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SOLUTION OF THE DETERMINISTIC EPIDEMIOLOGICAL MODEL IN TERMS OF THE HYPERGEOMETRIC FUNCTIONS $_0F_1$

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ABSTRACT

An Susceptible Infective model (*SI*) for a communicable disease is considered here and solved analytically. The bsic feature of the model is that it employes more realistic assumption regarding infective contact rate which is assumed to be variable and population dependent as distinguished from the classical models in which it was assumed to be constant. The model is solved in terms of the function $_0F_1$ which is a special case of the hypergeometric function $_2F_1$.
Key words . Confluent hypergeometric function, hypergeometric functions, Epidemiology, Susceptible Infective.

1. Introduction. Epidemiology as a scientific discipline has been fron *t*-runner in the field of studies generally termed as population dynamics. An epidemic usually described as occurrence of a disease in excess to normal expectancy. Since contagiousness is one of the main causes of spread of such epidemics, the term epidemiology has been applied to general study and devising measures of controlling all communicable disease. Preoccupation with prevention and cure of infectious diseases has been one of the basic features of researches in medical sciences, since the middle ages. This has resulted in invention of various vaccines and drugs and most of killer diseases such as Plague, Cholera, Smallpox etc. are more or less eradicated from almost entire globe. Still situation in certain under-developed nations is as grim as it was in the middle ages. Moreover the whole world is in the grip of fear of a new epidemic in the form of *AIDS* (Acquired Immune Deficiency Syndrome). Hence Mathematical modeling assumes greater significance in epidemiological studies. As in the case with most of the mathematical modeling such a study should begin with identifying all the significant parameters of the problem.

The spread of an infectious diesease in a host population depends on a number of disease related factors e.g. infectious agent mode of transmission, susceptibility and resistance of the host, incubation peirod, the infectious period etc. There are several extraneous considerations such as social, economic, geographic and demographic fators which also play significant role in this respect. A meaningful communicable disease model can be constructed only if these factor are incorporated in it. Depending on population size, the epidemiological models can be categorised in to two categories: Deterministic Models and Stochastic models.

The deterministic models involving differential equations were described and several threshold theorems were obtained by Kermack and Mckendriek [3,4]. A landmark was achieved in the field of mathematical theory of epidemiology with publication of the monumental work in the form of a book N.T.J. Bailey [1] dealing with both deterministic and stochastic models. A survey of results in epidemiology obtained till 1967 has been detailed by K. Dietz [2]. Recent decades have witnessed enormous growth in mathematical modeling for epidemics.

2. Mathematical Formulation of Model. The general model for a communicable disease (fatal or otherwise) in which an infected person does not recover is known as *SI* model. In this model at a time 't', total population '$N(t)$' is divided into two disjoint classes namely the infective class '$I(t)$' consisting of totality of infected individuals who can transmit disease and the suceptible class '$S(t)$' of individuals who can incur disease by contact the infected individuals.

The population size '$N(t)$' is so large that it can be considered as a continuous variable of time. The population in changing on account of immigration, births, emigration and deaths (due to disease in question or other causes). Let 'β' be the rate at which population is receiving new individuals due to immigration and birth and 'μ' be the rate at which individuals are being removed on account of emigration and death. Hence all the new entrants are assumed to be susceptible.

The population is assumed to be uniform or homogeneous. The daily contact rate '$C(N)$' is assumed to vary linearly with population and is taken as 'δN'. The rate at which total number of susceptible are infected by class I of infective can be taken as

$$(I\delta N)S/N = \delta IS$$

The initial value problem for the *SI* model can be put as follows:

$$ds/dt = \beta N - \mu S - \delta IS,$$
$$dS/dt = \delta IS - \mu I,$$
$$N=I+S,$$
$$I(0)=I_0>0;\ S(0)>0;\ \text{and}\ N(0)>N_0>0. \tag{2.1}$$

The rate of total number of susceptible transferred to infective class is 'δIS'. Total population removed per unit time on account of emigration and death is 'μN'. It is divided into two classes namely suscepible class and infective class in proportion to number of individuals in those classes. All newborns and immigrants are susceptible.

The fact account for the first term 'βN' in expression for dS/dt.

3. Solution of the Model Equation. The solution of the model is given by system of equations (2.1) can be effected as follows. On adding first two equations of the system (2.1) and employing the relation $N=I+S$, we obtain

$$N' = (\beta-\mu)N. \tag{3.1}$$

Here prime denotes differentiation with respect to t.
This equation has solution

$$N = N_0 e^{(\beta-\mu)t}. \tag{3.2}$$

Employing this value of N in the first equation in (2.1), we get

$$S' = \beta N - \mu S - \delta S(N-S)$$

or

$$S' = \delta S^2 - \left(\mu + \delta N_0 e^{(\beta-\mu)t}\right) + \beta N_0 e^{(\beta-\mu)t}. \tag{3.3}$$

Let us choose 'u' such that

$$\delta S u + u' = 0.$$

Making this substitution in equation (3.3) and simplifying we obtain

$$u'' + \left(\mu + \delta N_0 e^{(\beta-\mu)t}\right)\mu' + \beta N_0 e^{(\beta-\mu)t}\mu = 0 \tag{3.4}$$

Using relations (3.1) and (3.2) equation (3.4) takes the form

$$(\beta-\mu)^2 N^2 \frac{d^2u}{dN^2} + (\mu+\delta N)N(\beta-\mu)\frac{du}{dN} + \delta\beta N u = 0. \tag{3.5}$$

Putting $M=\{\delta/(\beta-\mu)\}\ N$, equation (3.5) becomes

$$M\frac{d^2u}{dM^2} + \left\{\frac{\mu}{(\beta-\mu)} - M\right\}\frac{du}{dM} - \delta u = 0. \tag{3.6}$$

Differential equation (3.6) is the well known Confluent Hypergeometric equation [6], whose general solution is:

$$u = C_1\ {}_iF_1\left(\delta; \frac{\mu}{(\beta-\mu)}; M\right) + u + C_2 M^{1-\mu/(\beta-\mu)}\ {}_1F_1\left(1+\delta-\frac{\mu}{(\beta-\mu)}; 2-\frac{2}{(\beta-\mu)}; M\right) \tag{3.7}$$

where C_1 and C_2 are arbitrary constants and $\mu/(\beta-\mu)$ is non-integral.

In (3.6) put $\mu/(\beta-\mu) = 2\delta$, $M=2Z$ and $u=e^z w$, the result is [6],

$$zw'' + 2aw' - zw = 0 \tag{3.8}$$

of which one solution must be $w = e^{-z}\,{}_1F_1(\delta; 2\delta; 2z)$.

In equation (3.8) change the independent variable to $\sigma = z^2/4$ and thus arrive at the equation

$$\sigma^2\frac{d^2w}{d\sigma^2} + (\delta + 1/2)\sigma^2\frac{dw}{d\sigma} - \sigma w = 0 \tag{3.9}$$

or

$$[\theta(\theta+\delta+1/2-1)-\sigma]w=0, \theta=\sigma d/d\sigma \tag{3.10}$$

Equation (3.10) is a differential equation for the ${}_0F_1$ function with denominator parameter $(\delta+1/2)$ and argument $\sigma=z^2/4$. Hence if $\delta+1/2$ is non integral (That is if 2δ is not an odd integer, the general solution of equation (3.10) is

$$w=A\,{}_0F_1(-;\delta+1/2;z^2/4)+\beta(z^2)^{1/2-\delta}{}_0F_1(-;3/2-\delta;z^2/4) \quad ...(3.11)$$

Hence the solution of equation (3.6) can be obtained in the form

$$u=e^{M/2}\left\{A\,{}_0F_1\left(-,\frac{\mu}{2(\beta-\mu)}+\frac{1}{2};\frac{M^2}{16}\right)+B(M^2/4)^{\left(\frac{1}{2}-\frac{\mu}{2(\beta-\mu)}\right)}{}_0F_1\left(-,\frac{3}{2}-\frac{\mu}{2(\beta-\mu)};\frac{M^2}{16}\right)\right\} \tag{3.12}$$

where A and B are arbitrary constants.

4. Conclusion. The exact solution obtained the hyper form of hypergeometric functions can be employed to predict the number of infective in a population in the grip of a contagious disease in which infected population does not recover, if the numerical values of the parameters are known. Hence if the infective class is to be reduced to negligible proportions after a given time span, corresponding value of contact rate can be computed and if need be strong measures such as quarantine be imposed to bring the contact rate to the requisite level.

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