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ON A TACHIBANA SPACE WITH PARALLEL BOCHNER CURVATURE TENSOR

By

A.K. Singh and A.A. Samyal

Department of Mathematics

H.N.B. Garhwal University, Campus Badshahi Thaul

Tehri Garhwal 249199, Uttranchal Pradesh, India

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ABSTRACT

In the present paper, we consider the Tachibana space ($n>2$) with parallel Bochner curvature tensor. Several theorems will be investigated.

1. Introduction. An n $(=2m)$ dimensional Kaehlerian space K_n^c is a Riemannian space admitting a structure tensor F_i^h satisfying :

(1.1) $F_i^h F_h^j = -\delta_i^j$,

(1.2) $F_{ij} = -F_{ji}, \left(F_{ij} = F_j^h g_{hi}\right)$

and

(1.3) $\nabla_j F_i^h = 0$,

where ∇_j denotes the operator of covariant differentiation with respect to Christoffel symbol $\left\{ {h \atop ji} \right\}$ formed with g_{ji}, where g_{ji} is Riemannian metric satisfying

(1.4) $F_j^t F_i^s g_{ts} = g_{ji}$.

From (1.2) and (1.3), we obtain

(1.5) $\nabla_k F_{ji} = 0$.

Now, from (1.3) and the Ricci identity

$\nabla_k \nabla_j F_i^h - \nabla_j \nabla_k F_i^h = R_{kjs}^h F_i^s - R_{kji}^s F_s^h$,

we obtain

(1.6) $R_{kji}^s F_s^h = R_{kjs}^h F_i^s$.

The Riemannian curvature tensor, R_{ijk}^h, is given by

(1.7) $R_{ijk}^h = \partial_i \left\{ {h \atop jk} \right\} - \partial_j \left\{ {h \atop ik} \right\} + \left\{ {h \atop il} \right\}\left\{ {l \atop jk} \right\} - \left\{ {h \atop jl} \right\}\left\{ {l \atop ik} \right\}$,

where $\partial_i = \dfrac{\partial}{\partial x^i}$ and $R_{ik} = R_{ijk}^i$ is the Ricci tensor, while $R = R_{jk} g^{jk}$ is a scalar curvature.

S. Tachibana [3] has defined the Bochner curvature tensor (with respect to real local co-ordinates) as

(1.8) $K_{ijk}^h = R_{ijk}^h + \dfrac{1}{n+4}\Big(R_{ik}\delta_j^h - R_{jk}\delta_i^h + g_{ik}R_j^h - g_{jk}R_i^h + S_{ik}F_j^h - S_{jk}F_i^h + F_{ik}S_j^h - F_{jk}S_i^h$

$$+ 2S_{ij}F_k^h + 2F_{ij}S_k^h\Big) - \frac{R}{(n+2)(n+4)}\left(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h\right),$$

where $S_{jk} = F_j^h R_{hk.}$

An almost Tachibana space is first of all an almost Hermite space. That is, a $2n$-dimensional space with an almost complex structure F satisfying (1.1), (1.2) and finally has the property that the skew symmetric tensor F_{ih} is killing tensor:

(1.9) $\nabla_j F_{hi} + \nabla_i F_{jh} = 0$,

from which

(1.10) $\nabla_j F_i^h + \nabla_i F_j^h = 0$

and

(1.11) $F_i = -\nabla_j F_i^j$.

Now, the Nijenhuis tensor N_{ji}^h is written in the form:

(1.12) $N_{ji}^h = -4\left(\nabla_j F_i^t\right)F_t^h + 2G_{ji}^t F_t^h + F_j^t G_{ti}^h - F_i^t G_{tj}^h$,

and consequently, we have [5]

Theorem 1. In an almost Tachibana space, we obtain

(1.13) $N_{ji}^h = -4\left(\nabla_j F_i^t\right)F_t^h$,

and consequently $\nabla_j F_i^h$ is pure in j and i.

When the Nijenhuis tensor vanishes, the almost Tachibana space is called a Tachibana space. In this case, we have from (1.13) the following [5]

$$\nabla_j F_i^h = 0.$$

Hence we have

Theorem 2. A Tachibana space is a Kaehlerian space.

In this paper, we shall consider the Tachibana space ($n>2$) with parallel Bochner curvature tensor, i.e.,

(1.14) $\nabla_j K_{ijk}^h = 0.$

We shall call the Tachibana space satisfying (1.14) a TPB_n-space and the space satisfying $K_{ijk}^h = 0$ will be called an TVB_n-space

In order to avoid complicated calculations, we put

(1.15) $\pi_{ij} = \dfrac{1}{n+4}\left(R_{ij} - \dfrac{R}{2(n+2)}g_{ij}\right)$,

(1.16) $M_{ij} = F_i^h \pi_{hj} = \dfrac{1}{n+4}\left(S_{ij} - \dfrac{R}{2(n+2)}F_{ij}\right)$,

and

(1.17) $D_{ijk}^h = \pi_{ik}\delta_j^h - \pi_{jk}\delta_i^h + g_{ik}\pi_j^h - g_{jk}\pi_i^h + M_{ik}F_j^h - M_{jk}F_i^h + F_{ik}M_j^h - F_{jk}M_i^h + 2M_{ij}F_k^h + 2F_{ij}M_k^h.$

Thus, (1.8) can be written as

(1.18) $K^h_{ijk} = R^h_{ijk} + D^h_{ijk}$.

Moreover, π_{ij}, M_{ij} and D^h_{ijk} satisfy the following conditions:

(1.19) $\pi = g^{ab}\pi_{ab} = \frac{R}{2(n+2)}$,

(1.20) $M_{ij} = -M_{ji}$

and

(1.21) $D_{ijkh} = D_{khij}$, $(D_{khij} = D^l_{ijk} g_{lh})$.

A Tachibana space is said to be a recurrent space if its curvature tensor R^h_{ijk} satisfies

(1.22) $\nabla_l R^h_{ijk} - \lambda_l R^h_{ijk} = 0$,

for a non-zero recurrence vector λ_l. From (1.3), (1.8), (1.14) and (1.22), we have the following:

Theorem 3. Every recurrent TPB_n-space is a TVB_n-space.

Let a TPB_n-space be an Einstein space, then the Ricci tensor satisfies

(1.23) $R_{ij} = \frac{R}{n} g_{ij}, \nabla_l R_{ij} = 0$, from which, we get

(1.24) $S_{ij} = \frac{R}{n} F_{ij}, \nabla_l R_{ij} = 0$ and $\nabla_l S_{ij} = 0$.

Substituting (1.23) and (1.24) in (1.14), we get

$\nabla_l R^h_{ijk} = 0$.

Thus, we have the following rusult:

Theorem 4. Every Tachibana Einstein TPB_n - space is symmetric in the sense of Cartan.

2. Ricci Recurrent. TPB_n- space : A Tachibana space is said to be a Ricci recurrent space, if the Ricci tensor R_{ij} satisfies

$\nabla_l R_{ij} - \lambda_l R_{ij} = 0$,

for a non-zero recurrence vector λ_l.

We, now, have the following result.

Theorem 5. A necessary and sufficient condition for R_{ij} to be a recurrent tensor is that π_{ij} be a recurrent tensor.

Proof. Suppose that R_{ij} is a recurrent tensor, i.e.,

(2.1) $\nabla_l R_{ij} - \lambda_l R_{ij} = 0$.

Multiplying (2.1) with respect to g^{ij}, we have

(2.2) $\nabla_l R - \lambda_l R = 0$,

which by virtue of (1.15) and (2.1) gives

(2.3) $\nabla_l \pi_{ij} - \lambda_l \pi_{ij} = 0.$

Hence π_{ij} is a recurrent tensor.

Conversely, if we have (2.3), then by multiplication with g^{ij} it gives

(2.4) $\nabla_l \pi - \lambda_l \pi = 0.$

Consequently, from (1.19) and (1.15), we have

$$\nabla_l R_{ij} - \lambda_l R_{ij} = 0,$$

i.e., the Ricci tensor is recurrent one.
This completes the proof of theorem 5.

Theorem 6. In a TPB_n-space, if $\nabla_l R_{ij} - \lambda_l R_{ij} = 0,$ for a non-zero recurrence vector λ_l and a non-zero tensor R_{ij}, then the tensor D^h_{ijk} is not equal to zero.

Proof. Suppose on the contrary that $D^h_{ijk} = 0$. Since the space is a TPB_n-space, we have from (1.18) that

$$\nabla_l R^h_{ijk} = 0.$$

Contracting the above equation with respect to h and i, we get $\nabla_l R_{ik} = 0$, from which we have $\lambda_l R_{ik} = 0$, which is impossible, since $\lambda_l \neq 0$ and $R_{jk} \neq 0$. Hence, the tensor $D^h_{ijk} \neq 0$.

The following two Lemmas were proved by Walker [4] and Ruse, Walker and Willmore [2]:

Lemma 1. The curvature tensor R_{hijk} satisfies the identity

(2.5) $\nabla_l \nabla_m R_{hijk} - \nabla_m \nabla_l R_{hijk} + \nabla_h \nabla_i R_{jklm} - \nabla_l \nabla_h R_{jklm} + \nabla_j \nabla_k R_{lmhi} - \nabla_k \nabla_j R_{lmhi} = 0.$

Lemma 2. If $a_{\alpha\beta}$, b_α are the quantities satisfying

(2.6) $a_{\alpha\beta} = a_{\beta\alpha}, a_{\alpha\beta} b_\gamma + a_{\beta\gamma} b_\alpha + a_{\gamma\alpha} b_\beta = 0,$

for $\alpha, \beta, \gamma = 1, 2, \ldots, N$, then either all the $a_{\alpha\beta}$ are zero or all the b_α are zero.

Using the above Lemmas, we can prove the following result:

Theorem 7. In a TPB_n-space, if $\nabla_l R_{ij} - \lambda_l R_{ij} = 0$ for a non-zero vector λ_l and a non-zero tensor R_{ij}, then the recurrence vector λ_l is gradient.

Proof. If we suppose that $\nabla_l R_{ij} - \lambda_l R_{ij} = 0$, then by Theorem 5, we get $\nabla_l \pi_{ij} - \lambda_l \pi_{ij} = 0$ and consequently from (1.16) and (1.17), we have

$\nabla_l M_{ij} - \lambda_l M_{ij} = 0$ and

(2.7) $\nabla_l D^h_{ijk} - \lambda_l D^h_{ijk} = 0.$

From (1.14), (1.18) and (2.7), we get

(2.8) $\nabla_l R^h_{ijk} - \lambda_l D^h_{ijk} = 0$.

Differentiating (2.8) covariantly and using (2.7), we obtain

(2.9) $\nabla_m \nabla_l R^h_{ijk} + \nabla_m \lambda_l D^h_{ijk} + \lambda_l \lambda_m D^h_{ijk} = 0.$

Hence

$$\nabla_l \nabla_m R^h_{ijkh} + \nabla_m \nabla_l D^h_{ijkh} + \lambda_{lm} D_{ijkh} = 0,$$

where $\lambda_{lm} = \nabla_l \lambda_m - \nabla_m \lambda_l$.

The identity (2.5) gives

(2.10) $\lambda_{lm} D_{ijkh} + \lambda_{ij} D_{khlm} + \lambda_{kh} R_{lmij} = 0$

The equation (2.10) is of the form (2.6) because $D_{ijkh}=D_{khij}$. By Theorem 6 the tensor D_{ijkh} is not zero. Hence, by the Lemma 2, $\lambda_{lm} = 0$, i.e., $\nabla_l \lambda_m = \nabla_m \lambda_l$, which is the condition for λ_l to be gradient.

The next theorem is easy to prove.

Theorem 8. A TPB_n-space satisfying $R_{ij}=0$ is symmetric in the sense of Cartan.

The following Lemma has been proved by Matsumoto [1]:

Lemma 3. In a compact Tachibana space with parallel Bochner curvature tensor $g^{ij} \nabla_j R$ is a contravariant analytic vector.

Now, suppose that the TPB_n-space is Ricci recurrent space, then we have $\nabla_l R_{ij} - \lambda_l R_{ij} = 0$ and $\nabla_l R - \lambda_l R = 0$. Hence, from the above Lemma, $\lambda^i R$ is a contravariant analytic vector and we get

Theorem 9. If a compact TPB_n-space is Ricci recurrent, then the vector $\lambda^i R$ is a contravarient analytic vector.

3. Parallel Vector Fields in TPB_n–Space. Let us assume that a TPB_n-space admits a parallel vector field v^i, that is,

(3.1) $\nabla_l v^i = 0,$

whence we have

(3.2) $\nabla_l \bar{v}^i = 0,$ where $\bar{v}^i = F^i_j v^j$.

Making use of the Ricci and Bianchi is identities, we get

(3.3) (a) $v^a R_{hajk} = 0, v^a R_{aj} = 0,$

(b) $\bar{v}^a R_{hajk} = 0, \bar{v}^a R_{aj} = 0,$

(3.4) (a) $v^a \nabla_l R_{hajk} = 0, v^a \nabla_l R_{aj} = 0,$

(b) $\bar{v}^a \nabla_l R_{hajk} = 0, \bar{v}^a \nabla_l R_{aj} = 0,$

(3.5) (a) $v^a \nabla_a R_{hijk} = 0, v^a \nabla_a R_{ij} = 0, v^a \nabla_a R = 0,$

(b) $\bar{v}^a \nabla_a R_{hijk} = 0, \bar{v}^a \nabla_a R_{ij} = 0, \bar{v}^a \nabla_a R = 0.$

We know that (Tachibana [3])

(3.6) $\nabla_a K^a_{ijk} = \frac{n}{n+4} K_{ijk},$

where

(3.7) $K_{ijk} = \nabla_l R_{jk} - \nabla_j R_{ik} + \frac{1}{2(n+2)}\left(g_{ik}\delta^l_j - g_{jk}\delta^l_i + F_{i,k}F^l_j - F_{jk}F^l_i + 2F_{ij}F^l_k\right)\nabla_l R.$

Therefore, for a TPB_n-space, we have

(3.8) $\nabla_i R_{jk} - \nabla_j R_{ik} + \frac{1}{2(n+2)}\left(g_{ik}\delta^l_j - g_{jk}\delta^l_i + F_{ik}F^l_j - F_{jk}F^l_i + 2F_{ij}F^l_k\right)\nabla_l R = 0$.

As the differential form $S = \frac{1}{2} S_{ij} dx^i dx^j$ is closed (Yano [5], page 72), it follows that

(3.9) $F^l_k \nabla_i S_{ij} = \nabla_j R_{ik} - \nabla_i R_{jk}.$

Multiplying (3.8) by $F^k_m F^i_n$, we get

(3.10) $\nabla_m R_{nj} = F^k_m F^i_n \left(\nabla_j R_{ik} - \nabla_l R_{jk}\right)$

Multiplying (3.8) by $F^k_m F^i_n$ and taking account of (3.10), we get

(3.11) $\nabla_m R_{nj} = \left(g_{mn}\nabla_j R + g_{mj}\nabla_n R - F_{mn}F^l_j \nabla_l R - F_{mj}F^l_n \nabla_l R + 2g_{nj}\nabla_m R\right)/2(n+2)$

Multiplying (3.11) by $v^n v^j$ and using (3.3), (3.4) and (3.5), we get

$V^2 \Delta_m R = 0,$ (where v is the magnitude of v^i) from which it follows that R is constant.

We know that (Matsumoto [1])

Lemma 4. If a Kaehlerian (Tachibana) space with parallel Bochner curvature tensor has constant scalar curvature, then it is a symmetric space.

Using the above Lemma, we obtain

Theorem 10. If a TPB_n-space admits a parallel vector field v^i, then at least one of the following two cases occurs :

(i) the space is symmetric,

(ii) v^i is a null vector.

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