{2}{\pi}\int_\delta^\pi\phi(t)K(n,t)dt\right|.$$

In order to prove the theorem, it is now required prove that

$$\sum_{n=2}^{\infty}\left|\frac{2}{\pi}\int_\delta^\pi\phi(t)K(n,t)dt\right|<\infty \tag{2.4}$$

But by virtue of the lemma, it is a sufficient for our purpose to show that

$$\sum_{n=2}^{\infty}|K(n,t)|dt\leq A, \tag{2.5}$$

for $0 < \delta \le t \le \pi$, where A is a positive constant, not necessarily the same at each occurrence.

Now

$$\sum_{n=2}^{m}|K(n,t)| = \sum_{n=2}^{m}\frac{1}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}(P_np_{n-k-1} - P_{n-k-1}p_n)P_{k+1}p_{k+1}\lambda_{k+1}\cos(k+1)t\right|$$

$$= \sum_{n=2}^{m}\frac{1}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}M(n,k,t)\right|, \quad \text{say.} \tag{2.6}$$

Applying Abel's transformation, we get

$$\sum_{k=0}^{n-1}M(n,k,t) = \sum_{k=0}^{n-2}\left[\Delta\{(P_np_{n-k-1} - P_{n-k-1}p_n)P_{k+1}p_{k+1}\lambda_{k+1}\}\sum_{\upsilon=0}^{k}\cos(\upsilon+1)t\right]$$

$$+(P_np_0 - P_0p_n)P_np_n\lambda_n\sum_{\upsilon=0}^{n-1}\cos(\upsilon+1)t \quad . \tag{2.7}$$

Therefore, for $0 < \delta \le t \le \pi$, we have

$$\left|\sum_{k=0}^{n-1}M(n,k,t)\right| \le A\sum_{k=0}^{n-2}[\Delta\{(P_np_{n-k-1} - P_{n-k-1}p_n)P_{k+1}p_{k+1}\lambda_{k+1}\}] + AP_n^2p_n\lambda_n \ . \tag{2.8}$$

Clearly

$$\sum_{n=2}^{m}\frac{1}{P_nP_{n-1}}.AP_n^2p_n\lambda_n \le A\sum_{n=2}^{m}p_n\lambda_n = o(1) \tag{2.9}$$

as $m \to \infty$, since $\sum p_n\lambda_n$ has been assumed to be convergent, so that the sequence of its partial sums is bounded.

Further

$$\sum_{k=0}^{n-2}|\Delta|\{(P_np_{n-k} - P_{n-k-1}p_n)P_{k+1}p_{k+1}\lambda_{k+1}\}|$$

$$\le \sum_{k=0}^{n-2}|\Delta(P_np_{n-k-1} - P_{n-k-1}p_n)|P_{k+1}p_{k+1}\lambda_{k+1} + \sum_{k=0}^{n-2}|(P_np_{n-k-1} - P_{n-k-2}p_n)\Delta(P_{k+1}p_{k+1}\lambda_{k+1})| \tag{2.10}$$

Hence

$$\sum_{n=2}^{m}\frac{1}{P_nP_{n-1}}\sum_{k=0}^{n-2}\Delta|(P_np_{n-k-1} - P_{n-k-1}p_n)|P_{k+1}p_{k+1}\lambda_{k+1}$$

$$= \sum_{k=0}^{m-2}(P_{k+1}p_{k+1}\lambda_{k+1})\sum_{n=k+2}^{m}\frac{|\Delta(P_np_{n-k-1} - P_{n-k-1}p_n)|}{P_nP_{n-1}}$$

$$\le \sum_{k=0}^{m-2}(P_{k+1}p_{k+1}\lambda_{k+1})\sum_{n=k+2}^{m}\left[\frac{|\Delta p_{n-k-1}|}{p_{n-1}} + \frac{p_{n-k-1}p_n}{P_nP_{n-1}}\right]$$

$$= o\left[\sum_{k=0}^{m-2}(p_{k+1}\lambda_{k+1})\sum_{n=k+2}^{m}|\Delta p_{n-k-1}|\right] + o\left[\sum_{k=0}^{m-2}(P_{k+1}p_{k+1}\lambda_{k+1})\sum_{n=k+2}^{m}\frac{p_n}{p_n p_{n-1}}\right]$$

$$= o\left[\sum_{k=0}^{m-2}p_{k+1}\lambda_{k+1}\right]$$

$=o(1)$, as $m\to\infty$. (2.11)

Again

$$\sum_{k=0}^{n-2}\left|(P_n p_{n-k-2} - P_{n-k-2}p_n)\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|$$

$$\leq \sum_{k=0}^{n-2}(P_n - P_{n-k-2})p_n\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right| + \sum_{k=0}^{n-2}(p_{n-k-2} - p_n)P_n\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right| \quad (2.12)$$

But

$$\sum_{n=2}^{m}\frac{1}{P_n P_{n-1}}\sum_{k=0}^{n-2}(P_n - P_{n-k-2})\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|$$

$$= \sum_{k=0}^{n-2}\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|\sum_{n=k+2}^{m}(P_n - P_{n-k-2})\Delta(1/p_{n-1})$$

$$\leq A\sum_{k=0}^{m-2}\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|P_{k+2}\sum_{n=k+2}^{m}\Delta(1/p_{n-1})$$

$$= o\left[\sum_{k=0}^{m-2}\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|\right]$$

$$= o\left[\sum_{k=0}^{m-2}\left|p_{k+1}^2\lambda_{k+1}\right|\right] + o\left[\sum_{k=0}^{m-2}P_{k+1}p_{k+1}\Delta\lambda_{k+1}\right] + o\left[\sum_{k=0}^{m-2}P_{k+1}\lambda_{k+1}\Delta p_{k+1}\right]$$

$=o(1)+o(1)$, (using (1.2) and (1.3))

$=o(1)$ (2.13)

as $m\to\infty$, since $(P_n - P_{n-k-2})$ decreases as n increases.

Also

$$\sum_{n=2}^{m}\frac{1}{P_n P_{n-1}}\sum_{k=0}^{n-2}(p_{n-k-2} - p_n)P_n\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|$$

$$= \sum_{n=2}^{m}\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}(p_{n-k-2} - p_n)\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|$$

$$\leq \sum_{n=2}^{m}\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}(p_{n-k-2} - p_{n-k-1})(k+2)\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|$$

$$= \sum_{k=0}^{m-2}(k+2)\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|\sum_{n=k+2}^{m}\left(\frac{p_{n-k-2} - p_{n-k-1}}{P_{n-1}}\right)$$

$$\leq \sum_{k=0}^{m-2} \frac{k+2}{P_{k+1}} \left|\Delta\left(P_{k+1} p_{k+1} \lambda_{k+1}\right)\right| \sum_{n=k+2}^{m} \left(p_{n-k-2} - p_{n-k-1}\right)$$

$$= o\left[\sum_{k=0}^{m-2} \frac{(k+2)p_{k+1}}{P_{k+1}} \left(p_{k+1}^{2} \lambda_{k+1}\right)\right] + o\left[\sum_{k=0}^{m-2} P_{k+1} p_{k+1} \Delta\lambda_{k+1}\right] + o\left[\sum_{k=0}^{m-2} P_{k+1} \Delta p_{k+1} \lambda_{k+1}\right]$$

$=o(1)$ as $m \to \infty$ in (2.13); [sinc $np_n \leq P_n$]. (2.14)

Now the required result in (2.5) follows with the help of the results from (2.6) to (2.14) and this completes the proof of our main thorem.

REFERENCES

[1] L.S. Bosanqnet and H. Kastelman, The absolute convergence of a series of integrals, *Proc. Lond. Math. Soc.*, **45** (1939) 88-97.

[2] G.H. Hardy, *Divergent Series*, Oxfrod (1959).

[3] S.N. Lal , On the absolute harmonic summability of the factored Fourier series. *Proc. Amer. Math. Soc.*, **14** (1963), 311-319.

[4] F.M. Mears, Some multiplication theorems for the Nörlund mean. *Bull. Amer. Math. Soc.*, **41** (1935), 875-880.

[5] M. Riesz, Sur l' equivalence de certain methods le sommation, *Proc. Lond. Math. Soc*, **22** (2), (1924), 412-419.

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SOME COMMON FIXED POINT THEOREMS IN FUZZY 3-METRIC SPACES

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ABSTRACT

The aim of this paper is to prove some common fixed point theorems in fuzzy 3-metric spaces by removing the assuption of continuity, relaxing the compatibility or compatibility of type (α) or compatibility of type (β), to weak compatibility and replacing the completeness of the space with a set of alternative condition.

Keywords . Fuzzy metric spaces, coincidence point, common fixed point, compatible maps, weakly compatible maps.

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1. Introduction and Preliminaries . The concept of fuzzy sets was introduced initially by Zadeh [39] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and applications. Especially, Deng [5], Erceg [6], Kaleva and Seikkala [20], Kramosil and Michalek [22] have introduced the concept of fuzzy metric spaces in different ways. Recently many authors have also studied the fixed point theory in these fuzzy metric spaces ([1], [2], [7], [9], [10], [18],[19],[22], [32],[33],[35]). Mishra et al. [23] proved common fixed point theorems on complete fuzzy metric spaces, which generalized, extended and fuzzified several known fixed point theorems for contractive type maps on metric and other spaces. They assumed continuity of

one map in each of the two pairs of compatible maps and also the commutativity of continuous maps. Cho [4] and Jung et al. [18] extended and generalized several fixed point theorems on metric spaces, Menger probabilistic metric spaces, uniform spaces and proved common fixed point theorems on complete fuzzy metric spaces. The result of Cho [4] was extended by Sharma [34] and Sharma and Deshpande [35].

Jung et al. [18], Hadzic [10], Jungck et al. [16] and Singh et al. [31] have proved common fixed point theorems for mappings under the condition of continuity and compatibility of type (α) in complete fuzzy metric, probabilistic metric and metric spaces.

In this paper we remove the assumption of continuity, relaxing compatibility or compatibility of type (α) or compatibility of type (β) to weak compatibility and replacing the completenss of space with set of alternative conditions. We also remove the assumption of commutativity of continuous maps in case of two pairs of maps. We extend the results of Sharma and Deshpande [36].

Now we begin with some definitions.

Definition 1. (Sharma [33]) A binary operation$*:[0,1]^4\to[0,1]$ is called a continuous t-norm if $([0,1],*)$ is an abelian topological monoid with unit 1 such that $a_1*b_1*c_1*d_1\le a_2*b_2*c_2*d_2$ whenever $a_1\le a_2$, $b_1\le b_2, c_1\le c_2$ and $d_1\le d_2$ for all a_1,a_2,b_1,b_2, c_1, c_2, and d_1, d_2 are in [0,1].

Definition 2. The 3-tuple $(X,M,*)$ is called a fuzzy 3-metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^4\times[0,\infty)$ satisfying the following conditions for all $x,y,z,u,w\in X$ and $t_1,t_2,t_3,t_4>0$:

(*FM-1*) $M(x,y,z,w,0)=0$,

(*FM-2*) $M(x,y,z,w,t)=1$, for all $t>0$,

(Only when the three simplex $<x,y,z,w>$ degenerate)

(*FM-3*) $M(x,y,z,w,t)=M(x,w,z,y,t)=M(y,z,w,x,t)=M(z,w,x,y,t)=\ldots$

(*FM-4*) $M(x,y,z,w,t_1+t_2+t_3+t_4)\ge M(x,y,z,u,t_1)*M(x,y,u,w,t_2)*M(x,u,z,w,t_3)*M(u,y,z,w,t_4)$

(*FM-5*) $M(x,y,z,w,.):[0,1)\to[0,1]$ is left continuous.

Definition 3. Let $(X,M,*)$ be a fuzzy 3-metric space:

(1) A sequence $\{x_n\}$ in fuzzy 3-metric space X is said to be convergent to a point $x\in X$, if

$$\lim_{n\to\infty} M(x_n,x,a,b,t)=1$$

for all $a,b\in X$ and $t>0$.

(2) A sequence $\{x_n\}$ in fuzzy 3-metric space X is called a Cauchy sequence, if

$$\lim_{n\to\infty} M(x_{n+p},x_n,a,b,t)=1$$

for all $a,b \in X$ and $t>0, p>0$.

(3) A fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 4. A pair of mappings A and S is said to be weakly compatible in fuzzy 3-metric space if they commute at coincidence points.

Example 1. Let (X,d) be 3-metric space define $a*b=ab$ or $a*b=min\{a,b\}$ and for all $x,y,a,b \in X$ and $t>0$

$$M(x,y,b,t)=t/(t+d(x,y,a,b)) . \tag{1.b}$$

Then $(X,M,*)$ is fuzzy 3-metric space. We call the fuzzy metric M induced by the metric d the standard fuzzy metric.

Remark (Sharma [33]) Since * is continuous, it follows from (*FM-4*) that the limit of the sequence is fuzzy 3-metric space is uniquely determined. *Let*$(X,M,*)$ is a fuzzy 3-metric space with the following condition :

(*FM-6*) $\lim_{t\to\infty} M(x,y,z,w,t)=1$ for all $x,y,z,w \in X$.

Example 2. Define $A,S : [0,3] \to [0,3]$ by

$$A(x)\begin{bmatrix} x \text{ if } x \in [0,1) \\ x \text{ if } x \in [1,3] \end{bmatrix} \text{ and } S(x)=\begin{bmatrix} 3-x & \text{if } x \in [0,1) \\ 3 & \text{if } x \in [1,3) \end{bmatrix}$$

Then for any $x \in [1,3]$, $ASx=SAx$, showing that A,S are weakly compatible maps on $[0,3]$.

Example 3. *Let* $X=[0,2]$ with the metric d defined by $d(x,y) = |x-y|$. For each $t \in (0,\infty)$ define

$$M(x,y,t)=t/(t+d(x,y)), M(x,y,0)=0,\ x,y \in X.$$

Clearly $M(X,M,*)$ is a fuzzy metric space on X where * is defined by $a*b=ab$ or $a*b=min\ \{a,b\}$. Define $A,B : X \to X$ by $Ax=x$ if $x \in [0,1/3)$, $Ax=1/3$ if $x \geq 1/3$ and $Bx=x/(1+x)$, $x \in [0,2]$.

Consider the sequence $\{x_n=1/2+1/n;\ n\geq 1\}$ in X. Then

$Lim_{n\to\infty} Ax_n=1/3$ and $Lim_{n\to\infty} Bx_n=1/3$. But

$Lim_{n\to\infty} M(ABx_n,BAx_n,t)=t/(t+|1/3-1/4|)\neq 1$.

Thus A and B are non compatible, but A and B are commuting at their coincidence point $x=0$, that is, weakly compatible at $x=0$. Also

$Lim_{n\to\infty} M(ABx_n,BBx_n,t)=t/(t+|1/3-1/4|)\neq 1$.

and

$Lim_{n\to\infty} M(BAx_n,AAx_n,t)=t/(t+|1/4-1/3|)\neq 1$.

Further

$Lim_{n\to\infty} M(AAx_n,BBx_n,t)=t/(t+|1/3-1/4|)\neq 1$.

Thus A and B are not compatible of type (β).

In view of this example, we observe that

(i) Weakly compatible maps need not to compatible.

(ii) Weakly compatible maps need not be compatible of type (α).

(iii) Weakly compatible maps need not be compatible of type (β).

Lemma 1 : (Sharma [33]). For all $x,y,z,w \in X$, $M(x,y,z,w,.)$ is non decreasing.

Lemma 2. (Sharma [33]). Let $\{y_n\}$ be sequence in a fuzzy 3-metric space $(X,M,*)$ with the condition (*FM-6*). If there exists a number $k \in (0,1)$ such that

$$M(y_{n+2},y_{n+1},a,b,kt) \geq M(y_{n+1},y_n,a,b,t)$$

for all $t>0$ and $a,b \in X$ and $n=1,2,...$ then $\{y_n\}$ is a Cauchy sequence in X.

Lemma 3. (Sharma [33]). If for all $x,y,a,b \in X$, $t>0$ and for a number $k \in (0,1)$,

$$M(x,y,a,b,kt) \geq M(x,y,a,b,t), \text{ then } x=y.$$

2. Main results. Sharma and Deshpande [36] proved the following

Theorem A. Let $(X,M*)$ be a fuzzy metric space with $t*t \geq t$ for all $t \in [0,1]$ and the condition (*FM-6*). Let A, B, S, T be mappings from X into itself such that

(1) $A(X) \subset T(X)$, $B(X) \subset S(X)$,

(2) there exists a constant $k \in (0,1)$ such that

$$M(Ax,By,kt) \geq M(Ty,By,t)*M(Sx,Ax,t)*M(Sx,By,\alpha t)*M(Ty,Ax,(2-\alpha)t)*M(Ty,Sx,t)$$

for all $x,y \in X$, $\alpha \in (0,2)$ and $t>0$.

(3) One of $A(X)$, $B(X)$, $S(X)$ or $T(X)$ is a complete subspace of X, then

(i) A and S have a coincidence point

(ii) B and T have a coincidence point

Further if

(4) the pairs $\{A,S\}$ and $\{B,T\}$ are weakly compatible, then

(iii) A,B,S and T have a unique fixed point in X.

For fuzzy 2-metric space, Sharma and Tiwari [37] proved

Theorem B. Let $(X,M,*)$ be a fuzzy 2-metric space with $t*t \geq t$ for all $t \in [0,1]$ and the condition (*FM-6*). Let A, B, S and T be mappings from X into itself such that

(1) $A(X) \subset T(X)$, $B(X) \subset S(X)$,

(2) there exists a constant $k \in (0,1)$ such that

$$M(Ax,By,a,kt) \geq M(Ty,By,a,t)*M(Sx,Ax,a,t)*M(Sx,By,a,\alpha t)$$
$$*M(Ty,Ax,a,(2-\alpha)t)*M(Ty,Sx,a,t)$$

for all $x,y,a \in X$, $\alpha \in (0,2)$ and $t>0$.

(3) One of $A(X)$, $B(X)$, $S(X)$ or $T(X)$ is a complete subspace of X, then

(i) A and S have a coincidence point

(ii) B and T have a coincidence point

Further if

(4) the pairs $\{A,S\}$ and $\{B,T\}$ are weakly compatible, then

(iii) A,B,S and T have a unique fixed point in X.

Theorem 1. Let $(X,M,*)$ be a fuzzy 3-metric space with $t*t \geq t$ for all $t \in [0,1]$ and the

condition (*FM-6*). Let A, B, S and T be mappings from X into itself satisying the conditions (1),(3) and

(2.1) there exists a constant $k\in(0,1)$ such that

$$M(Ax,By,a,b,kt) \geq M(Ty,By,a,b,t)*M(Sx,Ax,a,b,t)*M(Sx,By,a,b,\alpha t)$$
$$*M(Ty,Ax,a,b,(2-\alpha)t)*M(Ty,Sx,a,b,t)$$

for all $x,y,a,b\in X$, $\alpha\in(0,2)$ and $t>0$.

(i) A and S have a coincidence point

(ii) B and T have a coincidence point

Further if

(2.2) the pairs $\{A,S\}$ and $\{B,T\}$ are weakly compatible, then

(iii) A,B,S and T have a unique fixed point in X.

Proof. By (1), since $A(X)\subset T(X)$, so for any arbitrary $x_0\in X$, there exists a point $x_1\in X$ such that $Ax_0=Tx_1$. Since $B(X)\subset S(X)$, for the point x_1 we can choose a point $x_2\in X$ such that $Bx_1=Sx_2$ and so on. Inductively we can define a sequence $\{y_n\}$ in X such that

$$y_{2n}=Tx_{2n+1}=Ax_{2n} \text{ and } y_{2n+1}=Sx_{2n+2}=Bx_{2n+1} \quad \text{for } n=0,1,2,...$$

By (2.1) for all $t>0$ and $\alpha=1-q$ with $q\in(0,1)$, we have

$$\begin{aligned}
M(Ax_{2n+2},Bx_{2n+1},a,b,kt) &\geq M(Tx_{2n+1},Bx_{2n+1},a,b,t)*M(Sx_{2n+2},Ax_{2n+2},a,b,t)\\
&\quad *M(Sx_{2n+2},Bx_{2n+1},a,b,\alpha t)*M(Tx_{2n+1},Ax_{2n+2},a,b,(2-\alpha)t)\\
&\quad *M(Tx_{2n+1},Sx_{2n+2},a,b,t)\\
M(y_{2n+2},y_{2n+1},a,b,kt) &\geq M(y_{2n},y_{2n+1},a,b,t)*M(y_{2n+1},y_{2n+2},a,b,t)\\
&\quad *M(y_{2n+1},y_{2n+1},a,b,\alpha t)*M(y_{2n},y_{2n+2},a,b,(2-\alpha)t)\\
&\quad *M(y_{2n},y_{2n+1},a,b,t)\\
&\geq M(y_{2n},y_{2n+1},a,b,t)*M(y_{2n+1},y_{2n+2},a,b,t)*1\\
&\quad *M(y_{2n},y_{2n+2},a,b(1+q)t)*M(y_{2n},y_{2n+1},a,b,t)\\
&\geq M(y_{2n},y_{2n+1},a,b,t)*M(y_{2n+1},y_{2n+2},a,b,t)*1\\
&\quad *M(y_{2n},y_{2n+2},a,b(1+q)t)*M(y_{2n},y_{2n+1},a,b,t)\\
&\geq M(y_{2n},y_{2n+1},a,b,t)*M(y_{2n+1},y_{2n+2},a,b,t)*M(y_{2n},y_{2n+2},a,b(1+q)t)\\
&\geq M(y_{2n},y_{2n+1},a,b,t)*M(y_{2n+1},y_{2n+2},a,b,t)*M(y_{2n},y_{2n+2},a,b,tq+t/3+t/3+t/3)
\end{aligned}$$

(Following Sharma [33] and using (*FM-4*))

$$\begin{aligned}
&\geq M(y_{2n},y_{2n+1},a,b,t)*M(y_{2n+1},y_{2n+2},a,b,t)*M(y_{2n},y_{2n+2},a,y_{2n+1},qt)\\
&\quad *M(y_{2n},y_{2n+2},y_{2n+1},b,t/3)*M(y_{2n},y_{2n+1},a,b,t/3)*M(y_{2n+1},y_{2n+2},a,b,t/3)\\
&\geq M(y_{2n},y_{2n+1},a,b,t)*M(y_{2n+1},y_{2n+2},a,b,t)*M(y_0,y_2,a,y_1,t/4q^{2n})\\
&\quad *M(y_0,y_2,y_1,y_1,t/4q^{2n})*M(y_0,y_1,a,y_1,t/4q^{2n})*M(y_1,y_2,a,y_1,t/4q^{2n})\\
&\quad *M(y_0,y_2,y_1,b,t/3q^{2n})*M(y_{2n},y_{2n+1},a,b,t/3)*M(y_{2n+1},y_{2n+2},a,b,t/3).
\end{aligned}$$

Thus since $M(y_0,y_2,y_1,b,t/3q^{2n})\to 1$ and $M(y_0,y_2,a,y_1,t/4q^{2n})\to 1$ as $n\to\infty$.

Thus we have

(2.3) $M(y_{2n+1},y_{2n+2},a,b,kt)\geq M(y_{2n},y_{2n+1},a,b,t)*M(y_{2n+1},y_{2n+2},a,b,t)$.

Similarly we have

(2.4) $M(y_{2n+2},y_{2n+3},a,b,kt) \geq M(y_{2n+1},y_{2n+2},a,b,t)*M(y_{2n+2},y_{2n+3},a,b,t)$.

From (2.3) and (2.4) it follows that

$M(y_{n+1},y_{n+2},a,b,kt) \geq M(y_n,y_{n+1},a,b,t)*M(y_{n+1},y_{n+2},a,b,t)$

for $n=1,2,\ldots$ and also for positive integer n,p

$M(y_{n+1},y_{n+2},a,b,kt) \geq M(y_n,y_{n+1},a,b,t)*M(y_{n+1},y_{n+2},a,b,t/k^P)$.

Thus since $M(y_{n+1},y_{n+2},a,b,t/k^P) \to 1$ as $p \to \infty$ we have

$$M(y_{n+1},y_{n+2},a,b,kt) \geq M(y_n,y_{n+1},a,b,t).$$

So by Lemma 2, $\{y_n\}$ is a Cauchy sequence in X.

Now suppose $S(X)$ is complete. Note that the subsequence $\{y_{2n+1}\}$ is contained in $S(X)$ and has a limit in $S(X)$, call it z. Let $u=S^{-1}z$, thus $Su=z$. We shall use the fact that the subsequence $\{y_{2n}\}$ also converges to z.

By (2.2) with $\alpha=1$, we have

$$\begin{aligned}
M(Au,y_{2n+1},a,b,kt) &= M(Au,Bx_{2n+1},a,b,kt) \\
&\geq M(Tx_{2n+1},Bx_{2n+1},a,b,t)*M(Su,Au,a,b,t)*M(Su,Bx_{2n+1},a,b,t) \\
&\quad *M(Tx_{2n+1},Au,a,b,t)\ *M(Tx_{2n+1},Su,\ a,b,t) \\
&= M(y_{2n},y_{2n+1},a,b,t)*M(Su,Au,a,b,t)*M(Su,y_{2n+1},a,b,t)*M(y_{2n},Au,a,b,t) \\
&\quad *M(y_{2n},Su,a,b,t),
\end{aligned}$$

which implies that as $n \to \infty$,

$$\begin{aligned}
M(Au,z,a,b,kt) &\geq 1*M(z,Au,a,b,t)*1*M(z,Au,a,b,t)*1 \\
&\geq M(Au,z,a,b,t).
\end{aligned}$$

Therefore by Lemma 3, we have $Au=z$. Thus $Au=z=Su$ i.e. u is a coincidence point of A and S. This proves (i).

Since $A(X) \subset T(X)$, $Au=z$ implies that $z \in T(X)$. Let $v=T^{-1}z$, then $Tv=z$. It can be easily verified by using similar arguments of the previous part of the proof that $Bv=z$. Thus $Bv=z=Tv$, i.e. v is a coincidence point of B and T. This proves (ii).

If we assume that $T(X)$ is complete, then argument analogous to the previous completeness argument estabilishes (i) and (ii). The remaining two cases pertain essentially to the previous cases. Indeed if $B(X)$ is complete, then by (1), $z \in B(X) \subset S(X)$. Similarly if $A(X)$ is complete, then $z \in A(X) \subset T(X)$. Thus (i) and (ii) are completely established.

Since the par $\{A,S\}$ is weakly compatible therefore A and S commute at their coincidence point i.e. $ASu=SAu$ or $Az=Sz$. Similarly

$BTv=TBv$ or $Bz=Tz$.

Now we prove $Az=z$ by (2.1) with $\alpha=1$, we have

$$\begin{aligned}
M(Az,y_{2n+1},a,b,kt) &= M(Az,Bx_{2n+1},a,b,kt) \\
&\geq M(Tx_{2n+1},Bx_{2n+1},a,b,t)*M(Sz,Az,a,b,t)*M(Sz,Bx_{2n+1},a,b,t)*M(Tx_{2n+1},Az,a,b,t) \\
&\quad *M(Tx_{2n+1},Sz,a,b,t) \\
&= M(y_{2n},y_{2n+1},a,b,t)*M(Az,Az,a,b,t)*M(Az,y_{2n+1},a,b,t)*M(y_{2n},Az,a,b,t)*M(y_{2n},Az,a,b,t)
\end{aligned}$$

Taking limit as $n\to\infty$, we have

$$M(Az,z,a,b,kt) \geq 1*1*M(Az,z,a,b,t)*M(z,Az,a,b,t)*M(z,Az,a,b,t)$$
$$\geq M(Az,z,a,b,t).$$

Therefore by Lemma 3, we have $Az=z$. Thus $Az=z=Sz$.

Similarly we have $Bz=z=Tz$. This means that z is a common fixed point of A,B,S and T.

For uniqueness of common fixed point let $w\neq z$ be another common fixed point of A,B,S and T. Then by (2.1) with $\alpha=1$, we have

$$M(z,w,a,b,kt) = M(Az,Bw,a,b,kt)$$
$$\geq M(Tw,Bw,a,b,t)*M(Sz,Az,a,b,t)*M(Sz,Bw,a,b,t)*M(Tw,Az,a,b,t)*M(Tw,Sz,a,b,t)$$
$$\geq 1*1*M(z,w,a,b,t)*M(w,z,a,b,t)*M(w,z,a,b,t)$$
$$\geq M(z,w,a,b,t).$$

Therefore by Lemma 1.3, we have $z=w$. This completes the proof.

Theorem 2. Let $(X,M,*)$ be a fuzzy 3-metric space with $t*t\geq t$ for all $t\in[0,1]$ and the condition (*FM-6*). Let A,B,S,T and P be mappings from X into itself such that

(2.5) $P(X)\subset AB(X)$, $P(X)\subset ST(X)$,

(2.6) there exists a constant $k\in(0,1)$ such that

$$M(Px,Py,a,b,kt) \geq M(ABy,Py,a,b,t)*M(STx,Px,a,b,t)*M(STx,Py,a,b,\alpha t)$$
$$*M(ABy,Px,a,b,(2-\alpha)t)*M(ABy,STx,a,b,t)$$

for all $x,y,a,b\in X$, $\alpha\in(0,2)$ and $t>0$.

(2.7) If one of $P(X)$, $AB(X)$ or $ST(X)$ is a complete subspace of X, then

(i) P and AB have a coincidence point

(ii) P and ST have a coincidence point

Further if

(2.8) $PB=BP$; $AB=BA$; $PT=TP$ and $ST=TS$,

(2.9) the pairs $\{P,AB\}$ and $\{P,ST\}$ are weakly compatible then

(iii) A,B,S,T and P have a unique common fixed point in X.

Proof. By (2.5) since $P(X)\subset AB(X)$, for any point $x_0\in X$, there exists a point $x_1\in X$ such that $Px_0=ABx_1$. Since $P(X)\subset ST(X)$, for the point x_1 we can choose a point $x_2\in X$ such that $Px_1=Sx_2$ and so on. Inductively we can define a sequence $\{y_n\}$ in X such that

$y_{2n}=Px_{2n}=ABx_{2n+1}$ and $y_{2n+1}=Px_{2n+1}=STx_{2n+2}$ for $n=0,1,2,\ldots$

By (2.6) for all $t>0$ and $\alpha=1-q$ with $q\in(0,1)$, we have

$$M(Px_{2n+2},Px_{2n+1},a,b,kt)$$
$$\geq M(ABx_{2n+1},Px_{2n+1},a,b,t)*M(STx_{2n+2},Px_{2n+2},a,b,t)$$
$$*M(STx_{2n+2},Px_{2n+1},a,b,\alpha t)*M(ABx_{2n+1},Px_{2n+2},a,b,(2-\alpha)t)$$
$$*M(ABx_{2n+1},STx_{2n+2},a,b,t),$$

$$M(y_{2n+2},y_{2n+1},a,b,kt)$$
$$\geq M(y_{2n},y_{2n+1},a,b,t)*M(y_{2n+1},y_{2n+2},a,b,t)*M(y_{2n+1},y_{2n+1},a,b,\alpha t)$$

$$*M(y_{2n},y_{2n+2},a,b,(1+q)t)*M(y_{2n},y_{2n+1},a,b,t)$$
$$\geq M(y_{2n},y_{2n+1},a,b,t)*M(y_{2n+1},y_{2n+2},a,b,t)*1*M(y_{2n},y_{2n+2},a,b,(1+q)t)$$
$$*M(y_{2n},y_{2n+1},a,b,t)$$
$$\geq M(y_{2n},y_{2n+1},a,b,t)*M(y_{2n+1},y_{2n+2},a,b,t)*M(y_{2n},y_{2n+2},a,b(1+q)t) .$$
$$\geq M(y_{2n},y_{2n+1},a,b,t)*M(y_{2n+1},y_{2n+2},a,b,t)*M(y_{2n},y_{2n+2},a,b,tq+t/3+t/3+t/3)$$

(Following Sharma [33] and using (*FM-4*))

$$\geq M(y_{2n},y_{2n+1},a,b,t)*M(y_{2n+1},y_{2n+2},a,b,t)*M(y_{2n},y_{2n+2},a,y_{2n+2},qt)$$
$$*M(y_{2n},y_{2n+2},y_{2n+1},b,t/3)*M(y_{2n},y_{2n+1},a,b,t/3)*M(y_{2n+1},y_{2n+2},a,b,t/3)$$
$$\geq M(y_{2n},y_{2n+1},a,b,t)*M(y_{2n+1},y_{2n+2},a,b,t)*M(y_0,y_2,a,y_1,t/4q^{2n})$$
$$*M(y_0,y_2,y_1,y_1,t/4q^{2n})*M(y_0,y_1,a,y_1,t/4q^{2n})*M(y_1,y_2,a,y_1,t/4q^{2n})$$
$$*M(y_0, y_2, y_1, b, t/3, q^{2n})*M(y_{2n},y_{2n+1},a,b,t/3)*M(y_{2n+1},y_{2n+2},a,b,t/3).$$

Thus since $M(y_0,y_2,y_1,b,t/3q^{2n})\to 1$ and $M(y_0,y_2,a,y_1,t/4q^{2n})\to 1$ as $n\to\infty$.
Thus we have

(2.10) $M(y_{2n+1},y_{2n+2},a,b,kt)\geq M(y_{2n},y_{2n+1},a,b,t)*M(y_{2n+1},y_{2n+2},a,b,t)$.

Similarly we have

(2.11) $M(y_{2n+2},y_{2n+3},a,b,kt)\geq M(y_{2n+1},y_{2n+2},a,b,t)*M(y_{2n+2},y_{2n+3},a,b,t)$.

From (2.10) and (2.11) it follows that

$M(y_{n+1},y_{n+2},a,b,kt)\geq M(y_n,y_{n+1},a,b,t)*M(y_{n+1},y_{n+2},a,b,t)$

for $n=1,2,....$ and also for positive integer n,p

$M(y_{n+1},y_{n+2},a,b,kt)\geq M(y_n,y_{n+1},a,b,t)*M(y_{n+1},y_{n+2},a,b,t/k^P)$.

Thus since $M(y_{n+1},y_{n+2},a,b,t/k^P)\to 1$ as $p\to\infty$ we have

$$M(y_{n+1},y_{n+2},a,b,kt)\geq M(y_n,y_{n+1},a,b,t) .$$

So by Lemma 2, $\{y_n\}$ is a Cauchy sequence in X.

Now suppose $ST(X)$ is complete. Note that the subsequence $\{y_{2n+1}\}$ is contained in $ST(X)$ and has a limit in $ST(X)$, call it z.

Let $u=(ST)^{-1}z$, thus $STu=z$. We shall use the fact that the subsequence $\{y_{2n}\}$ also converges to z.

By (2.6) with $\alpha=1$, we have

$$M(Pu,y_{2n+1},a,b,kt)=M(Pu,Px_{2n+1},a,b,kt)$$
$$\geq M(ABx_{2n+1},Px_{2n+1},a,b,t)*M(STu,Pu,a,b,t)*M(STu,Px_{2n+1},a,b,t)$$
$$*M(ABx_{2n+1},Pu,a,b,t) *M(ABx_{2n+1},STu, a,b,t)$$
$$=M(y_{2n},y_{2n+1},a,b,t)*M(z,Pu,a,b,t)*M(z,y_{2n+1},a,b,t)*M(y_{2n},Pu,a,b,t)$$
$$*M(y_{2n},z,a,b,t),$$

which implies that as $n\to\infty$

$$M(Pu,z,a,b,kt) \geq 1*M(z,Pu,a,b,t)*1*M(z,Pu,a,b,t)*1$$
$$\geq M(Pu,z,a,b,t) .$$

Therefore by Lemma 3, we have $Pu=z$. Since $STu=z$, thus $Pu=z=STu$ i.e. u is coincidence point of P and ST. This proves (i).

Since $P(X)\subset AB(X)$, $Pu=z$ implies that $z\in AB(X)$.

Let $v=(AB)^{-1}z$, then $ABv=z$. It can be easily be verified by using similar argument of the previouis part of the proof that $Pv=z$. If we assume that $AB(X)$ is complete then argument analogous to the previous completeness argument estabilishes (i) and (ii).

The remaining one case pertain essentially to the previous cases. Indeed if $P(X)$ is complete, then by (2.5), $z\in P(X)\subset ST(X)$ or $z\in P(X)\subset AB(X)$. Thus (i) and (ii) are completely established. Since the pair $\{P,ST\}$ is weakly compatible therefore P and ST commute at their coincidence point i.e. $P(STu)=(ST)Pu$ or $Pz=STz$. Similarly $P(ABv)=(AB)Pv$ or $Pz=ABz$.

Now we prove that $Pz=z$ by (2.6) with $\alpha=1$, we have

$$\begin{aligned}M(Pz,y_{2n+1},a,b,kt) &= M(Pz,Px_{2n+1},a,b,kt)\\ &\geq M(ABx_{2n+1},Px_{2n+1},a,b,t)*M(STz,Pz,a,b,t)*M(STz,\ Px_{2n+1},a,b,t)\\ &\quad *M(ABx_{2n+1},Pz,a,b,t)*M(ABx_{2n+1},STz,a,b,t)\\ &= M(y_{2n},y_{2n+1},a,b,t)*M(Pz,Pz,a,b,t)*M(Pz,y_{2n+1},a,b,t)*M(y_{2n},Pz,a,b,t)\\ &\quad *M(y_{2n},Pz,a,b,t).\end{aligned}$$

Taking limit as $n\to\infty$, we have

$$\begin{aligned}M(Pz,z,a,b,kt) &\geq 1*1*M(Pz,z,a,b,t)*M(z,Pz,a,b,t)*M(z,Pz,a,b,t)\\ &\geq M(Pz,z,a,b,t)\ .\end{aligned}$$

Therefore by Lemma 3, we have $Pz=z$. Thus $ABz=z=Pz=STz$.

Now we shall show that $Bz=z$. in fact by (2.6) with $\alpha=1$ and (2.8), we have

$$\begin{aligned}M(z,Bz,a,b,kt) &= M(Pz,BPz,a,b,kt)=M(Pz,PBz,a,b,kt)\\ &\geq M(AB(Bz),PBz,a,b,t)*M(STz,Pz,a,b,t)*M(STz,PBz,a,b,t)*M(AB(Bz),Pz,a,b,t)\\ &\quad *M(AB(Bz),STz,a,b,t)\\ &= 1*1*M(z,Bz,a,b,t)*M(Bz,z,a,b,t)*M(Bz,z,a,b,t)\\ &\geq M(z,Bz,a,b,t),\end{aligned}$$

which implies by Lemma 3, that $Bz=z$. Since $ABz=z$, therefore $Az=z$. Finally we know that $Tz=z$. Indeed by (2.6) with $\alpha=1$ and (2.8)

$$\begin{aligned}M(Tz,z,a,b,kt) &= M(TPz,Pz,a,b,kt)=M(PTz,Pz,a,b,kt)\\ &\geq M(ABz,Pz,a,b,t)*M(ST(Tz),PTz,a,b,t)*M(ST(Tz),Pz,a,b,t)*M(ABz,P(Tz),a,b,t)\\ &\quad *M(ABz,ST(Tz),a,b,t)\\ &= 1*1*M(Tz,z,a,b,t)*M(z,Tz,a,b,t)*M(z,Tz,a,b,t)\\ &\geq M(tz,z,a,b,t),\end{aligned}$$

which implies by Lemma 3, that $Tz=z$. Since $STz=z$, we have $z=Stz=Sz$. Therefore by combining the above results, we have $Az=Bz=Sz=Tz=Pz=z$, that is z is a common fixed point of A,B,S,T and P.

For uniqueness of common fixed point let $w\neq z$ be another common fixed point of A,B,S,T, and P.

Then by (2.6) with $\alpha=1$, we have

$$M(z,w,a,b,kt)\geq M(ABw,Pw,a,b,t)\ *M(STz,Pz,a,b,t)\ *M(STz,Pw,a,b,\alpha t)$$

$$*M(ABw,Pz,a,b,t)\ *M(ABw,STz,a,b,t)$$
$$\geq 1*1*M(z,w,a,b,t)*M(w,z,a,b,t)*M(w,z,a,b,t)$$
$$\geq M(z,w,a,b,t).$$

Therefore by Lemma 3, we have $z=w$. This completes the proof.

Theorem 3. Let $(X,M,*)$ be a fuzzy 3-metric space with $t*t\geq t$ for all $t\in[0,1]$ and the condition $(FM\text{-}6)$. Let A,B,S,T,P and Q be mappings from X into itself such that

(2.12) $P(X)\subset AB(X)$, $Q(X)\subset ST(X)$,

(2.13) there exists a constant $k\in(0,1)$ such that

$$M(Px,Qy,a,b,kt) \geq M(ABy,Qy,a,b,t)*M(STx,Px,a,b,t)*M(STx,Qy,a,b,\alpha t)$$
$$*M(ABy,Px,a,b,(2-\alpha)t)*M(ABy,STx,a,b,t)$$

for all $x,y,a,b\in X$, $\alpha\in(0,2)$ and $t>0$.

(2.14) If one of $P(X)$, $Q(X)$ $AB(X)$ or $ST(X)$ is a complete subspace of X, then

(i) P and ST have a coincidence point

(ii) Q and AB have a coincidence point

Further if

(2.15) $AB=BA$, $QB=BQ$, $PT=TP$ and $ST=TS$,

(2.16) the pairs $\{Q,AB\}$ and $\{P,ST\}$ are weakly compatible then

(iii) A,B,S,T,P and Q have a unique common fixed point in X.

Proof. By (2.12) since $P(X)\subset AB(X)$, for any point $x_0\in X$, there exists a point $x_1\in X$ such that $Px_0=ABx_1$. Since $Q(X)\subset ST(X)$, for the point x_1 we can choose a point $x_2\in X$ such that $Qx_1=STx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$y_{2n}=Px_{2n}=ABx_{2n+1}$ and $y_{2n+1}=Qx_{2n+1}=STx_{2n+2}$ for $n=0,1,2,...$

for all $t>0$ and $\alpha=1-q$ with $q\in(0,1)$. As proved in Theorem 1 and Theorem 2, we can prove that $\{y_n\}$ is a Cauchy sequence in X. Now suppose $ST(X)$ is complete. Note that the subsequence $\{y_{2n+1}\}$ is contained in $ST(X)$ and has a limit in $ST(X)$, call it z. Let $u=(ST)^{-1}z$. then $STu=z$ we shall use the fact that the subsequence $\{y_{2n}\}$ also converges to z.

By (2.13) with $\alpha=1$, we have

$$M(Pu,Qx_{2n+1},a,b,kt)\geq M(ABx_{2n+1},Qx_{2n+1},a,b,t)*M(STu,Pu,a,b,t)$$
$$*M(STu,Qx_{2n+1},a,b,t)*M(ABx_{2n+1},Pu,a,b,t)\ *M(ABx_{2n+1},STu,\ a,b,t)$$
$$=M(y_{2n},y_{2n+1},a,b,t)*M(STu,Pu,a,b,t)*M(STu,y_{2n+1},a,b,t)*M(y_{2n},Pu,a,b,t)*M(y_{2n},STu,a,b,t),$$

which implies that as $n\to\infty$

$$M(Pu,z,a,b,kt) \geq M(Pu,z,a,b,t).$$

Therefore by Lemma 3, we have $Pu=z$. Since $STu=z$, thus $Pu=z=STu$ i.e. u is a coincidence point of P and ST. This proves (i). Since $P(X)\subset AB(X)$, $Pu=z$ implies that $z\in AB(X)$. Let $v=(AB)^{-1}z$, then $ABv=z$.

By (2.13) with $\alpha=1$, we have

$M(Pz,y_{2n+1},a,b,kt) = M(Pz,Qx_{2n+1},a,b,kt)$
$\geq M(ABx_{2n+1},Qx_{2n+1},a,b,t)*M(STz,Pz,a,b,t)*M(STz,Qx_{2n+1},a,b,t)$
$*M(ABx_{2n+1},Pz,a,b,t)*M(ABx_{2n+1},STz,a,b,t)$
$\geq M(y_{2n},y_{2n+1},a,b,t)*M(STz,Pz,a,b,t)*M(STz,y_{2n+1},a,b,t)*M(y_{2n},Pz,a,b,t)$
$*M(y_{2n},STz,a,b,t)$.

Taking limit as $n\to\infty$, we have
$M(Pz,z,a,b,kt) \geq M(Pz,z,a,b,t)$.
Therefore by Lemma 3, we have $Pz=z=STz$.
Now we show that $Qz=z$. in fact by (2.13) with $\alpha=1$ and (2.15) we have
$M(y_{2n},Qz,a,b,kt) = M(Px_{2n},Qz,a,b,kt)$
$\geq M(ABz,Qz,a,b,t)*M(STx_{2n},Px_{2n},a,b,t)*M(STx_{2n},Qz,a,b,t)*M(ABz,Px_{2n},a,b,t)$
$*M(ABz,STx_{2n},a,b,t)$.
Taking the limit $n\to\infty$, we have
$M(z,Qz,a,b,t) \geq M(Qz,z,a,b,t)$.
Therefore by Lemma 3, we have $Qz=z=ABz$. Thus $Pz=Qz=ABz=STz=z$.
By putting $x=z$ and $y=Bz$ with $\alpha=1$ in (2.13), using (2.15) and Lemma 3, it is easy to see that $Bz=z$. Since $ABz=z$, therefore $Az=z$.
Similarly by putting $x=Tz$ and $y=z$ and $\alpha=1$ in (2.13), using (2.15) and Lemma 3, it is easy to prove that $Tz=z$. Since $STz=z$, we have $Sz=z$. Therefore, by combining the above results, we have

$$Az=Bz=Sz=Tz=Qz=z$$

that is z is the common fixed point of A, B, S, T, P and Q.
It is easy to prove uniqueness.

Theorem 4. Let $(X,M,*)$ be a fuzzy 3-metric space with $t*t\geq t$ for all $t\in[0,1]$ and the condition (*FM-6*). Let A,B,S,T and $\{P_i\}_{i\in I}$ be mappings from X into itself

(2.17) $\cup_{i\in I}P_i(X)\subset AB(X)$, $\cup_{i\in I}P_i(X)\subset ST(X)$, where I is an index set,

(2.18) there exists a constant $k\in(0,1)$ such that

$$M(P_ix,P_iy,a,b,kt) \geq M(ABy,P_iy,a,b,t)*M(STx,P_ix,a,b,t)*M(STx,P_iy,a,b,\alpha t)$$
$$*M(ABy,P_ix,a,b,(2-\alpha)t)*M(ABy,STx,a,b,t)$$

for all $x,y,a,b\in X$, $\alpha\in(0,2)$ $i\in I$ and $t>0$,

(2.19) If one of $AB(X)$ or $ST(X)$ or $P_i(X)(i\in I)$ is a complete subspace of X, then

(i) for all $i\in I$, P_i and AB have a coincidence point ,

(ii) for all $i\in I$, P_i and ST have a coincidence point .

Further if

(2.20) For all $i\in I$, $P_iB=BP_i$; $AB=BA$; $P_iT=TP_i$ and $ST=TS$,

(2.21) For all $i\in I$, the pairs $\{P_i,AB\}$ and $\{P_i,ST\}$ are weakly compatible then

(iii) A,B,S,T,P and $\{P_i\}_{i\in I}$ have a unique common fixed point in X.

If we put $B=T=I_x$ (the identity map on X) in Theorem 2, we get the following result :

Corollary 1. Let $(X,M,*)$ be a fuzzy 3-metric space with $t*t\geq t$ for all $t\in[0,1]$ and the condition (*FM-6*). Let A,S and P be mappings from X into itself such that

(2.22) $P(X)\subset A(X)$, $P(X)\subset S(X)$

(2.23) there exists a constant $k\in(0,1)$ such that

$$M(Px,Py,a,b,kt) \geq M(Ay,Py,a,b,t)*M(Sx,Py,a,b,\alpha t)*M(Ay,Py,a,b,(2-\alpha)t) *M(Ay,Sx,a,b,t)$$

for all $x,y,a,b\in X$, $\alpha\in(0,2)$ and $t>0$.

(2.24) If one of $P(X)$ or $S(X)$ is a complete subspace of X, then

(i) P and A have a coincidence point,

(ii) P and S have a coincidence point.

Further if

(2.25) The pairs $\{P,A\}$ and $\{P,S\}$ are weakly compatible then

(iii) A,S and P have a unique common fixed point in X.

If we put $A=B=S=T=I_x$ (the identity map on X) in Theorem 2.2, we get the following result :

Corollary 2. Let $(X,M,*)$ be a fuzzy 3-metric space with $t*t\geq t$ for all $t\in[0,1]$ and the condition (*FM-6*). Let P be mappings from X into itself such that

(2.26) there exists a constant $k\in(0,1)$ such that

$$M(Px,Py,a,b,kt) \geq M(y,Py,a,b,t)*M(x,Px,a,b,t)*M(x,Py,a,b,\alpha t) *M(y,Px,a,b,(2-\alpha)t) *M(y,x,a,b,t)$$

for all $x,y,a,b\in X$, $\alpha\in(0,2)$ and $t>0$.

If $P(X)$ is complete subspace of X then P has a unique common fixed point in X.

By using theorem 1, we get the following result.

Theorem 5. Let $(X,M,*)$ be a fuzzy 3-metric space with $t*t\geq t$ for all $t\in[0,1]$ and the condition (*FM-6*). Let A,B and T be mappings from X into itself such that

(2.27) $A(X)\cup B(X)\subset T(X)$

(2.18) there exists a constant $k\in(0,1)$ such that

$$M(Ax,By,a,b,kt) \geq M(Ty,By,a,b,t)*M(Tx,Ax,a,b,t)*M(Tx,By,a,b,\alpha t) *M(Ty,Ax,a,b,(2-\alpha)t)*M(Ty,Tx,a,b,t)$$

for all $x,y,a,b\in X$, $\alpha\in(0,2)$ $i\in I$ and $t>0$.

(2.29) One of $A(X)$, $B(X)$ or $T(X)$ is a complete subspace of X, then A,B, T have a coincidence point.

We establish Theorem 1 for sequence of mappings in the following manner.

Theorem 6. Let $(X,M,*)$ be a fuzzy 3-metric space with $t*t\geq t$ for all $t\in[0,1]$ and the condition (*FM-6*). Let S,T, $A_i:X\to X$, $i=0,1,2...$ such that

(2.30) $A_0(X)\subset T(X)$, $A_i(X)\subset S(X)$, $i\in N$

(2.31) there exists a constant $k\in(0,1)$ such that

$$M(A_0x,A_iy,a,b,kt) \geq M(Ty,A_iy,a,b,t)*M(Sx,A_0x,a,b,t)*M(Sx,A_iy,a,b,\alpha t) *M(Ty,A_0x,a,b,(2-\alpha)t)*M(Ty,Sx,a,b,t)$$

for all $x,y,a,b \in X$, $\alpha \in (0,2)$ and $t>0$.

(2.32) The pairs $\{A_0,S\}$ and $\{A_i,T\}$ $(i \in N)$ are weakly compatible,

(2.33) if one of $S(X)$, $T(X)$ or $A_0(X)$ is a complete subspace of X or altrnatively A_i, $i \in N$ are complete subspace of X.

Then S,T, and $A_i, i=0,1,2,...$ have a unique common fixed point.

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REFERENCES

[1] R. Badard, Fixed point theorems for fuzzy numbers, *Fuzzy Sets and Systems*, **13** (1984), 291-302.

[2] R. Chugh and S.Kumar, *Proc. Indian Acad. Sci., Math. Sci.*, **111 (2)** (2001), 241-247.

[3] S.S. Chang, Y.J. Cho, B.S. Lec, J.S. Jung and S.M. Kang, Coincidence points and minimization theorems in fuzzy metric spaces, *Fuzzy Sets and Systems*, **88**(1), 5(4) (1997), 949-962.

[4] Y.J. Cho, Fixed point in fuzzy metric space, *Journal of Fuzzy Mathematics*, **5(4)** (1997), 949-962.

[5] Z.K. Deng, Fuzzy pseudo metric spaces, *J. Math., Anal. Appl.*, **86** (1982), 75-84.

[6] M.A. Erceg, Metric spcaes in fuzzy set theory, *J. Math. Anal. Appl.* **69** (1970), 205-230.

[7] J.X. Fang, On fixed points theorems in fuzzy metric spaces, *Fuzzy Sets and Systems*, **64** (1994), 395-399.

[8] A. George and P. Veeramani, On some results in fuzzy metric space, *Fuzzy Sets and Systems*, **64** (1994), 395-399.

[9] M. Grabiec, Fixed point in fuzzy metric spaces, *Fuzzy Sets and Systems*, **27** (1988), 385-389.

[10] O. Hadzic, Fixed point theorems for multivalued mappings in classes of fuzzy metric spaces, *Fuzzy Sets and Systems*, **29** (1989), 115-125.

[11] O. Hadzic, *Fixed Point Theorems in Probabilistic Metric Spaces*, Serbian Aacademy of Sciences and Arts, Institute of Mathematics, University of Navisad, Yogoslovia, 1995.

[12] I. Istratescu, A fixed point theorem for mappings with probabilistic contractive iterated, *Rev. Roumaric. Math. Pure Appl.*, **26** (1981), 431-435.

[13] K. Iseki, Some applications of Banach type contraction principles *Math. Sem. Notes, Kobe Univ.*, **4** Arts Institute of Mathematics, Univ of Novisad, Yugoslovia, 1995.

[14] G. Jungck, Commuting mapping and fixed points, *Amer. Math. Monthly*, **83** (1976), 261-263.

[15] G. Jungck, Common fixed points for commuting and compatible mappings on compacta, *Proc. Amer. Math. Soc.*, **103** (1988), 977-983.

[16] G. Jungck, P.P. Murthy and Y.J. Cho, Compatible mappings of type (A) and common fixed points, *Math. Japonica*, **38** (1983), 381-390.

[17] G. Jungck and B.E. Rhoades, Fixed point for set valued functions without continuity, *Indian J. Pure Appl. Math.*, **29** (3) (1998), 227-238.

[18] J.S. Jung, Y.J. Cho and J.K. Kim, Minimization theorems for fixed point theorems in fuzzy metric spaces and applications, *Fuzzy Sets and Systems*, **61** (1994), 199-207.

[19] J.S. Jung, Y.J. Cho, S.S. Chang and S.M. Kang, Coincidence theorems for set-valued mappings and ekelands variational principle in fuzzy metric spaces, *Fuzzy Sets and Systems*, **79** (1996), 239-250.

[20] O. Kaleva and S. Sheikkala, On fuzzy metric space, *Fuzzy Sets and Systems*, **12** (1984), 215-229.

[21] R. Kannan, Some results on fixed points, *Bull. Cal. Math. Soc.*, **60** (1968), 71-76.

[22] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, *Kybernetica* **11** (1975), 336-344.

[23] S. N. Mishra, N. Sharma, and S.L. Singh, Common fixed point of maps of fuzzy metric spaces, *Internet. J. Math. Sci.*, **17** (1994), 253-258.

[24] H.K. Pathak, Y.J. Cho, S.S. Chang and S.M. Kang, Compatible mappings of type (P) and fixed point theorems in metric spaces and probabilistic metric spaces, *Novi Sad J. Math.*, **26** No.2 (1996), 87-109.

[25] B.E. Rhoades, A comparison of various definitions of contractive mappings, *Trans. Amer. Math. Soc.*, **226** (1977), 257-290.

[26] B.E. Rhoades, Contractive definition revisited, *Contemp. Math.*, **21** (1983), 313-334.

[27] B. Schweizer and A. Sklar, Statistical metric space, *Pacific J. Math.*, **10** (1960), 189-205.

[28] S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math.* (Beograd) (N.S.) **32** (46) (1982), 149-155.

[29] S. Sessa, B.E. Rhoades and M.S. Khan, On common fixed point of compatible mappings in metric space and Banach space, *Internat. J. Math. and Math. Sci.*, **11** (2) (1988), 375-392.

[30] S.L. Singh, Some common fixed point theorems in L-space, *Math. Sem. Notes Kobe Univ.*, **7** (1979), 91-97.

[31] S.L. Singh and S. Kasahara, On some recent results on common fixed points, *Indian J. Pure Appl. Math.*, **13** (1982), 757-761, Corrigendum, **14** (1983), 1075.

[32] S. L. Singh and B. Ram, Common fixed point of commuting mappings in 2-metric space, *Math., Sem. Notes Kob Univ.*, **10** (1982), 197-208.

[33] S. Sharma, On fuzzy metric spaces, *South East Asian Bulletin of Math., Springer Verlag*, **26** (2002), 133-145.

[34] S. Sharma, Common fixed point theorem in fuzzy metric spaces *Fuzzy Sets and Systems*, **125** (2001), 345-352.

[35] S. Sharma and B. Deshpande, Commond fixed point of compatible of type (β) on fizzyk metric spaces, *Demonstratio Math.*, **35** No.1 (2002), 165-174.

[36] S. Sharma and B. Deshpande, Discontinuity and weak compatibility in fixed point consideration on non-complete fuzzy metric spaces, *J. Fuzzy Maths.*, **11** No. 2 (2003), 671-686.

[37] S. Sharma and J. K. Tiwari, Some common fixed point theorems in fuzzy 2-metric spaces, *Pak. J. C.S.I.R.* (accepted for publication)

[38] B.M.L. Tiwari and S.L. Singh, A note on recent generalization of Jungck contraction principle, *J, U.P.G.C. Soc.* **3** (1986), 13-18.

[39] L.A. Zadeh, *Fuzzy sets, Inform control* **8** (1965), 338-353.

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AVAILABILITY AND MAXIMIZING SURVIVABILITY OF CABLE T.V. TRANSMISSION SYSTEM

By

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ABSTRACT

In this short communication, we have computed availability and Maximizing Survivability for cable T.V. transmission system which is one of the example of multi-state acyclic network (MAN). The system is designed to transmit from the cable T.V. operator's satellite link to users' televisions through amplifier units. The user's televisions are connected by cables, which are the system components. The system is operational if user's televisions are connected by cables, which are the system components. The system is operational if user's televisions are connected with amplifier units and the cable T.V. operator's satellite link.

Key words. Multi-state acyclic network, Multi-state, Universal generating function; Survivability, Availability.

1. Introduction. The MAN consists of a number of positions in which multi-state elements are allocated. Each network has root position where the signal source is located. The number of leaf node (position, like television) that can only receive a signals and a number of non-leaf nodes (intermediate nodes, amplifier unit) that can retransmit the received signal to the leaf node. The system is designed to transmit from teh cable T.V. operator's satellite link to users' televisions. The users' televisions are connected by cables and amplifier units, which are the system components. The system is operational if all user televisions are connected amplifier units to cable T.V. operator's satellite link otherwise system fails. Cable T.V. network is the generalization of tree structured multi-state system. Cable T.V. network has operator's satellite link, which is the root position where the signal source is allocated and amplifier units, which are the non-leaf positions. The sink position (leaf node) is situated at user's televisions. The aim of the cable T.V. network is the transmission of signal from source to the sink (last) position. Malinowski and Preuss [1] discovered that the acyclic multi-state transmission network is a generalization of the tree-structured multi-state systems. The universal generating function technique is used for solving availability and

Maximizing Survivability of cable T.V. transmission system that is suggested by Levitin and Lisnianski [1,4].

We have taken one example of cable T.V. network for solving the availability and Survivability using the universal generating function technique with $n=4$ and $n^*=1$, where n is the total number of nodes (positions) and n^* is the number of leaf nodes. the resulting polynomial contains $2^{n^*}-1$ terms. The paper organized as follows: section 2 is devoted to example of cable T.V. transmission system for estimating the availability and survivability of MAN. Section 3 present the conclusion.

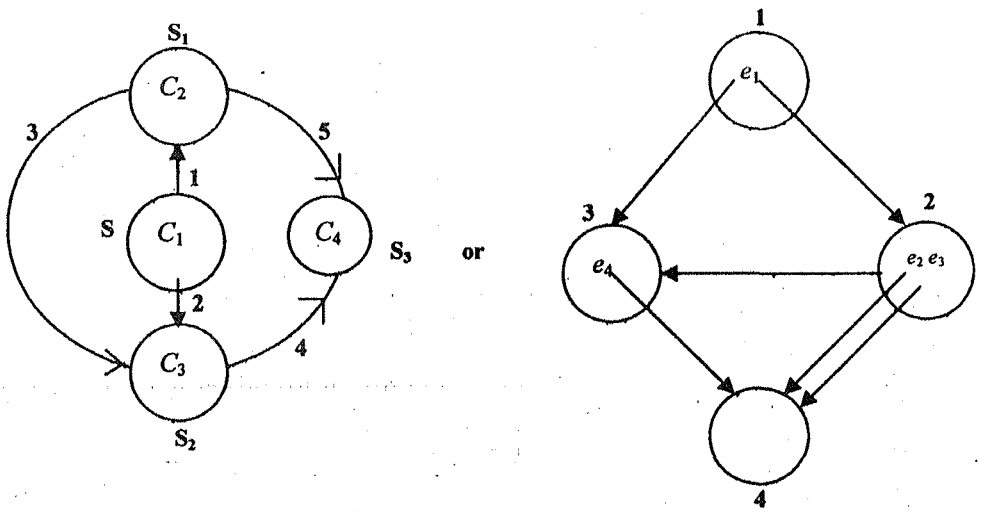

Fig. Cable T.V. Transimission System. (Simplest MAN)

2. Example.

The system is designed to transmit from the cable T.V. operator's satellite link S to users' televisions, S_3 through amplifier units S_1,S_2. the users' televisions are connected by cables numbered 4,5 and amplifier units numbered 1,2,3 which are the system components. The system is operational if all user televisions are connected from amplifier untis 1,2,3 to the cable T.V. operator's satellite link otherwise system fails.

Consider multi-state acyclic network (MAN) with $n=4$ and $n^*=1$ presented in above figure. Assume that *ME* e_1 is located at C_1 and provides a connection between C_1 and any subset of $\{C_2,C_3\}$. *ME* e_2 is located at C_2 and provides a conncection between C_2 and any subst of $\{C_3,C_4\}$. *ME* e_3 is also located at C_2 and provides a connection between C_2 and C_4. *ME* e_4 is located at C_3 and provides a connection between C_3 and C_4. Using the universal generating function technique, the u-functions of individual *MEs* located at nodes C_1,C_2 and C_3 are :

$u_{11}(z) = p_{1\phi}\, z^{0000} + p_{1\{2\}} z^{0100} + p_{1\{3\}} z^{0010} + p_{1\{2,3\}} z^{0110}$,

$u_{22}(z) = p_{2\phi}\, z^{0000} + p_{2\{3\}} z^{0010} + p_{2\{4\}} z^{0001} + p_{2\{3,4\}} z^{0011}$,

$u_{32}(z) = p_{3\phi}\, z^{0000} + p_{3\{4\}} z^{0001}$,
$u_{43}(z) = p_{4\phi}\, z^{0000} + p_{4\{4\}} z^{0001}$.

The u-functions of groups of *MEs* located at the same nodes are,

$U_1(z) = u_{11}(z)$,
$U_3(z) = u_{43}(z)$,
$U_2(z) = \Omega_{max}\,(u_{11}(z), u_{32}(z))$

$$= p_{2\phi}\, p_{3\phi} z^{0000} + p_{2\{3\}} p_{3\phi} z^{0010} + p_{2\{4\}} p_{3\phi} z^{0001} + p_{2\{3,4\}} p_{3\phi} z^{0011} +$$
$$p_{2\phi}\, p_{3\{4\}} z^{0001} + p_{2\{3\}} p_{3\{4\}} z^{0011} + p_{2\{4\}} p_{3\{4\}} z^{0001} + p_{2\{3,4\}} p_{3\{4\}}{}^{0011}$$
$$= q_{2\phi} z^{0000} + q_{2\{3\}} z^{0010} + q_{2\{4\}} z^{0001} + q_{2\{3,4\}} z^{0011},$$

where $q_{2\phi} = p_{2\phi}\, p_{3\phi} q_{2\{3\}} = p_{2\{3\}}\, p_{3\phi} q_{2\{4\}} = p_{2\{4\}}\, p_{3\phi} + p_{2\phi}\, p_{3\{4\}} + p_{2\{4\}} p_{3\{4\}}$

$q_{2\{3,4\}} = p_{2\{3,4\}}\, p_{3\phi} + p_{2\{3\}}\, p_{3\{4\}} + p_{2\{3,4\}} p_{3\{4\}}$

Since $p_{3\phi} + p_{3\{4\}} = 1$, the equations for $q_{2\{4\}} = p_{2\{4\}} + p_{2\phi}\, p_{3\{4\}}$

$q_{2\{3,4\}} = p_{2\{3,4\}} + p_{2\{3\}}\, p_{3\{4\}}$.

2.1. Availability of MAN. Following the consecutive procedure, we obtain:

$$\widetilde{U}_1(z) = U_1(z).$$

$$\varphi\left(\widetilde{U}_1(z)\right) = p_{1\{2\}} z^{0100} + p_{1\{3\}} z^{0010} + p_{1\{2,3\}} z^{0110},$$

$$\widetilde{U}_2(z) = \Omega_\phi\left(\varphi\left(\widetilde{U}_1(z)\right), U_2(z)\right),$$

$$= p_{1\{2\}} q_{2\phi} z^{0100} + p_{1\{2\}} q_{2\{3\}} z^{0110} + p_{1\{2\}} q_{2\{4\}} z^{0101} + p_{1\{2\}} q_{2\{3,4\}} z^{0111} + p_{1\{3\}} z^{0010}$$

$$+ p_{1\{2,3\}} q_{2\phi} z^{0110} + p_{1\{2,3\}} q_{2\{3\}} z^{0110} + p_{1\{2,3\}} q_{2\{4\}} z^{0111} + p_{1\{2,3\}} q_{2\{3,4\}} z^{0111},$$

and after simplification :

$$\varphi\left(\widetilde{U}_2(z)\right) = \left(p_{1\{2\}} q_{2\{3\}} + p_{1\{3\}} + p_{1\{2,3\}} q_{2\phi} + p_{1\{2,3\}} q_{2\{3\}}\right) z^{0010}$$

$$+ \left(p_{1\{2\}} q_{2\{3,4\}} + p_{1\{2,3\}} q_{2\{4\}} + p_{1\{2,3\}} q_{2\{3,4\}}\right) z^{0011} + p_{1\{2\}} q_{2\{4\}} z^{0001}.$$

(Note that operator φ reduces the number of different terms in this u-function from nine to three.)

$$\widetilde{U}_3(z) = \Omega_\phi\left(\varphi\left(\widetilde{U}_2(z)\right), U_3(z)\right)$$

$$= \left(p_{1\{2\}} q_{2\{3\}} + p_{1\{3\}} + p_{1\{2,3\}} q_{2\phi} + p_{1\{2,3\}} q_{2\{3\}}\right) p_{4\phi} z^{0010}$$

$$+ \left(p_{1\{2\}} q_{2\{3,4\}} + p_{1\{2,3\}} q_{2\{4\}} + p_{1\{2,3\}} q_{2\{3,4\}}\right) p_{4\phi} z^{0011} + p_{1\{2\}} q_{2\{4\}} p_{4\phi} z^{0001}$$

$$+ \left(p_{1\{2\}} q_{2\{3\}} + p_{1\{3\}} + p_{1\{2,3\}} q_{2\phi} + p_{1\{2,3\}} q_{2\{3\}}\right) p_{4\{4\}} z^{0011}$$

$$+ \left(p_{1\{2\}} q_{2\{3,4\}} + p_{1\{2,3\}} q_{2\{4\}} + p_{1\{2,3\}} q_{2\{3,4\}}\right) p_{4\{4\}} z^{0011} + p_{1\{2\}} q_{2\{4\}} p_{4\{4\}} z^{0001}$$

and after simplification:

$$\varphi\left(\widetilde{U}_3(z)\right)=\left[\left(p_{1\{2\}}q_{2\{3,4\}}+p_{1\{2,3\}}q_{2\{4\}}+p_{1\{2,3\}}q_{2\{3,4\}}\right)p_{4\phi}+p_{1\{2\}}q_{2\{4\}}p_{4\phi}\right.$$
$$+\left(p_{1\{2\}}q_{2\{3\}}+p_{1\{3\}}+p_{1\{2,3\}}q_{2\phi}+p_{1\{2,3\}}q_{2\{3\}}\right)p_{4\{4\}}$$
$$\left.+\left(p_{1\{2\}}q_{2\{3,4\}}+p_{1\{2,3\}}q_{2\{4\}}+p_{1\{2,3\}}q_{2\{3,4\}}\right)p_{4\{4\}}+p_{1\{2\}}q_{2\{4\}}+p_{4\{4\}}\right]z^{0001}.$$

The coefficient in the term with the vector $\widetilde{G}^{(3)}=0001$ in $\varphi\left(\widetilde{U}_3(z)\right)$ is the probability that the signal reaches C_4, which is equal to MAN availability:

$$A_4=\left(p_{1\{2\}}q_{2\{3,4\}}+p_{1\{2,3\}}q_{2\{4\}}+p_{1\{2,3\}}q_{2\{3,4\}}\right)p_{4\phi}+p_{1\{2\}}q_{2\{4\}}p_{4\phi}$$
$$+\left(p_{1\{2\}}q_{2\{3\}}+p_{1\{3\}}+p_{1\{2,3\}}q_{2\phi}+p_{1\{2,3\}}q_{2\{3\}}\right)p_{4\{4\}}$$
$$+\left(p_{1\{2\}}q_{2\{3,4\}}+p_{1\{2,3\}}q_{2\{4\}}+p_{1\{2,3\}}q_{2\{3,4\}}\right)p_{4\{4\}}+p_{1\{2\}}q_{2\{4\}}p_{4\{4\}}$$

2.2 Survivability of MAN. In this section, the first *ME* is located at position *1* second and third *ME* are located at posion 2 and fourth *ME* is located at position 3.

Following the consecutive procedure, we obtain:

$$\hat{U}_1(z)=\theta\{U_1(z)\}=\left(\beta+\alpha p_{1\phi}\right)z^{0000}+\alpha p_{1\{2\}}z^{0100}+\alpha p_{1(3)}z^{0010}+\alpha p_{1(2,3)}z^{0110},$$

$$\hat{U}_2(z)=\theta\{U_2(z)\}=\left(\beta+\alpha p_{2\phi}\right)z^{0000}+\alpha q_{2\{3\}}z^{0010}+\alpha q_{2(4)}z^{0001}+\alpha q_{2(3,4)}z^{0011},$$

$$\hat{U}_3(z)=\theta\{U_3(z)\}=\left(\beta+\alpha p_{4\phi}\right)z^{0000}+\alpha p_{4\{4\}}z^{0001}.$$

$$\overline{U}_1(z)=\hat{U}_1(z),$$

$$\varphi\left(\overline{U}_1(z)\right)=\alpha p_{1\{2\}}z^{0100}+\alpha p_{1\{3\}}z^{0010}\alpha p_{1\{2,3\}}z^{0110},$$

$$\overline{U}_2(z)=\psi\left(\varphi\left(\overline{U}_1(z)\right),\left(\hat{U}_2(z)\right)\right)$$
$$=\Psi\left(\alpha p_{1\{2\}}z^{0100}+\alpha p_{1\{3\}}z^{0010}+\alpha p_{1\{2,3\}}z^{0110},\left(\beta+\alpha q_{2\phi}\right)z^{0000}+\alpha q_{2\{3\}}z^{0010}+\alpha q_{2\{4\}}z^{0001}+\alpha q_{2\{3,4\}}z^{0011}\right)$$
$$=\alpha p_{1\{2\}}\left(\beta+\alpha q_{2\phi}\right)z^{0100}+\left\{\alpha p_{1\{3\}}\left(\beta+\alpha q_{2\phi}\right)+\alpha^2 p_{1\{3\}}q_{2\{3\}}+\alpha^2 p_{1\{2\}}q_{2\{3\}}\right\}z^{0010}$$
$$+\left\{\alpha p_{1\{2,3\}}\left(\beta+\alpha q_{2\phi}\right)+\alpha^2 p_{1\{2,3\}}q_{2\{3\}}\right\}z^{0110}+\alpha^2 p_{1\{2\}}q_{2\{4\}}z^{0101}+\left\{\alpha^2 p_{1\{3\}}q_{2\{4\}}+\alpha^2 p_{1\{3\}}q_{2\{3,4\}}\right\}z^{0011}$$
$$+\left\{\alpha^2 p_{1\{2,3\}}q_{2\{4\}}+\alpha^2 p_{1\{2\}}q_{2\{3,4\}}+\alpha^2 p_{1\{2,3\}}q_{2\{3,4\}}\right\}z^{0111}$$
$$=\alpha p_{1\{2\}}\left(\beta+\alpha q_{2\phi}\right)z^{0100}+\left\{\alpha p_{1\{3\}}+\alpha^2 p_{1\{3\}}\left(q_{2\phi}+q_{2\{3\}}\right)+\alpha^2 p_{1\{2\}}q_{2\{3\}}\right\}z^{0010}$$
$$+\left\{\alpha\beta p_{1\{2,3\}}+\alpha^2 p_{1\{2,3\}}\left(q_{2\phi}+q_{2\{3\}}\right)\right\}z^{0110}+\alpha^2 p_{1\{2\}}q_{2\{4\}}z^{0101}+\left\{\alpha^2 p_{1\{2\}}\left(q_{2\{4\}}+q_{2\{3,4\}}\right)\right\}z^{0011}$$
$$+\left\{\alpha^2 p_{1\{2,3\}}\left(q_{2\{4\}}+q_{2\{3,4\}}\right)+\alpha^2 p_{1\{2\}}q_{2\{3,4\}}\right\}z^{0110},$$

$$= \alpha p_{1\{2\}}\left(\beta + \alpha q_{2\phi}\right)z^{0100} + \left\{\alpha\beta p_{1\{3\}} + \alpha^2 p_{1\{3\}}\left(q_{2\phi} + q_{2\{3\}}\right) + \alpha^2 p_{1\{2\}}p_{2\{3\}} + \alpha\beta p_{1\{2,3\}}\right.$$

$$\left. + \alpha^2 p_{1\{2,3\}}\left(q_{2\phi} + q_{2\{3\}}\right)\right\}z^{0010} + \alpha^2 p_{1\{2\}}q_{2\{4\}}z^{0101} + \left\{\alpha^2 p_{1\{2,3\}}\left(q_{2\{4\}} + q_{2\{3,4\}}\right) + \alpha^2 p_{1\{2\}}q_{2\{3,4\}}\right\}z^{0011},$$

$$\varphi\left(\overline{U}_2(z)\right) = \left\{\alpha\beta p_{1\{3\}} + \alpha^2 p_{1\{3\}}\left(q_{2\phi} + q_{2\{3\}}\right) + \alpha^2 p_{1\{2\}}p_{2\{3\}} + \alpha\beta p_{1\{2,3\}} + \alpha^2 p_{1\{2,3\}}\left(q_{2\phi} + q_{2\{3\}}\right)\right\}z^{0010}$$

$$+ \alpha^2 p_{1\{2\}}q_{2\{4\}}z^{0101} + \left\{\alpha^2 p_{1\{2,3\}}\left(q_{2\{4\}} + q_{2\{3,4\}}\right) + \alpha^2 p_{1\{2\}}q_{2\{3,4\}}\right\}z^{0011}.$$

$$\overline{U}_3(z) = \psi\left(\varphi\left(\overline{U}_2(z)\right), \hat{U}_3(z)\right)$$

$$= \Psi\left[\left\{\alpha\beta p_{1\{3\}} + \alpha^2 p_{1\{3\}}\left(q_{2\phi} + q_{2\{3\}}\right) + \alpha^2 p_{1\{2\}}p_{2\{3\}} + \alpha\beta p_{1\{2,3\}} + \alpha^2 p_{1\{2,3\}}\left(q_{2\phi} + q_{2\{3\}}\right)\right\}z^{0010} + \right.$$

$$\left. \alpha^2 p_{1\{2\}}q_{2\{4\}}z^{0101} + \left\{\alpha^2 p_{1\{2,3\}}\left(q_{2\{4\}} + q_{2\{3,4\}}\right) + \alpha^2 p_{1\{2\}}q_{2\{3,4\}}\right\}z^{0011}, \left(\beta + \alpha p_{4\phi}\right)z^{0000} + \alpha p_{4\{4\}}z^{0001}\right]$$

$$= \left[\left\{\alpha\beta p_{1\{3\}} + \alpha^2 p_{1\{3\}}\left(q_{2\phi} + q_{2\{3\}}\right) + \alpha^2 p_{1\{2\}}p_{2\{3\}} + \alpha\beta p_{1\{2,3\}} + \alpha^2 p_{1\{2,3\}}\left(q_{2\phi} + q_{2\{3\}}\right)\right\}\left(\beta + \alpha p_{4\phi}\right)z^{0010}\right.$$

$$+ \alpha^2 p_{1\{2\}}q_{2\{4\}}\left(\beta + \alpha p_{4\phi}\right)z^{0001} + \left\{\alpha^2 p_{1\{2,3\}}\left(q_{2\{4\}} + q_{2\{3,4\}}\right) + \alpha^2 p_{1\{2\}}q_{2\{3,4\}} + \alpha^3 p_{1\{2\}}q_{2\{4\}}p_{4\{4\}}\right\}z^{0001} +$$

$$\left. + \left\{\alpha\beta p_{1\{3\}} + \alpha^2 p_{1\{3\}}\left(q_{2\phi} + q_{2\{3\}}\right) + \alpha^2 p_{1\{2\}}p_{2\{3\}} + \alpha\beta p_{1\{2,3\}} + \alpha^2 p_{1\{2,3\}}\left(q_{2\{4\}} + q_{2\{3,4\}}\right) + \alpha^2 p_{1\{2\}}q_{2\{3,4\}}\right\}z^{0011}\right]$$

$$= \left[\left\{\alpha\beta p_{1\{3\}} + \alpha^2 p_{1\{3\}}\left(q_{2\phi} + q_{2\{3\}}\right) + \alpha^2 p_{1\{2\}}p_{2\{3\}} + \alpha\beta p_{1\{2,3\}} + \alpha^2 p_{1\{2,3\}}\left(q_{2\phi} + q_{2\{3\}}\right)\right\}\left(\beta + \alpha p_{4\phi}\right)\right]z^{0010}$$

$$+ \left[\alpha^2 p_{1\{2\}}q_{2\{4\}}\left(\beta + \alpha p_{4\phi}\right) + \alpha^2 p_{1\{2,3\}}\left(q_{2\{4\}} + q_{2\{3,4\}}\right) + \alpha^2 p_{1\{2\}}q_{2\{3,4\}} + \alpha^3 p_{1\{2\}}q_{2\{4\}}p_{4\{4\}} + \alpha\beta p_{1\{3\}} + \right.$$

$$+ \alpha^2 p_{1\{3\}}\left(q_{2\phi} + q_{2\{3\}}\right) + \alpha^2 p_{1\{2\}}p_{2\{3\}} + \alpha\beta p_{1\{2,3\}}\alpha^2 p_{1\{2,3\}}\left(q_{2\phi} + q_{2\{3\}}\right) + \alpha^2 p_{1\{2,3\}}\left(q_{2\{4\}} + q_{2\{3,4\}}\right)$$

$$\left. + \alpha^2 p_{1\{2\}}q_{2\{3,4\}}\right]z^{0001},$$

$$\varphi\left(\overline{U}_3(z)\right) = \alpha^2 p_{1(2)}q_{2\{4\}}\left(\beta + \alpha p_{4\phi}\right) + \alpha^2 p_{1\{2,3\}}\left(q_{2\{4\}} + q_{2\{3,4\}}\right) + \alpha^2 p_{1\{2\}}q_{2\{3,4\}} + \alpha^3 p_{1\{2\}}q_{2\{4\}}p_{4\{4\}} +$$

$$\alpha\beta p_{1\{3\}} + \alpha^2 p_{1\{3\}}\left(q_{2\phi} + q_{2\{3\}}\right) + \alpha^2 p_{1\{2\}}p_{2\{3\}} + \alpha\beta p_{1\{2,3\}} + \alpha^2 p_{1\{2,3\}}\left(q_{2\phi} + q_{2\{3\}}\right)$$

$$+ \alpha^2 p_{1\{2,3\}}\left(q_{2\{4\}} + q_{2\{3,4\}}\right) + \alpha^2 p_{1\{2\}}q_{2\{3,4\}}.$$

Taking into account than the MAN surviability

$$S = \alpha^2 p_{1\{2\}}q_{2\{4\}}\left(\beta + \alpha p_{4\phi}\right) + \alpha^2 p_{1\{2,3\}}\left(q_{2\{4\}} + q_{2\{3,4\}}\right) + \alpha^2 p_{1\{2\}}q_{2\{3,4\}} + \alpha^3 p_{1\{2\}}q_{2\{4\}}p_{4\{4\}} + \alpha\beta p_{1\{3\}} +$$

$$\alpha^2 p_{1\{3\}}\left(q_{2\phi} + q_{2\{3\}}\right) + \alpha^2 p_{1\{2\}}p_{2\{3\}} + \alpha\beta p_{1\{2,3\}} + \alpha^2 p_{1\{2,3\}}\left(q_{2\phi} + q_{2\{3\}}\right)$$

$$+ \alpha^2 p_{1\{2,3\}}\left(q_{2\{4\}} + q_{2\{3,4\}}\right) + \alpha^2 p_{1\{2\}}q_{2\{3,4\}},$$

Since $\beta = 1 - \alpha$.

$$S = \alpha^2 p_{1\{2\}}q_{2\{4\}}\left((1-\alpha) + \alpha p_{4\phi}\right) + \alpha^2 p_{1\{2,3\}}\left(q_{2\{4\}} + q_{2\{3,4\}}\right) + \alpha^2 p_{1\{2\}}q_{2\{3,4\}} + \alpha^3 p_{1\{2\}}q_{2\{4\}}p_{4\{4\}}$$

$$+ \alpha(1-\alpha)p_{1\{3\}} + \alpha^2 p_{1\{3\}}\left(q_{2\phi} + q_{2\{3\}}\right) + \alpha^2 p_{1\{2\}}p_{2\{3\}} + \alpha(1-\alpha)p_{1\{2,3\}} + \alpha^2 p_{1\{2,3\}}\left(q_{2\phi} + q_{2\{3\}}\right)$$

$$+\alpha^2 p_{1\{2,3\}}\left(q_{2\{4\}}+q_{2\{3,4\}}\right)+\alpha^2 p_{1\{2\}}q_{2\{3,4\}}.$$

3. Conclusion. The short communication suggests an availability and survivability of cable T.V. transmission system which are generalization of the tree-structured multi-state networks and multistate acyclic networks. The method is based on universal generating function technique extended for representing random binary vectors. The resulting polynomial contains 2^{n^*}-1 terms. Therefore, the suggested method can be applied for MAN with moderate values of n^*. The suggested method allows one to obtain probability that the signal generated at the cable T.V. operator's satellite link of a MAN reaches users' televisions.

REFERENCES

[1] A. Lisnianski and G. Levitin, Multi-State System Reliability, New Jersey, 2003.

[2] G. Levtin, Optimal allocation of multi-state retransmitters in acyclic transmission network, *Reliab Engng Syst. Safety,* **75** (2002), 73-82.

[3] G. Levitin, Reliability evaluation for acyclic consecutively connected networks with multi-state elements, *Reliab Engng Syst. Safety,* **73** (2001), 137-143.

[4] G. Livitin, Maximizing survivability of acyclic transmission networks with multi-state retransmitters and vulnerable nodes, *Reliab Engng Syst. Safety,* **77** (2002), 189-199.

[5] J. Malinowski and W. Preuss, Reliability evaluation for tree-structured systems with multistate components, *Microelectron Reliab,* **36** (1996), 9-17.

[6] R.S. Gaur and S. Chaudhary, A study of an algorithm for reliability of ACCN, *Proceedings of National Conference on Modern Mathematics,* Agra, (2003).

[7] R.S. Gaur and S. Chaudhary, Reliability evaluation of acyclic multi-state transmission networks (AMTNs) using the universal generating function technique, *IEEE on Reliab Trans.* (under communication)

[8] R.S. Gaur and S. Chaudhary, Reliability enhancement and fixed transmission times of acyclic transmission multi-state networks (ATMNs). Proceedings of the *Ramanujan Mathematical Society,* Agra, 2004.

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FRACTIONAL DERIVATIVES OF OUR HYPERGEOMETRIC FUNCTIONS OF FOUR VARIABLES

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ABSTRACT

In the present paper, we derive fractional derivatives of hypergeometric functions of four variables recently introduced by Chandel and Sharma [8,9]

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1. Introduction. Motivated by the earlier work of Srivastava and Goyal [25], Srivastava, Chandel and Vishwakarma [26] derived a number of key formulas for fractional derivatives of the multivariable H-function of Srivastava and Panda [20,21,22] for multivariable H-function also see Srivastava, Gupta and Goyal [23]). Each of these formulas can be shown to yield interesting new results for various classes of generalized hypergeometric functions of several variables of Srivastava and Daoust [19] (Also see Srivastava and Manocha [24]pp 64-65]), Lauricella [16], Exton [11,14], Chandel [1], Chandel and Gupta [2] and Karlsson [15].

Also Chandel and Vishwakarma [5] derived fractional derivatives involving confluent hypergeometric forms of Karlsson's multiple hypergeometric function ${}^{(k)}F_{CD}^{(n)}$ [15], introduced by Chandel and Vishwakarma [3,4]. Further Chandel and Vishwakarma [6] derived the multidimensional fractional derivatives involving the multiple hypergeometric functions of Lauricella [16] with their confluent forms, Exton [11,14], Chandel [1], Chandel and Gupta [2] and their confluent forms and of Karlsson [15].

Further for special interest Chandel and Vishwakarma [7] applied same techniques in order to derive fractional derivatives involving hypergeometric functions of four variables of Exton [12, 13, 14] and those functions of Sharma and Parihar [18], which are not included in the Exton [12,13,14].

In the present paper for special interest, we shall derive fractional derivatives involving hypergeometric functions of four variables (recently

introduced and studied by the authors Chandel and Sharma [8,9].

2. Use of One Fractional Derivatives Operator. Making an appeal to the formula [17, p. 67]

$$D_x^{\mu}\{x^{\lambda}\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} x^{\lambda-\mu}, \text{ Re } (\lambda) > 1,$$

we derive the following fractional derivatives involving hypergeometric functions of four variables due to authors Chandel and Sharma [8,9]:

$$(2.1)\ D_x^{\lambda-\mu}\left\{x^{\lambda-1} H_{A_1}^{(4)}(\mu,b,c,d;e,e',e'';z_1x,z_2x,z_3x,z_4)\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} H_{A_1}^{(4)}(\lambda,b,c,d;e,e',e'';z_1x,z_2x,z_3x,z_4),\ Re(\lambda) > 0;$$

$$(2.2)\ D_x^{\lambda-\mu}\left\{x^{\lambda-1} H_{A_1}^{(4)}(a,\mu,c,d;e,e',e'';z_1x,z_2,z_3,z_4x)\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} H_{A_1}^{(4)}(a,\lambda,c,d;e,e',e'';z_1x,z_2,z_3,z_4x),\ Re(\lambda) > 0;$$

$$(2.3)\ D_x^{\lambda-\mu}\left\{x^{\lambda-1} H_{A_1}^{(4)}(a,b,\mu,d;e,e',e'';z_1,z_2,z_3x,z_4x)\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} H_{A_1}^{(4)}(a,b,\lambda,d;e,e',e'';z_1,z_2,z_3x,z_4x),\ Re(\lambda) > 0;$$

$$(2.4)\ D_x^{\lambda-\mu}\left\{x^{\lambda-1} H_{A_1}^{(4)}(a,b,c,\mu;e,e',e'';z_1,z_2x,z_3,z_4)\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} H_{A_1}^{(4)}(a,b,c,\lambda;e,e',e'';z_1,z_2x,z_3,z_4),\ Re(\lambda) > 0;$$

$$(2.5)\ D_x^{\lambda-\mu}\left\{x^{\lambda-1} H_{A_1}^{(4)}(a,b,c,d;\lambda,e',e'';z_1x,z_2,z_3x,z_4)\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} H_{A_1}^{(4)}(a,b,c,d;\mu,e',e'';z_1x,z_2,z_3x,z_4), Re(\lambda) > 0;$$

$$(2.6)\ D_x^{\lambda-\mu}\left\{x^{\lambda-1} H_{A_1}^{(4)}(a,b,c,d;e,\lambda,e'';z_1,z_2x,z_3,z_4)\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} H_{A_1}^{(4)}(a,b,c,d;e,\mu,e'';z_1,z_2x,z_3,z_4), Re(\lambda) > 0;$$

$$(2.7)\ D_x^{\lambda-\mu}\left\{x^{\lambda-1} H_{A_1}^{(4)}(a,b,c,d;e,e',\lambda;z_1,z_2,z_3,z_4x)\right\}$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} x^{\mu-1} H_{A_1}^{(4)}(a,b,c,d;e,e',\mu;z_1,z_2,z_3,z_4x),\ Re(\lambda) > 0;$$

(2.8) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}H_{B_1}^{(4)}(\mu,b,c,d;e_1,e_2,e_3,e_4;z_1x,z_2x,z_3,z_4x)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}H_{B_1}^{(4)}(\lambda,b,c,d;e_1,e_2,e_3,e_4;z_1x,z_2x,z_3,z_4x),\ Re(\lambda)>0;$$

(2.9) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}H_{B_1}^{(4)}(a,\mu,c,d;e_1,e_2,e_3,e_4;z_1x,z_2,z_3x,z_4)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}H_{B_1}^{(4)}(a,\lambda,c,d;e_1,e_2,e_3,e_4;z_1x,z_2,z_3x,z_4),\ Re(\lambda)>0;$$

(2.10) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}H_{B_1}^{(4)}(a,b,\mu,d;e_1,e_2,e_3,e_4;z_1,z_2,z_3x,z_4x)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}H_{B_1}^{(4)}(a,b,\lambda,d;e_1,e_2,e_3,e_4;z_1,z_2,z_3x,z_4x),\ Re(\lambda)>0;$$

(2.11) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}H_{B_1}^{(4)}(a,b,c,\mu;e_1,e_2,e_3,e_4;z_1,z_2x,z_3,z_4)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}H_{B_1}^{(4)}(a,b,c,\lambda;e_1,e_2,e_3,e_4;z_1,z_2x,z_3,z_4),\ Re(\lambda)>0;$$

(2.12) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}H_{B_1}^{(4)}(a,b,c,d;\lambda,e_2,e_3,e_4;z_1x,z_2,z_3,z_4)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}H_{B_1}^{(4)}(a,b,c,d;\mu,e_2,e_3,e_4;z_1x,z_2,z_3,z_4),\ Re(\lambda)>0;$$

(2.13) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}H_{B_1}^{(4)}(a,b,c,d;e_1,\lambda,e_3,e_4;z_1,z_2x,z_3,z_4)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}H_{B_1}^{(4)}(a,b,c,d;e_1,\mu,e_3,e_4;z_1,z_2x,z_3,z_4),\ Re(\lambda)>0;$$

(2.14) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}H_{B_1}^{(4)}(a,b,c,d;e_1,e_2,\lambda,e_4;z_1,z_2,z_3x,z_4)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}H_{B_1}^{(4)}(a,b,c,d;e_1,e_2,\mu,e_4;z_1,z_2,z_3x,z_4),\ Re(\lambda)>0;$$

(2.15) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}H_{B_1}^{(4)}(a,b,c,d;e_1,e_2,e_3,\lambda,;z_1,z_2,z_3,z_4x)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}H_{B_1}^{(4)}(a,b,c,d;e_1,e_2,e_3,\mu;z_1,z_2,z_3,z_4x),\ Re(\lambda)>0;$$

(2.17) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{A_1}^{(4)}(a,b,\mu,d;e,e';z_1,z_2x,z_3,z_4)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{A_1}^{(4)}(a,b,\lambda,d;e,e';z_1,z_2x,z_3,z_4),\ Re(\lambda)>0;$$

(2.18) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{A_1}^{(4)}(a,b,c,\mu;e,e';z_1,z_2,z_3,z_4x)\right\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{A_1}^{(4)}(a,b,c,\lambda;e,e';z_1,z_2,z_3,z_4x), Re(\lambda)>0;$$

(2.19) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{A_1}^{(4)}(a,b,c,d;e,\lambda;z_1,z_2,z_3,z_4x)\right\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{A_1}^{(4)}(a,b,c,d;e,\mu;z_1,z_2,z_3,z_4x), Re(\lambda)>0;$$

(2.20) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{A_2}^{(4)}(a,\mu,c,d;e,e';z_1x,z_2,z_3x,z_4x)\right\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{A_2}^{(4)}(a,\lambda,c,d;e,e';z_1x,z_2,z_3x,z_4x), Re(\lambda)>0;$$

(2.21) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{A_2}^{(4)}(a,b,\mu,d;e,e';z_1,z_2x,z_3,z_4)\right\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{A_2}^{(4)}(a,b,\lambda,d;e,e';z_1,z_2x,z_3,z_4), Re(\lambda)>0;$$

(2.22) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{A_2}^{(4)}(a,b,c,\mu;e,e';z_1,z_2,z_3,z_4x)\right\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{A_2}^{(4)}(a,b,c,\lambda;e,e';z_1,z_2,z_3,z_4x), Re(\lambda)>0;$$

(2.23) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{A_2}^{(4)}(a,b,c,d;e,\lambda;z_1,z_2,z_3,z_4x)\right\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{A_2}^{(4)}(a,b,c,d;e,\mu;z_1,z_2,z_3,z_4x), Re(\lambda)>0;$$

(2.24) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{A_3}^{(4)}(a,\mu,c,d;e,e';z_1x,z_2,z_3x,z_4x)\right\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{A_3}^{(4)}(a,\lambda,c,d;e,e';z_1x,z_2,z_3x,z_4), Re(\lambda)>0;$$

(2.25) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{A_3}^{(4)}(a,b,\mu,d;e,e';z_1,z_2x,z_3,z_4x)\right\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{A_3}^{(4)}(a,b,\lambda,d;e,e';z_1,z_2x,z_3,z_4x), Re(\lambda)>0;$$

(2.26) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{A_3}^{(4)}(a,b,c,\mu;e,e';z_1,z_2,z_3,z_4x)\right\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{A_3}^{(4)}(a,b,c,\lambda;e,e';z_1,z_2,z_3,z_4x), Re(\lambda)>0;$$

(2.27) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{A_3}^{(4)}(a,b,c,d;e,\lambda;z_1,z_2,z_3,z_4x)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{A_3}^{(4)}(a,b,c,d;e,\mu;z_1,z_2,z_3,z_4x),\ Re(\lambda)>0;$$

(2.28) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{B_1}^{(4)}(a,\mu,b_2,b_3,b_4;e,e';z_1x,z_2,z_3,z_4)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{B_1}^{(4)}(a,\lambda,b_2,b_3,b_4;e,e';z_1x,z_2,z_3,z_4),\ Re(\lambda)>0;$$

(2.29) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{B_1}^{(4)}(a,b_1,\mu,b_3,b_4;e,e';z_1,z_2x,z_3,z_4)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{B_1}^{(4)}(a,b_1,\lambda,b_3,b_4;e,e';z_1,z_2x,z_3,z_4),\ Re(\lambda)>0;$$

(2.30) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{B_1}^{(4)}(a,b_1,b_2,\mu,b_4;e,e';z_1,z_2,z_3x,z_4)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{B_1}^{(4)}(a,b_1,b_2,\lambda,b_4;e,e';z_1,z_2,z_3x,z_4),\ Re(\lambda)>0;$$

(2.31) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{B_1}^{(4)}(a,b_1,b_2,b_3,\mu;e,e';z_1,z_2,z_3,z_4x)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{B_1}^{(4)}(a,b_1,b_2,b_3,\lambda;e,e';z_1,z_2,z_3,z_4x),\ Re(\lambda)>0;$$

(2.32) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{B_1}^{(4)}(a,b_1,b_2,b_3,b_4;e,\lambda;z_1,z_2,z_3,z_4x)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{B_1}^{(4)}(a,b_1,b_2,b_3,b_4;e,\mu;z_1,z_2,z_3,z_4x),\ Re(\lambda)>0;$$

(2.33) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{B_2}^{(4)}(a,\mu,b_2,b_3,b_4;e,e';z_1x,z_2,z_3,z_4x)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{B_2}^{(4)}(a,\lambda,b_2,b_3,b_4;e,e';z_1x,z_2,z_3,z_4x),\ Re(\lambda)>0;$$

(2.34) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{B_2}^{(4)}(a,b_1,\mu,b_3,b_4;e,e';z_1,z_2x,z_3,z_4)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{B_2}^{(4)}(a,b_1,\lambda,b_3,b_4;e,e';z_1,z_2x,z_3,z_4),\ Re(\lambda)>0;$$

(2.35) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{B_2}^{(4)}(a,b_1,b_2,\mu,b_4;e,e';z_1,z_2,z_3x,z_4)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{B_2}^{(4)}(a,b_1,b_2,\lambda,b_4;e,e';z_1,z_2,z_3x,z_4),\ Re(\lambda)>0;$$

(2.36) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{B_2}^{(4)}(a,b_1,b_2,b_3,\mu;e,e';z_1,z_2,z_3,z_4x)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{B_2}^{(4)}(a,b_1,b_2,b_3,\lambda;e,e';z_1,z_2,z_3,z_4x),\ Re(\lambda)>0;$$

(2.37) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{B_2}^{(4)}(a,b_1,b_2,b_3,b_4;e,\lambda;z_1,z_2,z_3,z_4x)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{B_2}^{(4)}(a,b_1,b_2,b_3,b_4;e,\mu;z_1,z_2,z_3,z_4x),\ Re(\lambda)>0;$$

(2.38) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{B_3}^{(4)}(a,\mu,b_2,b_3,b_4;e,e';z_1,z_2x,z_3,z_4x)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{B_3}^{(4)}(a,\lambda,b_2,b_3,b_4;e,e';z_1,z_2x,z_3,z_4x),\ Re(\lambda)>0;$$

(2.39) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{B_3}^{(4)}(a,b_1,\mu,b_3,b_4;e,e';z_1x,z_2,z_3,z_4)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{B_3}^{(4)}(a,b_1,\lambda,b_3,b_4;e,e';z_1x,z_2,z_3,z_4),\ Re(\lambda)>0;$$

(2.40) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{B_3}^{(4)}(a,b_1,b_2,\mu,b_4;e,e';z_1,z_2,z_3x,z_4)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{B_3}^{(4)}(a,b_1,b_2,\lambda,b_4;e,e';z_1,z_2,z_3x,z_4),\ Re(\lambda)>0;$$

(2.41) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{B_3}^{(4)}(a,b_1,b_2,b_3,\mu;e,e';z_1,z_2,z_3,z_4x)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{B_3}^{(4)}(a,b_1,b_2,b_3,\lambda;e,e';z_1,z_2,z_3,z_4x),\ Re(\lambda)>0;$$

(2.42) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{B_3}^{(4)}(a,b_1,b_2,b_3,b_4;e,\lambda;z_1,z_2,z_3,z_4x)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{B_3}^{(4)}(a,b_1,b_2,b_3,b_4;e,\mu;z_1,z_2,z_3,z_4x),\ Re(\lambda)>0;$$

(2.43) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{C_1}^{(4)}(\mu,b,c,d;e,e';z_1x,z_2,z_3x,z_4x)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{C_1}^{(4)}(\lambda,b,c,d;e,e';z_1x,z_2,z_3x,z_4x),\ Re(\lambda)>0;$$

(2.44) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{C_1}^{(4)}(a,\mu,c,d;e,e';z_1x,z_2x,z_3,z_4)\right\}$

$$=\frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{C_1}^{(4)}(a,\lambda,c,d;e,e';z_1x,z_2x,z_3,z_4),\ Re(\lambda)>0;$$

(2.45) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{C_1}^{(4)}(a,b,c,\mu;e,e';z_1,z_2,z_3,z_4x)\right\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{C_1}^{(4)}(a,b,c,\lambda;e,e';z_1,z_2,z_3,z_4x),\ Re(\lambda)>0;$$

(2.46) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{C_1}^{(4)}(a,b,c,d;e,\lambda;z_1,z_2,z_3,z_4x)\right\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{C_1}^{(4)}(a,b,c,d;e,\mu;z_1,z_2,z_3,z_4x),\ Re(\lambda)>0;$$

(2.47) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{C_2}^{(4)}(\mu,b,c,d;e,e';z_1x,z_2x,z_3,z_4x)\right\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{C_2}^{(4)}(\lambda,b,c,d;e,e';z_1x,z_2x,z_3,z_4x),\ Re(\lambda)>0;$$

(2.48) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{C_2}^{(4)}(a,\mu,c,d;e,e';z_1x,z_2,z_3x,z_4)\right\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{C_2}^{(4)}(a,\lambda,c,d;e,e';z_1x,z_2,z_3x,z_4),\ Re(\lambda)>0;$$

(2.49) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{C_2}^{(4)}(a,b,c,\mu;e,e';z_1,z_2,z_3,z_4x)\right\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{C_2}^{(4)}(a,b,c,\lambda;e,e';z_1,z_2,z_3,z_4x),\ Re(\lambda)>0;$$

(2.50) $D_x^{\lambda-\mu}\left\{x^{\lambda-1}G_{C_2}^{(4)}(a,b,c,d;e,\lambda;z_1,z_2,z_3,z_4x)\right\}$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)}x^{\mu-1}G_{C_2}^{(4)}(a,b,c,d;e,\mu;z_1,z_2,z_3,z_4x),\ Re(\lambda)>0;$$

3. Use of Two Fractional Derivative Operators.

(3.1) $D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}H_{A_1}^{(4)}(\mu,\mu',c,d;e,e',e'';z_1xy,z_2x,z_3x,z_4y)\right\}$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}H_{A_1}^{(4)}(\lambda,\lambda',c,d;e,e',e'';z_1xy,z_2x,z_3x,z_4y),$$

$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.2) $D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}H_{A_1}^{(4)}(\mu,b,\mu',d;e,e',e'';z_1x,z_2x,z_3xy,z_4y)\right\}$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}H_{A_1}^{(4)}(\lambda,b,\lambda',d;e,e',e'';z_1x,z_2x,z_3xy,z_4y),$$

$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.3) $D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}H_{A_1}^{(4)}(a,\mu,\mu',d;e,e',e'';z_1x,z_2,z_3y,z_4xy)\right\}$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} H_{A_1}^{(4)}(a,\lambda,\lambda',d;e,e',e'';z_1x,z_2,z_3y,z_4xy),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.4) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} H_{A_1}^{(4)}(\mu,b,c,\mu';e,e',e'';z_1x,z_2xy,z_3x,z_4) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} H_{A_1}^{(4)}(\lambda,b,c,\lambda';e,e',e'';z_1x,z_2xy,z_3x,z_4),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.5) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} H_{A_1}^{(4)}(a,\mu,c,\mu';e,e',e'';z_1x,z_2y,z_3,z_4x) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} H_{A_1}^{(4)}(a,\lambda,c,\lambda';e,e',e'';z_1x,z_2y,z_3,z_4x),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.6) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} H_{A_1}^{(4)}(a,b,\mu,\mu';e,e',e'';z_1,z_2y,z_3x,z_4x) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} H_{A_1}^{(4)}(a,b,\lambda,\lambda';e,e',e'';z_1,z_2y,z_3x,z_4x),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.7) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} H_{A_1}^{(4)}(a,b,c,d;\lambda,\lambda',e'';z_1x,z_2y,z_3x,z_4) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} H_{A_1}^{(4)}(a,b,c,d;\mu,\mu',e'';z_1x,z_2y,z_3x,z_4),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.8) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} H_{A_1}^{(4)}(a,b,c,d;\lambda,e',\lambda';z_1x,z_2,z_3x,z_4y) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} H_{A_1}^{(4)}(a,b,c,d;\mu,e',\mu';z_1x,z_2,z_3x,z_4y),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.9) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} H_{B_1}^{(4)}(\mu,\mu',c,d;e_1,e_2,e_3,e_4;z_1xy,z_2x,z_3y,z_4x) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} H_{B_1}^{(4)}(\lambda,\lambda',c,d;e_1,e_2,e_3,e_4;z_1xy,z_2x,z_3y,z_4x),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.10) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} H_{B_1}^{(4)}(\mu,b,\mu',d;e_1,e_2,e_3,e_4;z_1x,z_2x,z_3y,z_4xy) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} H_{B_1}^{(4)}(\lambda,b,\lambda',d;e_1,e_2,e_3,e_4;z_1x,z_2x,z_3y,z_4xy),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.11) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}H_{B_1}^{(4)}(a,\mu,\mu',d;e_1,e_2,e_3,e_4;z_1x,z_2,z_3xy,z_4y)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}H_{B_1}^{(4)}(a,\lambda,\lambda',d;e_1,e_2,e_3,e_4;z_1x,z_2,z_3xy,z_4y),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.12) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}H_{B_1}^{(4)}(a,b,c.d;\lambda,\lambda',e_3,e_4;z_1x,z_2y,z_3,z_4)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}H_{B_1}^{(4)}(a,b,c,d,;\mu,\mu',e_3,e_4;z_1x,z_2y,z_3,z_4),$$
$$Re(\lambda)>0,\ Re(\lambda')>0.$$

Five more results similar to (3.12) can also be obtained but we have left them due to lack of space.

(3.13) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}x^{\lambda-1}y^{\lambda'-1}G_{A_1}^{(4)}(a,\mu,\mu',d;e,e';z_1x,z_2y,z_3x,z_4)$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{A_1}^{(4)}(a,\lambda,\lambda',d;e,e';z_1x,z_2y,z_3x,z_4),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.14) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{A_1}^{(4)}(a,\mu,c,\mu';e,e';z_1x,z_2,z_3x,z_4y)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{A_1}^{(4)}(a,\lambda,c,\lambda';e,e';z_1x,z_2,z_3x,z_4y),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.15) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{A_1}^{(4)}(a,\mu,c,d;e,\lambda';z_1x,z_2,z_3x,z_4y)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{A_1}^{(4)}(a,\lambda,c,d;e,\mu';z_1x,z_2,z_3x,z_4y),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.16) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{A_2}^{(4)}(a,\mu,\mu',d;e,e';z_1x,z_2y,z_3x,z_4x)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{A_2}^{(4)}(a,\lambda,\lambda',d;e,e';z_1x,z_2y,z_3x,z_4x),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.17) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{A_2}^{(4)}(a,\mu,c,\mu';e,e';z_1x,z_2,z_3x,z_4xy)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{A_2}^{(4)}(a,\lambda,c,\lambda';e,e';z_1x,z_2,z_3x,z_4xy),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.18) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{A_2}^{(4)}(a,\mu,c,d;e,\lambda';z_1x,z_2,z_3x,z_4xy)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{A_2}^{(4)}(a,\lambda,c,d;e,\mu';z_1x,z_2,z_3x,z_4xy),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.19) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} G_{A_2}^{(4)}(a,b,\mu,d;e,\lambda';z_1,z_2x,z_3,z_4y) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{A_2}^{(4)}(a,b,\lambda,d;e,\mu';z_1,z_2x,z_3,z_4y),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.20) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} G_{A_2}^{(4)}(a,b,c,\mu;e,\lambda';z_1,z_2,z_3,z_4xy) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{A_2}^{(4)}(a,b,c,\lambda;e,\mu';z_1,z_2,z_3,z_4xy),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.21) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} G_{A_3}^{(4)}(a,\mu,\mu',d;e,e';z_1x,z_2y,z_3x,z_4y) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{A_3}^{(4)}(a,\lambda,\lambda',d;e,e';z_1x,z_2y,z_3x,z_4y),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.22) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} G_{A_3}^{(4)}(a,\mu,c,\mu';e,e';z_1x,z_2,z_3x,z_4y) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{A3}^{(4)}(a,\lambda,c,\lambda';e,e';z_1x,z_2,z_3x,z_4y),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.23) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} G_{A_3}^{(4)}(a,b,\mu,\mu';e,e';z_1,z_2x,z_3,z_4xy) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{A_3}^{(4)}(a,b,\lambda,\lambda';e,e';z_1,z_2x,z_3,z_4xy),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.24) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} G_{A_3}^{(4)}(a,,\mu,c,d;e,\lambda';z_1x,z_2,z_3x,z_4y) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{A_3}^{(4)}(a,\lambda,c,d;e,\mu';z_1x,z_2,z_3x,z_4y),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.25) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} G_{A_3}^{(4)}(a,b,\mu,d;e,\lambda';z_1,z_2x,z_3,z_4yx) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{A_3}^{(4)}(a,b,\lambda,d;e,\mu';z_1,z_2x,z_3,z_4xy),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.26) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{A_3}^{(4)}(a,b,c,\mu;e,\lambda';z_1,z_2,z_3,z_4xy)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{A_3}^{(4)}(a,b,c,\lambda;e,\mu';z_1,z_2,z_3,z_4xy),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.27) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}x^{\lambda-1}y^{\lambda'-1}G_{B_1}^{(4)}(a,\mu,\mu',b_3,b_4;e,e';z_1x,z_2y,z_3,z_4)$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{B_1}^{(4)}(a,\lambda,\lambda',b_3,b_4;e,e';z_1x,z_2y,z_3,z_4),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

We can also obtain five more relations similar to (3.27).

(3.28) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{B_1}^{(4)}(a,\mu,b_2,b_3,b_4';e,\lambda;z_1x,z_2,z_3,z_4y)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{B_1}^{(4)}(a,\lambda,b_2,b_3,b_4;e,\mu';z_1x,z_2,z_3,z_4y)$$
$$Re(\lambda)>0,\ Re(\lambda')>0.$$

We can also obtain three more relations similar to (3.28).

(3.29) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{B_2}^{(4)}(a,\mu,\mu',b_3,b_4;e,e';z_1x,z_2y,z_3,z_4x)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{B_2}^{(4)}(a,\lambda,\lambda',b_3,b_4;e,e';z_1x,z_2y,z_3,z_4x),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.30) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{B_2}^{(4)}(a,\mu,b_2,\mu',b_4;e,e';z_1x,z_2,z_3y,z_4x)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{B_2}^{(4)}(a,\lambda,b_2,\lambda',b_4;e,e';z_1x,z_2,z_3y,z_4x),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.31) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{B_2}^{(4)}(a,\mu,b_2,b_3,\mu';e,e';z_1x,z_2,z_3,z_4xy)\right\}$$
$$\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{B_2}^{(4)}(a,\lambda,b_2,b_3,\lambda';e,e';z_1x,z_2,z_3y,z_4xy),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.32) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{B_2}^{(4)}(a,\mu,b_2,b_3,b_4;e,\lambda';z_1x,z_2,z_3,z_4xy)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{B_2}^{(4)}(a,\lambda,b_2,b_3,b_4;e,\mu';z_1x,z_2,z_3,z_4xy),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.33) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{B_2}^{(4)}(a,b_1,\lambda,b_3,b_4;e,\mu';z_1,z_2x,z_3,z_4y)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{B_2}^{(4)}(a, b_1, \mu, b_3, b_4; e, \lambda'; z_1, z_2 x, z_3, z_4 y),$$

$$Re(\lambda) > 0, Re(\lambda') > 0;$$

(3.34) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} G_{B_2}^{(4)}(a, b_1, b_2, \lambda, b_4; e, \mu'; z_1, z_2, z_3 x, z_4 y) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{B_2}^{(4)}(a, b_1, b_2, \mu, b_4; e, \lambda'; z_1, z_2, z_3 x, z_4 y),$$

$$Re(\lambda) > 0, Re(\lambda') > 0;$$

(3.35) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} G_{B_2}^{(4)}(a, b_1, b_2, b_3, \lambda; e, \mu'; z_1, z_2, z_3, z_4 xy) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{B_2}^{(4)}(a, b_1, b_2, b_3, \mu; e, \lambda'; z_1, z_2, z_3, z_4 xy),$$

$$Re(\lambda) > 0, Re(\lambda') > 0;$$

(3.36) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} G_{B_3}^{(4)}(a, \mu, \mu', b_3, b_4; e, e'; z_1 y, z_2 x, z_3, z_4 x) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{B_3}^{(4)}(a, \lambda, \lambda', b_3, b_4; e, e'; z_1 y, z_2 x, z_3, z_4 x),$$

$$Re(\lambda) > 0, Re(\lambda') > 0;$$

(3.37) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} G_{B_3}^{(4)}(a, \mu, b_2, \mu', b_4; e, e'; z_1, z_2 x, z_3 y, z_4 x) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{B_3}^{(4)}(a, \lambda, b_2, \lambda', b_4; e, e'; z_1, z_2 x, z_3 y, z_4 x),$$

$$Re(\lambda) > 0, Re(\lambda') > 0;$$

(3.38) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} G_{B_3}^{(4)}(a, \mu, b_2, b_3, \mu'; e, e'; z_1, z_2 x, z_3, z_4 xy) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{B_3}^{(4)}(a, \lambda, b_2, b_3, \lambda'; e, e'; z_1, z_2 x, z_3, z_4 xy),$$

$$Re(\lambda) > 0, Re(\lambda') > 0;$$

(3.39) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} G_{B_3}^{(4)}(a, \mu, b_2, b_3, b_4; e, \lambda'; z_1, z_2 x, z_3, z_4 xy) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{B_3}^{(4)}(a, \lambda, b_2, b_3, b_4; e, \mu'; z_1, z_2 x, z_3, z_4 xy),$$

$$Re(\lambda) > 0, Re(\lambda') > 0;$$

(3.40) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} \left\{ x^{\lambda-1} y^{\lambda'-1} G_{B_3}^{(4)}(a, b_1, \mu, b_3, b_4; e, \lambda'; z_1 x, z_2, z_3, z_4 y) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{B_3}^{(4)}(a, b_1, \lambda, b_3, b_4; e, \mu'; z_1 x, z_2, z_3, z_4 y),$$

$$Re(\lambda) > 0, Re(\lambda') > 0;$$

(3.41) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{B_3}^{(4)}(a,b_1,b_2,\mu,b_4;e,\lambda';z_1,z_2,z_3x,z_4y)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{B_3}^{(4)}(a,b_1,b_2,\lambda,b_4;e,\mu';z_1,z_2,z_3x,z_4y),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.42) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{B_3}^{(4)}(a,b_1,b_2,b_3,\mu;e,\lambda';z_1,z_2,z_3,z_4xy)\right\}$$
$$\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{B_3}^{(4)}(a,b_1,b_2,b_3,\lambda;e,\mu';z_1,z_2,z_3,z_4xy),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.43) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{C_1}^{(4)}(\mu,\mu',,c,d;e,e';z_1xy,z_2y,z_3x,z_4x)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{C_1}^{(4)}(\lambda,\lambda',c,d;e,e';z_1xy,z_2y,z_3x,z_4x),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.44) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{C_1}^{(4)}(\mu,b,c,\mu';e,e';z_1x,z_2,z_3x,z_4xy)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{C_1}^{(4)}(\lambda,b,c,\lambda';e,e';z_1x,z_2,z_3x,z_4xy),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.45) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{C_1}^{(4)}(a,\mu,c,\mu';e,e';z_1x,z_2x,z_3,z_4y)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{C_1}^{(4)}(a,\lambda,c,\lambda';e,e';z_1x,z_2x,z_3,z_4y),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.46) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{C_1}^{(4)}(\mu,b,c,d;e,\lambda';z_1x,z_2,z_3x,z_4xy)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{C_1}^{(4)}(\lambda,b,c,d;e,\mu';z_1x,z_2,z_3x,z_4xy),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.47) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{C_1}^{(4)}(a,\mu,c,d;e,\lambda';z_1x,z_2x,z_3,z_4y)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{C_1}^{(4)}(a,\lambda,c,d;e,\mu';z_1x,z_2x,z_3,z_4y),$$
$$Re(\lambda)>0,\ Re(\lambda')>0;$$

(3.48) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}\left\{x^{\lambda-1}y^{\lambda'-1}G_{C_1}^{(4)}(a,b,c,\mu;e,\lambda';z_1,z_2,z_3,z_4xy)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')}x^{\mu-1}y^{\mu'-1}G_{C_1}^{(4)}(a,b,c,\lambda;e,\mu';z_1,z_2,z_3,z_4xy),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.49) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'}\left\{x^{\lambda-1} y^{\lambda'-1} G_{C_1}^{(4)}(\mu,\mu',c,d;e,e';z_1xy,z_2x,z_3y,z_4x)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{C_1}^{(4)}(\lambda,\lambda',c,d;e,e';z_1xy,z_2x,z_3y,z_4x),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.50) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'}\left\{x^{\lambda-1} y^{\lambda'-1} G_{C_1}^{(4)}(\mu,b,c,\mu';e,e';z_1x,z_2x,z_3,z_4xy)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{C_1}^{(4)}(\lambda,b,c,\lambda';e,e';z_1x,z_2x,z_3,z_4xy),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.51) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'}\left\{x^{\lambda-1} y^{\lambda'-1} G_{C_1}^{(4)}(a,\mu,c,\mu';e,e';z_1x,z_2,z_3x,z_4y)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{C_1}^{(4)}(a,\lambda,c,\lambda';e,e';z_1x,z_2,z_3x,z_4y),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.52) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'}\left\{x^{\lambda-1} y^{\lambda'-1} G_{C_1}^{(4)}(\mu,b,c,d;e,\lambda';z_1x,z_2x,z_3,z_4xy)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{C_1}^{(4)}(\lambda,b,c,d;e,\mu';z_1x,z_2x,z_3,z_4xy),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.53) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'}\left\{x^{\lambda-1} y^{\lambda'-1} G_{C_1}^{(4)}(a,\mu,c,d;e,\lambda';z_1x,z_2,z_3x,z_4y)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{C_1}^{(4)}(a,\lambda,c,d;e,\mu';z_1x,z_2,z_3x,z_4y),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

(3.54) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'}\left\{x^{\lambda-1} y^{\lambda'-1} G_{C_1}^{(4)}(a,b,c,\mu;e,\lambda';z_1,z_2,z_3,z_4xy)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\mu')} x^{\mu-1} y^{\mu'-1} G_{C_1}^{(4)}(a,b,c,\lambda;e,\mu';z_1,z_2,z_3,z_4xy)$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0;$$

4. Use of Three Fractional Derivative Operators.

(4.1) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''}\left\{x^{\lambda-1} y^{\lambda'-1} y^{\lambda''-1} H_{A_1}^{(4)}(\mu,\mu',\mu'',d;e,e',e'';xyu_1,xu_2,xzu_3,yzu_4)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} H_{A_1}^{(4)}(\lambda,\lambda',\lambda'',d;e,e',e'';xyu_1,xu_2,xzu_3,yzu_4),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$$

(4.2) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''}\left\{x^{\lambda-1} y^{\lambda'-1} y^{\lambda''-1} H_{A_1}^{(4)}(\mu,\mu',c,d;\lambda'',e',e'';zxyu_1,xu_2,xzu_3,yu_4)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} H_{A_1}^{(4)}(\lambda, \lambda', c, d; \mu'', e', e''; xyzu_1, xu_2, zxu_3, yu_4),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$$

(4.3) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''} \left\{ x^{\lambda-1} y^{\lambda'-1} y^{\lambda''-1} H_{A_1}^{(4)}(\mu, b, \mu', d; \lambda'', e', e''; u_1 xz, xu_2, u_3 xyz, u_4 y) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} H_{A_1}^{(4)}(\lambda, b, \lambda', d; \mu'', e', e''; u_1 xz, u_2 x, u_3 xyz, u_4 y),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$$

(4.4) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''} \left\{ x^{\lambda-1} y^{\lambda'-1} y^{\lambda''-1} H_{A_1}^{(4)}(a, \mu, \mu', d; \lambda'', e', e''; u_1 xz, u_2, u_3 yz, u_4 yx) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} H_{A_1}^{(4)}(a, \lambda, \lambda', d; \mu'', e', e''; u_1 xz, u_2, u_3 yz, u_4 xy),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$$

(4.5) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''} \left\{ x^{\lambda-1} y^{\lambda'-1} y^{\lambda''-1} H_{B_1}^{(4)}(\mu, \mu', \mu'', d; e_1, e_2, e_3, e_4; u_1 xy, u_2 x, u_3 yz, u_4 xz) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} H_{B_1}^{(4)}(\lambda, \lambda', \lambda'', d; e_1, e_2, e_3, e_4; u_1 xy, u_2 x, u_3 yz, u_4 xz),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$$

(4.6) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''} \left\{ x^{\lambda-1} y^{\lambda'-1} y^{\lambda''-1} H_{B_1}^{(4)}(\mu, \mu', c, d; \lambda'', e_2, e_3, e_4; u_1 xyz, u_2 x, u_3 y, u_4 x) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} H_{B_1}^{(4)}(\lambda, \lambda', c, d; \mu'', e_2, e_3, e_4; u_1 xyz, u_2 x, u_3 y, u_4 x),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$$

(4.7) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''} \left\{ x^{\lambda-1} y^{\lambda'-1} y^{\lambda''-1} H_{B_1}^{(4)}(\mu, b, \mu', d; \lambda'', e_2, e_3, e_4; u_1 xz, u_2 x, u_3 y, u_4 xy) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} H_{B_1}^{(4)}(\lambda, \lambda', c, d; \mu'', e_2, e_3, e_4; u_1 xyz, u_2 x, u_3 y, u_4 x)$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$$

(4.8) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''} \left\{ x^{\lambda-1} y^{\lambda'-1} y^{\lambda''-1} H_{B_1}^{(4)}(a, \mu, \mu', d; \lambda'', e_2, e_3, e_4; u_1 xz, u_2, u_3 xy, u_4 y) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} H_{B_1}^{(4)}(a, \lambda, \lambda', d; \mu'', e_2, e_3, e_4; u_1 xz, u_2, u_3 xy, u_4 y),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$$

(4.9) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''} \left\{ x^{\lambda-1} y^{\lambda'-1} y^{\lambda''-1} G_{A_1}^{(4)}(a, \mu, \mu', \mu''; e, e'; u_1 x, u_2 y, u_3 x, u_4 z) \right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} G_{A_1}^{(4)}(a, \lambda, \lambda', \lambda''; e, e'; u_1 x, u_2 y, u_3 x, u_4 z),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$$

(4.10) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}D_z^{\lambda''-\mu''}\left\{x^{\lambda-1}y^{\lambda'-1}y^{\lambda''-1}G_{A_1}^{(4)}(a,\mu,\mu',d;e,\lambda'';u_1x,u_2y,u_3x,u_4z)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}x^{\mu-1}y^{\mu'-1}z^{\mu''-1}G_{A_1}^{(4)}(a,\lambda,\lambda',d;e,\mu'';u_1x,u_2y,u_3x,u_4z),$$
$$Re(\lambda)>0,\ Re(\lambda')>0, Re(\lambda'')>0;$$

(4.11) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}D_z^{\lambda''-\mu''}\left\{x^{\lambda-1}y^{\lambda'-1}z^{\lambda''-1}G_{A_2}^{(4)}(a,\mu,\mu',\mu'';e,e';u_1x,u_2y,u_3x,u_4xz)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}x^{\mu-1}y^{\mu'-1}z^{\mu''-1}G_{A_2}^{(4)}(a,\lambda,\lambda',\lambda'';e,e';u_1x,u_2y,u_3x,u_4xz),$$
$$Re(\lambda)>0,\ Re(\lambda')>0, Re(\lambda'')>0;$$

(4.12) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}D_z^{\lambda''-\mu''}\left\{x^{\lambda-1}y^{\lambda'-1}z^{\lambda''-1}G_{A_2}^{(4)}(a,\mu,\mu',d;e,\lambda'';u_1x,u_2y,u_3x,u_4xz)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}x^{\mu-1}y^{\mu'-1}z^{\mu''-1}G_{A_2}^{(4)}(a,\lambda,\lambda',d;e,\mu'';u_1x,u_2y,u_3x,u_4xz),$$
$$Re(\lambda)>0,\ Re(\lambda')>0, Re(\lambda'')>0;$$

(4.13) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}D_z^{\lambda''-\mu''}\left\{x^{\lambda-1}y^{\lambda'-1}z^{\lambda''-1}G_{A_3}^{(4)}(a,\mu,\mu',\mu'';e,e';u_1x,u_2y,u_3x,u_4yz)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}x^{\mu-1}y^{\mu'-1}z^{\mu''-1}G_{A_3}^{(4)}(a,\lambda,\lambda',\lambda'';e,e';u_1x,u_2y,u_3x,u_4yz),$$
$$Re(\lambda)>0,\ Re(\lambda')>0, Re(\lambda'')>0;$$

(4.14) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}D_z^{\lambda''-\mu''}\left\{x^{\lambda-1}y^{\lambda'-1}z^{\lambda''-1}G_{A_3}^{(4)}(a,\mu,\mu',d;e,\lambda'';u_1x,u_2y,u_3x,u_4yz)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}x^{\mu-1}y^{\mu'-1}z^{\mu''-1}G_{A_3}^{(4)}(a,\lambda,\lambda',d;e,\mu'';u_1x,u_2y,u_3x,u_4yz),$$
$$Re(\lambda)>0,\ Re(\lambda')>0, Re(\lambda'')>0;$$

(4.15) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}D_z^{\lambda''-\mu''}\left\{x^{\lambda-1}y^{\lambda'-1}z^{\lambda''-1}G_{B_1}^{(4)}(a,\mu,\mu',\mu'',b_4;e,e';u_1x,u_2y,u_3z,u_4)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}x^{\mu-1}y^{\mu'-1}z^{\mu''-1}G_{B_1}^{(4)}(a,\lambda,\lambda',\lambda'',b_4;e,e';u_1x,u_2y,u_3z,u_4),$$
$$Re(\lambda)>0,\ Re(\lambda')>0, Re(\lambda'')>0;$$

(4.15) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}D_z^{\lambda''-\mu''}\left\{x^{\lambda-1}y^{\lambda'-1}z^{\lambda''-1}G_{B_1}^{(4)}(a,\mu,\mu',\mu'',b_4;e,e';u_1x,u_2y,u_3z,u_4)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}x^{\mu-1}y^{\mu'-1}z^{\mu''-1}G_{B_1}^{(4)}(a,\lambda,\lambda',\lambda'',b_4;e,e';u_1x,u_2y,u_3z,u_4),$$
$$Re(\lambda)>0,\ Re(\lambda')>0, Re(\lambda'')>0;$$

(4.16) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}D_z^{\lambda''-\mu''}\left\{x^{\lambda-1}y^{\lambda'-1}z^{\lambda''-1}G_{B_1}^{(4)}(a,b_1,\mu,\mu',\mu'';e,e';u_1,u_2x,u_3y,u_4z)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}x^{\mu-1}y^{\mu'-1}z^{\mu''-1}G_{B_1}^{(4)}(a,b_1,\lambda,\lambda',\lambda'';e,e';u_1,u_2x,u_3y,u_4z),$$

$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$

(4.17) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''}\left\{x^{\lambda-1} y^{\lambda'-1} z^{\lambda''-1} G_{B_1}^{(4)}\left(a, b_1, \mu, \mu', b_4; e, \lambda''; u_1, u_2 x, u_3 x, u_4 z\right)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} G_{B_1}^{(4)}\left(a, b_1, \lambda, \lambda', b_4; e, \mu''; u_1, u_2 x, u_3 y, u_4 z\right),$$

$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$

(4.18) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''}\left\{x^{\lambda-1} y^{\lambda'-1} z^{\lambda''-1} G_{B_1}^{(4)}\left(a, b_1, b_2, \mu, \mu'; e, \lambda''; u_1, u_2, u_3 x, u_4 yz\right)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} G_{B_1}^{(4)}\left(a, b_1, b_2, \lambda, \lambda'; e, \mu''; u_1, u_2, u_3 x, u_4 yz\right),$$

$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$

(4.19) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''}\left\{x^{\lambda-1} y^{\lambda'-1} z^{\lambda''-1} G_{B_2}^{(4)}\left(a, \mu, \mu', \mu'', b_4; e, e'; u_1 x, u_2 y, u_3 z, u_4 x\right)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} G_{B_2}^{2}\left(a, \lambda, \lambda', \lambda'', b_4; e, e'; u_1 x, u_2 y, u_3 z, u_4 x\right),$$

$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$

(4.20) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''}\left\{x^{\lambda-1} y^{\lambda'-1} z^{\lambda''-1} G_{B_2}^{(4)}\left(a, \mu, \mu', b_3, b_4; e, \lambda''; u_1 x, u_2 y, u_3, u_4 xz\right)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} G_{B_2}^{(4)}\left(a, \lambda, \lambda', b_3, b_4; e, \mu''; u_1 x, u_2 y, u_3, u_4 xz\right),$$

$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$

(4.21) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''}\left\{x^{\lambda-1} y^{\lambda'-1} z^{\lambda''-1} G_{B_2}^{(4)}\left(a, \mu, b_2, \mu', b_4; e, \lambda''; u_1 x, u_2, u_3 y, u_4 xz\right)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} G_{B_2}^{(4)}\left(a, \mu, b_2, \mu', b_4; e, \lambda''; u_1 x, u_2, u_3 y, u_4 xz\right),$$

$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$

(4.22) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''}\left\{x^{\lambda-1} y^{\lambda'-1} z^{\lambda''-1} G_{B_2}^{(4)}\left(a, \mu, b_2, b_3, \mu'; e, \lambda''; u_1 x, u_2, u_3, u_4 xyz\right)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} G_{B_2}^{(4)}\left(a, \lambda, b_2, b_3, \lambda'; e, \mu''; u_1 x, u_2, u_3, u_4 xyz\right),$$

$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$

(4.23) $$D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''}\left\{x^{\lambda-1} y^{\lambda'-1} z^{\lambda''-1} G_{B_3}^{(4)}\left(a, \mu, \mu', \mu'', b_4; e, e'; u_1 y, u_2 x, u_3 z, u_4 x\right)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} G_{B_3}^{(4)}\left(a, \lambda, \lambda', \lambda'', b_4; e, e'; u_1 y, u_2 x, u_3 z, u_4 x\right)$$

$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$

(4.24) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}D_z^{\lambda''-\mu''}\left\{x^{\lambda-1}y^{\lambda'-1}z^{\lambda''-1}G_{B_3}^{(4)}(a,\mu,\mu',b_3,\mu'';e,e';u_1y,u_2x,u_3,u_4xz)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}x^{\mu-1}y^{\mu'-1}z^{\mu''-1}G_{B_3}^{(4)}(a,\lambda,\lambda',b_3,\lambda'';e,e';u_1y,u_2x,u_3,u_4xz),$$
$$Re(\lambda)>0,\ Re(\lambda')>0, Re(\lambda'')>0;$$

(4.25) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}D_z^{\lambda''-\mu''}\left\{x^{\lambda-1}y^{\lambda'-1}z^{\lambda''-1}G_{B_3}^{(4)}(a,\mu,\mu',b_3,b_4;e,\lambda'';u_1y,u_2x,u_3,u_4xz)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}x^{\mu-1}y^{\mu'-1}z^{\mu''-1}G_{B_3}^{(4)}(a,\lambda,\lambda',b_3,b_4;e,\mu'';u_1y,u_2x,u_3,u_4xz),$$
$$Re(\lambda)>0,\ Re(\lambda')>0, Re(\lambda'')>0;$$

(4.26) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}D_z^{\lambda''-\mu''}\left\{x^{\lambda-1}y^{\lambda'-1}z^{\lambda''-1}G_{C_1}^{(4)}(\mu,\mu',c,\mu'';e,e';u_1xy,u_2y,u_3x,u_4zx)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}x^{\mu-1}y^{\mu'-1}z^{\mu''-1}G_{C_1}^{(4)}(\lambda,\lambda',c,\lambda'',;e,e';u_1xy,u_2y,u_3x,u_4xz),$$
$$Re(\lambda)>0,\ Re(\lambda')>0, Re(\lambda'')>0;$$

(4.27) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}D_z^{\lambda''-\mu''}\left\{x^{\lambda-1}y^{\lambda'-1}z^{\lambda''-1}G_{C_1}^{(4)}(\mu,\mu',c,d;e,\lambda'';u_1xy,u_2y,u_3x,u_4zx)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}x^{\mu-1}y^{\mu'-1}z^{\mu''-1}G_{C_1}^{(4)}(\lambda,\lambda',c,d;e,\mu'';u_1xy,u_2y,u_3x,u_4xz),$$
$$Re(\lambda)>0,\ Re(\lambda')>0, Re(\lambda'')>0;$$

(4.28) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}D_z^{\lambda''-\mu''}\left\{x^{\lambda-1}y^{\lambda'-1}z^{\lambda''-1}G_{C_1}^{(4)}(\mu,b,c,\mu';e,\lambda'';u_1x,u_2,u_3x,u_4xyz)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}x^{\mu-1}y^{\mu'-1}z^{\mu''-1}G_{C_1}^{(4)}(\lambda,b,c,\lambda',;e,\mu'';u_1x,u_2,u_3x,u_4xyz),$$
$$Re(\lambda)>0,\ Re(\lambda')>0, Re(\lambda'')>0;$$

(4.29) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}D_z^{\lambda''-\mu''}\left\{x^{\lambda-1}y^{\lambda'-1}z^{\lambda''-1}G_{C_1}^{(4)}(a,\mu,c,\mu';e,\lambda'';u_1x,u_2x,u_3,u_4yz)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}x^{\mu-1}y^{\mu'-1}z^{\mu''-1}G_{C_1}^{(4)}(a,\lambda,c,\lambda';e,\mu'';u_1x,u_2x,u_3,u_4yz)$$
$$Re(\lambda)>0,\ Re(\lambda')>0, Re(\lambda'')>0;$$

(4.30) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}D_z^{\lambda''-\mu''}\left\{x^{\lambda-1}y^{\lambda'-1}z^{\lambda''-1}G_{C_2}^{(4)}(\mu,\mu',c,\mu'';e,e';u_1xy,u_2x,u_3y,u_4xz)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}x^{\mu-1}y^{\mu'-1}z^{\mu''-1}G_{C_2}^{(4)}(\lambda,\lambda',c,\lambda'';e,e';u_1xy,u_2x,u_3y,u_4xz),$$
$$Re(\lambda)>0,\ Re(\lambda')>0, Re(\lambda'')>0;$$

(4.31) $$D_x^{\lambda-\mu}D_y^{\lambda'-\mu'}D_z^{\lambda''-\mu''}\left\{x^{\lambda-1}y^{\lambda'-1}z^{\lambda''-1}G_{C_2}^{(4)}(\mu,\mu',c,d;e,\lambda'';u_1xy,u_2x,u_3y,u_4xz)\right\}$$
$$=\frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')}x^{\mu-1}y^{\mu'-1}z^{\mu''-1}G_{C_2}^{(4)}(\lambda,\lambda',c,d;e,\mu'';u_1xy,u_2x,u_3y,u_4xz),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$$

$$(4.32)\quad D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''}\left\{x^{\lambda-1} y^{\lambda'-1} z^{\lambda''-1} G_{C_2}^{(4)}(a,\mu,c,\mu';e,\lambda'';u_1x,u_2,u_3x,u_4yz)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} G_{C_2}^{(4)}(a,\lambda,c,\lambda';e,\mu'';u_1x,u_2,u_3x,u_4yz),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$$

$$(4.33)\quad D_x^{\lambda-\mu} D_y^{\lambda'-\mu'} D_z^{\lambda''-\mu''}\left\{x^{\lambda-1} y^{\lambda'-1} z^{\lambda''-1} G_{C_2}^{(4)}(\mu,b,c,\mu';e,\lambda'';u_1x,u_2x,u_3,u_4yz)\right\}$$

$$= \frac{\Gamma(\lambda)\Gamma(\lambda')\Gamma(\lambda'')}{\Gamma(\mu)\Gamma(\mu')\Gamma(\mu'')} x^{\mu-1} y^{\mu'-1} z^{\mu''-1} G_{C_2}^{(4)}(\lambda,b,c,\lambda';e,\mu'';u_1x,u_2x,u_3,u_4xyz),$$

$$Re(\lambda) > 0,\ Re(\lambda') > 0, Re(\lambda'') > 0;$$

Applying the same techniques, we have also derived fractional derivatives involving hypergeometric functions of four variables introduced by Chandel, Agrawal and Kumar [10] but we have not recorded them due to lac of space.

REFERENCES

[1] R.C.S. Chandel, On some multiple hypergometric functions related to Lauricella's functions, *Jñānābha* Sect A, **3** (1973),119-136. Errata and Addenda, ibid. **5** (1975), 177-180.

[2] R.C.S. Chandel and A.K. Gupta, Multiple hypergeometic functions related to Lauricella's functions, *Jñānābha* , **16** (1986),195-209.

[3] R.C.S. Chandel and P.K. Vishwakarma, Karlsson's multiple hypergeometric function and its confluent forms, *Jñānābha* , **19** (1989),173-185.

[4] R.C. S. Chandel and P.K. Vishwakarma, Fractional integration and integral representations of Karlsson's multiple hypergeometric function and its confluent forms, *Jñānābha* **20** (1990),101-110.

[5] R.C.S. Chandel and P.K. Vishwakrma, Fractional derivatives of confluent hypergemetric forms of Karlsson's multiple hypergeometric function, ${}^{(k)}F_{CD}^{(n)}$, *Pure Appl. Math. Sci.* **35** (1992), 31-39.

[6] R.C.S. Chandel and P.K. Vishwakarma, multidimensional fractional derivatives of the multiple hypergeometric functions of several variables, *Jñānābha* , **24** (1994),19-27.

[7] R.C.S. Chandel and P.K. Vishwakarma, Fractional derivatives of the multiple hypergeometric functions of four variables, *Jñānābha* , **26** (1996),83-86.

[8] R.C.S. Chandel and S. Sharma, Some new hypergeometric functions of four variables II, *Bull. Vijñāna Parishad of India,* **2** (1994), 44 .

[9] R.C.S. Chandel and S. Sharma, Hypergeometric functions of four variables, *Pure Appl. Math. Sci.* **58** (2003), 7-18.

[10] R.C.S. Chandel, R.D. Agrawal and H. Kumar, Hypergeometric functions of four variables and their integral representations, *Math. Eduction,* **26** (1992), 76-94.

[11] H. Exton, On two multiple hypergeometric funtions related to Lauricella's $F_D^{(n)}$, *Jñānābha* Sect. A, **2** (1992),59-73.

[12] H. Exton, Certain hypergeometric functionsof four variables, *Bull. Soc. Math. Gréce (N.S.)* **14** (1973), 132-140.

[13] H. Exton, Some integral representations and transformations of hypergeometric functions of four variables, *Bull. Soc. Math. Grece* (N.S.) **14** (1973), 132-140.

[14] H. Exton, *Multiple hypergeometric Functions and Applications,* Jhon Wiley and Sons, New York, London, Sydney and Toronto, 1976.

[15] P.W. Karlsson, On intermediate Lauricella functions, *Jñānābha* , **16** (1986),211-222.

[16] G. Lauricella, Sulle funzioni ipergeometriche a più variabili, *Rend. Circ. Mat. Palermo,* **7** (1893), 111-158.

[17] K.B. Oldham and J. Spanier, *The Fractional Calculus,* Academic Press, New York, London 1974.

[18] C. Sharma and C.L. Parihar, Hypergeometric functions of four variables I, *J. Indian, Acad. Math.* **11** (1989), 121-133.

[19] H. M. Srivastava and M.C. Daoust, Certain generalized Neumann expansions associated with the Kampé de Fériet function, *Nederl. Akad. Wetensch. Proc. Ser.* A **72** *Indag. Math.* **31** (1969), 449-457.

[20] H.M. Srivastava and R. Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials, *J. Reine. Angew. Math.* **283/284** (1976), 265-274.

[21] H.M. Srivastava and R. Panda, Expansion theorems for the *H*-function of several complex variables, *J. Reine Angew. Math.,* **288** (1976), 129-145.

[22] H.M. Srivastava and R. Panda, Some Expanssion theorems and generating relations for the *H*-function of several complex variables I and II, *Commet. Math. Univ. St. Paul.* **24** fasc **2** (1975), 119-137; **25** fasc **2** (1976), 167-197.

[23] H.M. Srivastava , K.C. Gupta and S.P. Goyal, *The H-Function of One and Two Variables with Applications,* South Asian Publisher, New Delhi/Madrass, 1982.

[24] H.M. Srivastava and H.L. Manocha, *A Treatise On Generating Functions,* John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[25] H.M. Srivastava and S.P. Goyal, Fractional dervatives of the *H*-function of several variables, *J. Math. Anal. Appl.,* **112** (1985), 641-651.

[26] H.M. Srivastava, R.C.S. Chandel and P.K. Vishwakarma, Fractional derivatives of certain generalized hypergeometric functions of several variables, *J. Math. Anal. Appl.* **184**

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SOME COMMON FIXED POINT THEOREMS IN COMPLETE METRIC SPACES

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ABSTRACT

In the present paper, some comman fixed point theorems have been proved which generalize the results recently proved in [3], [7] and [10].

1. Introduction. After the works of Khan et. al [3], Pathak and Sharma [6], Sastri and Babu [8], Sastri et. al. [10] etc. have obtained several common fixed point theorems in metric space. Sastri et. al. [10] have shown that the inequality used in Khan et. al. [3] does not hold for $x=y$ because a(0) is not defined there in while the unicity condition fails in Pathak and Sharma [6]. In their support they have provided examples [10].

Let R^+ denote the set of all non negative reals, N the set of all natural numbers and ϕ, the set continuous of ψ such that ψ is monotonically increasing ant $\psi(t)=0$ if and only if $t=0$.

Using this function ψ and altering the distance between points, Khan et. al. [3], Park [4,5], Pathak and Sharma [6], Sastri and Babu [8] Sastri et. al [10], etc. have obtained fixed point for one mapping and common fixed point for two mappings. Sastri et. al. [8] have proved the following

Theorem A. Let (X,d) be a bounced metric space and $T:X\to X$ such that

$$\psi(d(Tx,Ty)) \le k \max \{\psi(d(x,y)),\psi(d(x,Tx)),\psi(d(y,Ty))\}$$

and $0\le k\le 1$ for all x, y in X. Then T has a unique fixed point in X.

They have also proved the following

Theorem B. Let (X,d) be a bounded complete metric space and S and T be self maps of X such that $ST=TS$. Further, assume that S and T satisfy the following inequality

there exists $k\in(0,1)$ and $\psi\in\Phi$ such that

$$\psi(d(Sx,Ty)) \le k \max \{\psi(d(x,y)),\psi(d(x,Sx)),\psi(d(y,Ty))\}$$

for all x, y in X. Then one of S and T (and hence both) have a unique common fixed point in X.

Sessa [11] introduced the notion of weakly commutativity which is weaker than commutativity. Every commutative pair is weakly commutative but the converse is not always true [10].

Definition 1. Two mappings $f,g : X \to X$, where X is a metric space are said to be weakly commutative if and only if $d(fgx,gfx) \leq d(fx,gx)$ for all $x \in X$.

Later on Jungck [1] extended the notion of weakly commutative to compatibility and showed that every weakly commutative maps are compatible but the converse is not always ture.

Definition 2. Two mappings $f,g:X \to X$, where X is a complete metric space, are said to be compatible if and only if whenever $\{x_n\}$ is a sequence in X such that $fx_n, gx_n \to t$ then $d(gfx_n, fgx_n) \to 0$.

Jungck and Rhoades [2] established the following

Proposition A. Let f,g be compatible self maps of a metric space (X,d)

1. If $f(t)=g(t)$ then $fg(t)=gf(t)$,
2. Suppose that $\lim fx_n = \lim gx_n = t$ for some $t \in X$ and $x_n \in X$,

Then

(a) If f is a continuous at t, $\lim gfx_n = f(t)$

(b) If f and g are continuous at t then $f(t)=g(t)$ and $fg(t)=gf(t)$.

In this paper, we obtain a necessary and sufficient condition for a common fixed point of these compatible mappings which extend the results proved in [3],[7] and [10].

We now first prove the following

Theorem 1. Let (X,d) be a complete metric space and let f, g be continuous mappings of X into itself then a mapping $h:X \to f(X) \cap g(X)$ has a fixed point in X if and only if

$$\phi(d(hx,hy)) \leq k \max \{\phi(d(hx,fx)),\phi(d(hy,gy)),\phi(d(fx,gy)),\phi((d(hx,gy)/2+d(hy,fx))\} \quad \text{...(A)}$$

for all $x,y \in X$, $0 \leq k \leq 1$ and $\phi \in \Phi$ and f, g are compatible with h. Further f,g,h have a unique common fixed point in X.

Proof. We shall first show that the condition is sufficient. Let $x_0 \in X$ such that $hx_0 = fx$. Since $x_1 \in X$ and $h(X) \subset g(X)$, there exists a point $x_2 \in X$ such that $hx_1 = gx_2$. In this way a sequence $\{x_n\}$ is constructed. So that $hx_{2n} = fx_{2n+1}$ and $hx_{2n+1} = gx_{2n+2}$, $n=0,1,2...$.

Define $d_n = d(hx_n, hx_{n+1})$. From (a)

$$\begin{aligned}\phi(d_{2n}) &= \phi(d(hx_{2n}, hx_{2n+1})) = \phi(d(hx_{2n+1}, hx_{2n})) \\ &\leq k \max \{\phi(d(hx_{2n+1}, fx_{2n+1})), \phi(d(hx_{2n}, gx_{2n})), \phi(d(hx_{2n+1}, gx_{2n})),\end{aligned}$$

$$\phi\{d(hx_{2n+1},gx_{2n})+d(hx_{2n},fx_{2n+1})\}/2\}$$
$$=k \max \{\phi(d(hx_{2n+1},hx_{2n})),\phi(d(hx_{2n},hx_{2n-1})),0,$$
$$\phi\{(d(hx_{2n+1},hx_{2n-1})/2\}\}$$
$$\leq k \max \{\phi(d_{2n}),\phi(d_{2n-1}),0, \phi\{d(hx_{2n-1},hx_{2n}),(hx_{2n},hx_{2n+1})\}/2\}$$
$$=k \max \{\phi(d_{2n}),\phi(d_{2n-1}),0, \phi\{(d_{2n-1}+d_{2n})/2\}$$

If $d_{2n}>d_{2n-1}$, for any n. Then

$$\phi(d_{2n}) \leq k\phi(d_{2n}) <\phi(d_{2n}) \text{ a contradiction.}$$

Therefore

$$\phi(d_{2n}) \leq k\phi(d_{2n-1}) <\phi(d_{2n-1}) \qquad ...(1)$$

and because ϕ is non decreasing, $d_{2n}<d_{2n-1}$.

Similarly we can show that $d_{2n+1}<d_{2n}$, so that for each n, $d_{n+1}<d_n$ and $\{d_n\}$ strictly decreasing sequence of reals. Further as

$$\phi(d_n) \leq k\phi(d_{n-1})\leq k^2\phi(d_{n-2})\leq ... \leq k^{n-1}\phi(d_1)\to 0 \text{ as } n\to\infty.$$

Since $\{d_n\}$ is a decreasing sequence of reals, it converges. Call the limit p. Suppose $p>0$, then since ϕ is continuous $\lim \phi(d_n)=\phi(p)=0$ therefore $p=0$. We now wish to show that $\{hx_n\}$ is a Cauchy sequence. Assume that it is not Cauchy.

Then for every positive number ε and for every poritive integer k there exit two positive integers $2m(k)$ and $2n(k)$ such that $2m(k)>2n(k)>k$ and $d(hx_{2m(k)},hx_{2n(k)})>\varepsilon$. Further let $2m(k)$ denote the smallest even integer for which $2m(k)>2n\ (k)>k$, $d(hx_{2m(k)},hx_{2n(k)})>\varepsilon$ and $d(hx_{2m(k)-2},hx_{2n(k)})\leq \varepsilon$.

Then $\phi(\varepsilon) <\phi(d(hx_{2n(k)},hx_{2m(k)}))$

$$\leq\phi(d(hx_{2n(k)},hx_{2m(k)-2})+d_{2m(k)-2}+d_{2m(k)-1})).$$

Taking limit as $n\to\infty$ we have

$$\phi(\varepsilon) = \lim \phi(d(hx_{2n(k)},hx_{2m(k)})) . \qquad ... (2)$$

Using triangle inequality

$$|d(hx_{2m(k)},hx_{2n(k)+1})-d(hx_{2m(k)},hx_{2n(k)})|\leq d_{2n(k)};$$
$$|d(hx_{2m(k)+1},hx_{2n(k)+1})-d(hx_{2m(k)},hx_{2n(k)+1})|\leq d_{2m(k)} \text{ and}$$
$$|d(hx_{2m(k)+1},hx_{2n(k)+2})-d(hx_{2m(k)+1},hx_{2n(k)+1})|\leq d_{2n(k)+1}.$$

These imply $\lim d(hx_{2m(k)},hx_{2n(k)+1})$

$$= \lim d(hx_{2m(k)},hx_{2n(k)}) = \lim d(hx_{2m(k)},hx_{2n(k)}) = \lim d(hx_{2m(k)+1},hx_{2n(k)+1})$$
$$= \lim d(hx_{2m(k)+1},hx_{2n(k)+2}).$$

Since ϕ is continuous, $\lim \phi\ (d(hx_{2m(k)},hx_{2n(k)+1})) = \lim d(hx_{2m(k)+1},hx_{2n(k)+2})$

$$= \lim d(hx_{2m(k)+1},hx_{2n(k)+2}) = \phi(\varepsilon) .$$

From (A),

$$\phi(d(hx_{2m(k)+1},hx_{2n(k)+2})) \leq k \max \{\phi(d(hx_{2m(k)+1},fx_{2m(k)+1}),$$
$$\phi(d(fx_{2m(k)+1},gx_{2n(k)+2})),\ \phi(d(hx_{2m(k)+2},gx_{2n(k)+2})),$$
$$\phi\{(d(hx_{2m(k)+1},gx_{2n(k)+2})+d(hx_{2n(k)+2},fx_{2m(k)+1}))/2\}$$
$$= k \max \{\phi(d_{2m(k)}, \phi(d(hx_{2m(k)},hx_{2n(k)+1})), \phi(d_{2n(k)+1}),$$
$$\phi\{(d(hx_{2m(k)+1},hx_{2n(k)+1}))/2\}.$$

Taking limit as $k \to \infty$, we get

$$\phi(\varepsilon) \leq k\phi(\varepsilon) < \phi(\varepsilon), \text{ a contradiction.}$$

So $\{hx_n\}$ is a Cauchy sequence and because. X is complete, it is convergent. Call the limit u. then

$$\lim hx_{2n+1} = \lim fx_{2n+1} = u. \text{ Since } f \text{ and } h \text{ are compatible,}$$

$$\lim d(fhx_{2n+1}, hfx_{2n+1}) = 0 \qquad \ldots (3)$$

Also $\lim hx_{2n+2} = \lim gx_{2n+2} = u$. Since g and h are compatible,

$$\lim d(hgx_{2n+2}, ghx_{2n+2}) = 0. \qquad \ldots (4)$$

We now show that the continuity of f will imply

$$fu = gu = hu.$$

From the triangle inequality

$$d(fhx_{2n+1}, ghx_{2n+2}) \leq d(fhx_{2n+1}, hfx_{2n+1}) + d(hfx_{2n+1}, hgx_{2n+2}) + d(hgx_{2n+2}, ghx_{2n+2}).$$

Taking limit as $n \to \infty$ using (3) and (4) and the continuity of f and g we have

$$d(fu, gu) \leq \lim (hfx_{2n+1}, hgx_{2n+2}).$$

From (1)

$$\phi(d(hfx_{2n+1}, hgx_{2n+2}) \leq k \max \{\phi(d(hfx_{2n+2}, ffx_{2n+1})), \phi(d(hgx_{2n+2}, ggx_{2n+2}), \phi(d(ffx_{2n+1}, ggx_{2n+2})), \phi(d(hgx_{2n+1}, ffx_{2n+2}))/2\} \qquad \ldots (5)$$

From (3) and continuity of f,

$$\lim d(hfx_{2n+1}, ffx_{2n+1}) \leq \lim d(hfx_{2n+1}, fhx_{2n+1}) + \lim d(fhx_{2n+1}, ffx_{2n+1}) = 0.$$

From (4) and continuity of g,

$$\lim d(hgx_{2n+2}, ggx_{2n+2}) \leq \lim d(hgx_{2n+2}, ghx_{2n+2}) + \lim d(ghx_{2n+2}, ggx_{2n+2}) = 0.$$

From the continuity of f and g,

$$\lim (d(ffx_{2n+1}, ggx_{2n+2}) = d(fu, gu).$$

From (3) and (4) and continuity of f and g,

$$\begin{aligned} &\lim \{d(hfx_{2n+1}, ggx_{2n+2})\} + d(hgx_{2n+2}, ffx_{2n+1})\}/2 \\ &\leq \lim \{d(hfx_{2n+1}, fhx_{2n+1}) + d(fhx_{2n+1}, ggx_{2n+2})\} + d(hgx_{2n+2}, ghx_{2n+2}) \\ &\quad + d(hgx_{2n+2}, ghx_{2n+2}) + d(ghx_{2n+2}, ffx_{2n+1})/2 \\ &= d(fu, gu). \end{aligned}$$

Taking limit in (5) as $n \to \infty$ and using the continuity of ϕ,

$$\phi(d(fu, gu) \leq k\phi(d(fu, gu))$$

which implies that

$$fu = gu.$$

In the similar way it can be shown that $fu = hu$.

So $fu = hu = gu$.

Now

$$\phi(d(hfx_{2n+1}, hx_{2n+2})) \leq k \max \{\phi(d(hfx_{2n+1}, ffx_{2n+1})), \phi(d(hx_{2n+2}, gx_{2n+2})), \phi\{d(hfx_{2n+1}, gx_{2n+2}) + d(hx_{2n+2}, ffx_{2n+1}))/2\}.$$

Taking limit as $n \to \infty$,

$$\lim \phi\{d(hfx_{2n+1},hx_{2n+2})\} \le k\ \phi(d(fu,u))$$

i.e $\phi(d(hu,u)) \le k\ \phi(d(fu,u)) = k\ \phi(d(hu,u))$,

which implies $u=hu=fu=gu$. Let v be the another common fixed point of f, g, h. Then from (A)

$$\phi(d(u,v))=\phi(d(hu,hv)) \le k \max \{\phi(d(hu,fu));\phi(d(hv,gv)), \phi(d(fu,gv)), \phi\{d(hu,gv)+d(hv,fu)/2\}$$
$$= k \max \{\phi(d(u,v)), \phi(d(u,v)), \phi(d(u,v)),\phi(d(u,v))\}$$

Therefore, $\phi(d(u,v)) \le k \max \phi(d(u,v))$

which implies $u=v$. This completes the necessary part of the proof.

The condition is sufficient. Let $fz=gz=z$ for some $z\in X$ and define h by $hx=z\forall x\in X$. Then h is continuous from X to $g(X) \cap f(X)$. Moreover for $x\in X$, $hfx=z$, $fhx=fz=z$ and $hgx=z$, $ghx=gz=z$. So h commutes with f and g hence the mapping h and f and h and g are compatible.

further

$$\phi(d(hx,hy))=\phi(d(z,z))=\phi(0)=0.$$

Therefore

$$\phi(d(hx,hy)) \le k \max \{\phi(d(hx,fx)), \phi(d(hy,gy)), \phi(d(fx,gy)), \phi(d(hx,gy)+d(hy,fx))/2\}\ \forall\ x,y\in X.$$

So, h satisfies the condition (A). This completes the proof of the theorem.

Theorem 2. Let (X,d) be complete metric space and let f,g and h be self maps on X such that for some positive integers m,n,p,

$$\phi(d(h^px,h^py)) \le k \max \{\phi(d(h^px,f^mx)), \phi(d(h^py,g^ny)), \phi(d(f^mx,g^ny)/2), \phi\{d(h^px,g^ny)+d(h^py,f^my))\}$$

for all $x,y\in X$, $0 \le k<1$ and $\phi \in \Phi$.

Let $fh=hf$, $gh=hg$ and f, g be continuous. Then f, g, h have a unique common fixed point in X.

Proof. Let $H=h^p$, $F=f^m$ and $G=g^m$. Since $fh=hf$, $gh=hg$ it follows that $FH=HF$; $GH=HG$ and so F, H and G, H are compatible. Also F, G, H satisfy the inequality of Theorem 3, hence it follows that F,G and H have a unique common fixed point z, say in X.

It now remains to show that z is also unique common fixed point of f,g and h. Now

$$\phi(d(z,hz))=\phi(d(Hz,H(hz)))$$
$$\le k \max \{\phi(d(Hz,Fz)), \phi(d(H(hz),G(hz)),\phi(d(Fz,G(hz)), \phi\{d(Hz,G(hz)/2)+d(Hhz,Fz))\}$$
$$= k \max \{\phi(d(z,z)), \phi(d(hz,hz)),\phi(d(z,hz), \phi((z,hz)+d(hz,z))/2\},$$

i.e. $\phi(d(z,hz))=k\phi(d(z,hz))$

implies $\phi(d(z,hz))=0$ *implies* $hz=z$.

Next we show that $hz=fz=gz$. Now,

$$\phi(d(hz,fz))=\phi(d(Hhz,Hfz))$$
$$\leq k \max \{\phi(d(Fhz,Hhz)), \phi(d(Gfz,Hfz))\phi(d(Fhz,Gfz)),$$
$$\phi\{ d(Hz,G(hz)+d(Hhz,Fz))/2\}$$
$$=k \max \{\phi(d(z,z)), \phi(d(hz,hz))\phi(d(z,hz)), \phi\{(d(z,hz))/2+d(hz,z))\}$$

i.e. $\phi(d(z,hz))\leq\phi(d(z,hz))$ implies $\phi(d(z,hz))=0$ implies $hz=z$.

Next we show that $hz=fz=gz$. Now,

$$\phi(d(hz,fz))=\phi(d(Hhz,Hfz))$$
$$\leq k \max \{\phi(d(Fhz,Hhz)), \phi(d(Gfz,Hfz))\phi(d(Gfz,hfz)),$$
$$\phi\{(d(Fhz,Hfz)+d(Gfz,Hhz))/2\}$$
$$=k \max \{\phi(d(FZ,HZ)), \phi(d(fz,fz))\phi(d(hz,fz)), \phi\{(d(hz,fz)/2+d(fz,hz))\}$$
$$=k\ \phi\ (d(fz,hz))$$

which implies

$$\phi(d(gz,hz))=0 \text{ implies } d(fz,\ hz)=0$$

implies

$$fz=hz.$$

Similarly we can show that $gz=hz$ and $gz=hz=fz=z$.

The unicity part of the theorem follows easily.

Fixed point for a sequence of mappings.

Theorem 3. Let (X,d) be a complete metric space and let $\{T_i\}$ be a sequence of continuous mappings on X into itself. Then a mapping $h:X\rightarrow X$ has a unique fixed point in X if and only if

$$\phi(d(hx,hy)) \leq k \max \{\phi(d(hx,T_ix)),\phi(d(hy,T_iy)),\phi(d(T_ix,T_iy)),$$
$$\phi(d(hx,T_iy)/2+d(hy,T_ix))\}$$

for all $x,y\in X$, $0 \leq k<1$; $i, j\in N$ and h is compatible with each $T_i\in \{T_i\}$ and h have a unique common fixed point in X.

Proof. Result follows from the Theorem 1.

REFERENCES

[1] G. Jungck, Compatible mappings and common fixed points, *International J. Math. Math Sci.*, **9** (1986), 771.

[2] G. Jungck and B.E. Rhoades, Fixed point theorems for Compatible mappings, *International J. Math and Math Sci.*, **16** (1993), 417.

[3] M.S. Khan, M. Swaleh and S. Sessa, Fixed point theorems altering distance between the points, *Bull. Anstr. Math. Soc.* **30** (1984), 1.

[4] S. Park, A unified approach to fixed points of contractive maps. *J. Korean Math Soc.*, **16** (1980), 95.

[5] S. Park, Fixed points and periodic points of contractive pairs of maps, *Proc. College Nat. Sci, Seol Nat. Univ.* **5** (1980), 9.

[6] H.K. Pathak and R. Sharma, A note on fixed point theorems of Khan, Swaleh and Sessa. *The Mathematics Education,* **28** (1994), 151.

[7] B.E. Rhoades, Kalishankar Tiwary and G.N. Singh, A common fixed point Theorems for compatible mappings, *Indian J. Pure Appl. Math.* **26** (1995), 403.

[8] K.P.R. Sastri and G.V.R. Babu, Fixed point theorems in metric space by altering distances, *Bull. Cal. Math. Soc.* **90** (1998), 175.

[9] K.P.R. Sastri and G.V.R. Babu, Some fixed point theorems by altering distance between the points, *Indian J. Pure Appl. Math.* **30**, (1999), 6.

[10] K.P.R. Sastri, G.V.R. Babu and D. Narayana Rao, Fixed point Theorems in Complete metricspace by using a continuous control functions, *Bull. Cal. Math. Soc.*, **9** (1999), 493.

[11] S. Sessa, On a weak Commutativity Condition of mappings in fixed point consideration, *Publ. Inst. Math.*, **32**, (1982), 149.

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