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VARIATION OF δ-COINCIDENCE SETS

By
Anjali Srivastava
School of Studies in Mathematics
Vikram University, Ujjain 456010, Madhya Pradesh, India

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ABSTRACT

David Gauld [2] proved that in many familiar cases the upper semi-finite topology on the set of closed subsets of a space is the largest topology making the coincidence function continuous, when the collection of functions is given the graph topology. For a $\delta>0$ and for continuous maps $f, g : X \to Y$, where X,Y are metric spaces, by taking the δ-coincidence set to consist of points x for which $f(x)$ and $g(x)$ lie at a distance less than or equal to δ, we obtain various results of Gauld in the new setting.

Key Words and Phrases . Graph topology, Upper semifinite topology, Coincidence set, δ-active homotopy, Property W (δ-strong).

1. Introduction. For metric spaces X,Y the collection $\mathscr{F}(X,Y)$ of continuous maps from a metric space X to a metric space Y is equipped with the graph toplogy: The family $\{<W> \mid W$ is an open set of $X\times Y\}$, where

$$<W> = \{f \in \mathscr{F}(X,Y) \mid \text{graph } \Gamma f \text{ of } f \text{ is contained in } W\}$$

forms a basis for this toplogy [4].

Let $h : X\to Y$ be a continuous map from a metric space X to a metric space Y. Then for a continuous map $f: X\to Y$, the conicidence set $\psi_h(f)$ consists of those points of X at which f and h agree. Because $\psi_h(f) \in \zeta(X)$, a map $\psi_h:\mathscr{F}(X,Y)\to\zeta X$ can be defined which sends $f \in \mathscr{F}(X,Y)$ to the coincidence set of f and h. In [2], Gauld studied the variation $\psi_h(f)$ with f. Also the continuity of the coincidence function $\chi : \mathscr{F}(X,Y)\times\mathscr{F}(X,Y)\to\zeta X$ sending (f,h) to their coincidence set is considered.

For a real number $\delta>0$, we let the δ-coincidence set $D(f,h)$ be the set

$$D(f,h)=\{x\in X \mid d(f(x),g(x))\leq\delta\}$$

and for an element $h\in \mathscr{F}(X,Y)$, define a map

$$D_h:\in \mathscr{F}(X,Y)\to\zeta X$$

which sends an element $f\in \mathscr{F}(X,Y)$ to the δ-coincidence set of f and h. The δ-coincidence function D is defined in a similar way. In section 2 of this paper we prove continuity of the map D_h and δ-coincidence function D. Finally we transfer some results of Gauld in this new setting in section 3 of this paper.

For terms and definition not explained here, we refer to [1,2].

2. Variation of coincidence Sets. Throghout the section X,Y are metric spaces. The section begins with the following definitions :

Definition 1. Let $f, h : X \to Y$ be two continuous maps where X,Y are metric spaces. The set $D_{f,h}=\{x \in X \mid d(f(x),g(x)) \leq \delta\}$
is called the δ-coincidence set of f and h.

Note that $D_{f,h}$ is a closed set of X.

The δ-coincidence set and the coincidence set of two maps, $f, h : X \to Y$ are different. For $\delta=0$, two sets are the same.

Definition 2. The map D: $\mathcal{C}(X,Y) \times \mathcal{C}(X,Y) \to \zeta X$ defined by

$$D(f,g) = \{x \in X \mid d(f(x),g(x)) \leq \delta\}$$

is called the δ-coincidence function.

If h is fixed then the restriction of the map D to $\mathcal{C}(X,Y) \times \{h\} = \mathcal{C}(X,Y)$ is denoted by D_h . For $\delta=0$, D and D_h are easily seen to be the maps χ and ψ_h as described in [2].

Theorem 1. Let $h: X \to Y$ be a continuous map from a metric space X to a metric space Y. If $\mathcal{C}(X,Y)$ is toplogized with the graph topology and ζX by the upper semi-finite topology then

$$D_h : \mathcal{C}(X,Y) \to \zeta X \text{ is continuous.}$$

Proof. Let V be an open set of X. Take $f \in D_h^{-1}[V]$. Then

$$D(f,h)=\{x \in X \mid d(f(x),\ h(x)) \leq \delta\} \subseteq V.$$

Take $U = V \times Y \cup \{(x,y) \in X \times Y \mid d(y,h(x)) > \delta\}$.

Then $f \in \langle U \rangle \subset D_h^{-1}[V]$.

3. Upper Semi-Finite Topology and Continuity of $D(D_h)$

Definition 3. Let X, Y be metric spaces. A homotopy $H: X \times I \to Y$ is called δ-active if $d(H(x,t)), H(x,0)) > \delta$ for all $x \in X$ and $t \in (0,1]$ where I denotes the unit closed interval of the real line R with the subspace toplogy.

Let $H: X \times I \to Y$ be a δ-active homotopy. We recall the map $\tau_H: \mathcal{C}(X,I) \to \mathcal{C}(X,Y)$ defined by $\tau_H(\alpha)=H(x, \alpha(x))$; where $\alpha \in \mathcal{C}(X,I)$ and $x \in X$ [2]. Denote by H_0, the restriction of H to the base $X \times \{0\}$. Identify $X \times \{0\}$ with X and note that $D_{H_0} o \tau_H = \sigma$, where $\sigma : \mathcal{C}(X,I) \to \zeta X$ maps α to its zero set. Since for a perfectly normal space X the upper semi-finite toplogy is the largest toplogy on ζX making σ continuous [See 2, Prop. 1.4], we conclude the following:

Let X be a perfectly normal space and $H: X \times I \to Y$ be a δ–active homotopy. Then the upper semi-finite topology is the largest topology on ζX making D_{H_0} continuous.

Call a path α in X, to be δ-non overlapping if $d\ (\alpha(t),\alpha(0)) > \delta$ for any $t \in (0,1]$.

A space X is said to have the property W (δ-strong) if there exists a δ-active deformation of X i.e. there is a δ-active homotopy $H:X\times I \to X$ satisfying that $H_0=I_x$.

We obtain the analogues of corollaries 2.6 and 2.7 of [2] as follows:

Let X be a perfectly normal space. Then the upper semi-finie topology on ζX is the largest topology making D_h continuous provided either of the following holds:

(a) Y has the property W (δ-strong).

(b) $h:X\to Y$ is a constant map and Y has a δ-non-overlapping path beginning from the image point of h.

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