

**EFFECTS OF MASS TRANSFER AND ROTATION ON FLOW PAST A  
PLATE WITH THE DEPENDENT SUCTION IN NON-HOMOGENEOUS  
POROUS MEDIUM**

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**ABSTRACT**

In the present paper effects of mass transfer and rotation on the unsteady free convection flow of an incompressible, homogeneous, electrically conducting, viscous fluid past an infinite porous vertical plate bounded by time dependent porous medium is studied. It is considered that the plate is subjected to time dependent suction velocity and the flow is under the influence of transversely applied uniform magnetic field. The effects of important parameters on primary and secondary velocity and corresponding skin-friction are discussed with the help of graph and tables.

**1. Introduction.** In nature and in industries, many transport processes exists, where the transfer of heat and mass takes place as a result of combined bouyancy effects of thermal diffusion and diffusion of chemical species. The phenomena of heat and mass transfer also takes place in chemical processing industry such as food processing and polymer production. In addition heat and mass transfer effects in flow of fluids have application in engineering and technology.

A comprehensive study on the theory of rotating fluids and theory of heat transfer has been respectively, presented by Greenspan [5] and Morgan [8]. Several authors including Morgan [9], Kapur and Malik [7], Rao and Gupta [12], Thornely [18] Nanda and Mohenti [11], Gupta and Goyal [6], Debnath [2], Acheson [1], Debnath and Mukherjee [3], Reismann [13], Seth and Maiti [14], Devi et al [4], Pillai [10], Singh and Kumar [15] and Singh et al. [16] have studied problems on flow in rotating system under different physical situations and boundary conditions.

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More recently, Singh et al. [17] have studied unsteady oscillatory flow of an incompressible, electrically conducting, viscous liquid, through a porous medium past an infinite vertical porous plate with constant suction and transverse uniform magnetic field. In this study the suction velocity and the permeability of the medium is assumed constant. In the proposed study the investigation of Singh et al. [17] is extended to observe the heat and mass transfer effects on the unsteady flow of an incompressible, homogeneous, electrically conducting, viscous fluid through a porous medium with time dependent permeability, time dependent suction velocity and thermal diffusion under the influence of uniform magnetic field.

**2. Formulation of the Problem.** Consider unsteady oscillatory free convection flow of an electrically conducting, homogeneous, incompressible, viscous liquid through a porous medium past an infinite porous vertical plate in the presence of uniform magnetic field. Under cartesian coordinate system  $(x,y,z)$ , the plate is considered in  $z$ -plane and the liquid and the plate both are in a state of rigid body rotation with uniform angular velocity  $\Omega$  about  $z$ -axis. The  $x$ -axis and  $y$ -axis are assumed in the plane of the vertical porous plate so that  $z$ -axis is normal to these axes at the point of intersection and the components of velocity in these directions are  $u, v, w$  respectively. In addition the suction velocity at the porous plate is considered  $w = -w_0(1 + \epsilon e^{int})$  and the porous medium  $k(t) = k_0(1 + \epsilon e^{int})$  where  $w_0, n$  are real and positive. Initially, when  $t \leq 0$  the plate and the fluid are assumed to be at the same temperature  $T_\infty$  and the foreign mass is assumed to be at low level and uniformly distributed in the flow region such that it is every where  $C_\infty$ . When,  $t > 0$ , the temperature of the plate is instantaneously raised to  $T_w$  and the species concentration is also raised to  $C_w$  and thereafter maintained constant. In addition, the analysis is based on the following assumptions.

- (i) The uniform magnetic field  $\vec{B} [= \mu_e H]$  so that  $\vec{H} = (0, 0, H_0)$  acts in the  $z$ -direction i.e. normal to the flow of fluid.
- (ii) The magnetic Reynolds number is very small so that induced magnetic field is negligible in comparison to applied magnetic field.
- (iii) No external electric field is applied in the flow region so that the effect of polarization of ionized fluid is negligible.
- (iv) The Hall effect and viscous dissipation effect have been ignored while electromagnetic body force (Lorentz force) is considered.
- (v) An absorption type heat source  $Q = Q_0(T - T_\infty)$  and usual Boussinesq's approximation is taken into account.

Under the above-stated assumptions, the governing equations of momentum, energy and diffusion are :

$$\frac{\partial u}{\partial t} - (1 + \epsilon e^{int}) \frac{\partial u}{\partial z} - 2Ev = G_r T + G_m C + \frac{\partial^2 u}{\partial z^2} \left[ M^2 + \frac{1}{k_0(1 + \epsilon e^{int})} \right] u, \quad \dots(2.1)$$

$$\frac{\partial v}{\partial t} - (1 + \epsilon e^{int}) \frac{\partial u}{\partial z} - 2Eu = \frac{\partial^2 v}{\partial z^2} - \left[ M^2 + \frac{1}{k_0(1 + \epsilon e^{int})} \right] v, \quad \dots(2.2)$$

$$P_r \frac{\partial T}{\partial t} - P_r (1 + \epsilon e^{int}) \frac{\partial T}{\partial z} = \frac{\partial^2 T}{\partial z^2} - \alpha_0 T, \quad \dots(2.3)$$

$$S_c \frac{\partial C}{\partial t} - S_c (1 + \epsilon e^{int}) \frac{\partial C}{\partial z} = \frac{\partial^2 C}{\partial z^2} + S_c S_0 \frac{\partial^2 T}{\partial z^2}. \quad \dots(2.4)$$

The boundary conditions relevant to the problem are :

$$\begin{aligned} W = 1 + \epsilon e^{int}, \quad T = 1 + \epsilon e^{int}, \quad C = 1 + \epsilon e^{int}, \quad \text{at} \quad z = 0 \quad \dots(2.5) \\ W \rightarrow 0, \quad T \rightarrow 0, \quad C \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty, \end{aligned}$$

The non-dimensional quantities introduced to obtain the above equations are :

$$u^* = u/U_0, \quad v^* = v/U_0, \quad z^* = w_0 z/\nu, \quad t^* = w_0^2 t/\nu, \quad K^* = w_0^2 K/\nu^2, \quad k_0^* = w_0^2 k_0/\nu^2,$$

$$W^* = (u/U_0 + iv/U_0), \quad n^* = \nu n/w_0^2, \quad T^* = (T - T_\infty)/(T_w - T_\infty),$$

$$P_r = \mu C_p / K \text{ (Prandtl number)} \quad S_c = \nu / D \text{ (Schmidt number)},$$

$$G_r = \nu g \beta^* (T_w - T_\infty) / w_0^2 U_0 \text{ (Grashof number)}, \quad S_0 = \frac{D_r (T_w - T_\infty)}{\nu (C_w - C_\infty)} \text{ (Soret number)},$$

$$G_m = \frac{\nu g \beta^* (C_w - C_\infty)}{w_0^2 U_0} \text{ (modified Grashof number)} \quad E = \nu \Omega / w_0^2 \text{ (Rotation parameter)},$$

$$M = \mu_e H_0 \sqrt{\sigma \nu / \rho} / w_0 \text{ (Hartmann number)} \quad \text{and} \quad \alpha_0 = \nu^2 Q_0 / K w_0^2 \text{ (Heat source parameter)}$$

where  $\nu$  is the kinematic coefficient of viscosity,  $\beta^*$  is the volumetric coefficient of thermal expansion,  $\beta$  is the volumetric coefficient of thermal expansion with concentration,  $\sigma$  is the electrical conductivity of the liquid,  $\rho$  is the density of the liquid,  $\mu_e$  is the magnetic permeability,  $H_0$  is the uniform magnetic field,  $k_0$  is the constant permeability of the porous medium,  $K$  is the thermal conductivity,  $C_p$  is the specific heat at constant pressure,  $T$  is the temperature,  $T_w$  is the plate temperature,  $C_w$  is the concentration at the plate,  $C_\infty$  is the concentration of species far away from the plate,  $T_\infty$  is the temperature far away from the plate,  $D$  is the thermal diffusivity and other symbols have their usual meanings.

Combining (2.1) and (2.2) and using  $u + iv = W$ , we obtain

$$\partial W / \partial t - (1 + \epsilon e^{int}) \partial W / \partial z + 2iEW + \left[ M^2 + \left[ k_0 (1 + \epsilon e^{int}) \right]^{-1} \right] W = \partial^2 W / \partial z^2 + G_r T + G_m C$$

**3. Solution of the Problem.** We assume the velocity, temperature and concentration of the liquid in the neighbourhood of the plate as:

$$W(z, t) = W_1(z) + \epsilon W_2(z) e^{int}, \quad \dots(3.1)$$

$$T(z, t) = T_1(z) + \epsilon T_2(z) e^{int}, \quad \dots(3.2)$$

$$C(z, t) = C_1(z) + \epsilon C_2(z) e^{int}. \quad \dots(3.3)$$

Using (3.1), (3.2) and (3.3) in (2.3), (2.4) and (2.6), we obtain

$$T'''_1(z) + P_r T''_1(z) - \alpha_0 T'_1(z) = 0 \quad \dots(3.4)$$

$$T'''_2(z) + P_r T''_2(z) - (inP_r + \alpha_0) T'_2(z) = -P_r T''_1(z), \quad \dots(3.5)$$

$$C''_1(z) + S_c C'_1(z) = -S_c S_0 T'''_1(z), \quad \dots(3.6)$$

$$C'''_2(z) + S_c C''_2(z) - inS_c C'_2(z) = -S_c C'_1(z) - S_c S_0 T'''_2(z), \quad \dots(3.7)$$

$$W''_1(z) + W'_1(z) - (M_1 + 2iE)W_1(z) = -G_r T'_1(z) - G_m C'_1(z), \quad \dots(3.8)$$

and

$$W''_2(z) + W'_2(z) - [(M_1 + i(2E + n))]W_2(z) = -G_r T'_2(z) - G_m C'_2(z) - W_1(z) - \frac{1}{k_0} W_1(z), \dots(3.9)$$

where  $M_1 = M^2 + 1/k_0$ .

Using (3.1), (3.2) and (3.3) in (2.5), the boundary conditions become

$$W_1 = 1, W_2 = 1, T_1 = 1, T_2 = 1, C_1 = 1, C_2 = 1 \text{ at } z = 0$$

$$W_1 = 0, W_2 = 0, T_1 = 0, T_2 = 0, C_1 = 0, C_2 = 0 \text{ as } z \rightarrow \infty \quad \dots(3.10)$$

The solutions of the coupled equations (3.4)–(3.9) under the boundary conditions (3.10) are

$$T_1(z) = e^{-H_1 z}, \quad \dots(3.11)$$

$$T_2(z) = (1 - F_1)e^{-H_2 z} + F_1 e^{-H_1 z}, \quad \dots(3.12)$$

$$C_1(z) = (1 - F_0)e^{-S_c z} + F_0 e^{-H_1 z}, \quad \dots(3.13)$$

$$C_2(z) = F_2 e^{-S_c z} - F_3 e^{-H_2 z} - F_4 e^{-H_1 z} + F_5 e^{-H_3 z}, \quad \dots(3.14)$$

$$W_1(z) = F_8 e^{-H_4 z} - F_7 e^{-S_c z} - F_6 e^{-H_1 z}, \quad \dots(3.15)$$

$$W_2(z) = F_{14} e^{-H_5 z} + F_9 e^{-H_1 z} + F_{10} e^{-H_2 z} - F_{11} e^{-H_3 z} + F_{12} e^{-H_4 z} - F_{13} e^{-S_c z}. \quad \dots(3.16)$$

From (3.15), the steady parts of the primary velocity  $u_1(z)$  and secondary velocity  $v_1(z)$  are

$$u_1(z) = e^{-P_3 z} (A_9 \cos Q_3 z + B_9 \sin Q_3 z) - A_8 e^{-S_c z} - A_7 e^{-H_1 z}, \quad \dots(3.17)$$

$$v_1(z) = e^{-P_3 z} (B_9 \cos Q_3 z - A_9 \sin Q_3 z) - B_8 e^{-S_c z} - B_7 e^{-H_1 z}. \quad \dots(3.18)$$

From (3.16), the time dependent parts of the primary velocity  $u_2(z)$  and the secondary velocity  $v_2(z)$  are :

$$\begin{aligned} u_2(z) = & e^{-P_4 z} (A_{14} \cos Q_4 z + B_{14} \sin Q_4 z) - e^{-P_3 z} (A_{12} \cos Q_3 z + B_{12} \sin Q_3 z) \\ & - e^{-P_2 z} (A_{11} \cos Q_3 z + B_{11} \sin Q_2 z) + e^{-P_1 z} (A_{10} \cos Q_1 z + B_{10} \sin Q_1 z) \\ & + A_9 e^{-H_1 z} - A_{13} e^{-S_c z}, \end{aligned} \quad \dots(3.19)$$

$$\begin{aligned} v_2(z) = & e^{-P_4 z} (B_{14} \cos Q_4 z - A_{14} \sin Q_4 z) + e^{-P_3 z} (B_{12} \cos Q_3 z - A_{12} \sin Q_3 z) \\ & - e^{-P_2 z} (B_{11} \cos Q_3 z - A_{11} \sin Q_2 z) + e^{-P_1 z} (B_{10} \cos Q_1 z - A_{10} \sin Q_1 z) \\ & + B_9 e^{-H_1 z} - B_{13} e^{-S_c z}. \end{aligned} \quad \dots(3.20)$$

Therefore the primary velocity  $u(z,t)$  and secondary velocity  $v(z,t)$  are

$$u(z,t) = u_1(z) + \epsilon [u_2(z) \cos nt - v_2(z) \sin nt] \quad \dots(3.21)$$

and  $v(z,t) = v_1(z) + \epsilon [v_2(z) \cos nt + u_2(z) \sin nt] \quad \dots(3.22)$

**4. Skin-Friction, Rate of Heat and Mass Transfer.** The non dimensional skin-friction ( $\tau$ ) at the plate at  $z=0$  is

$$\tau = (dW/dz)_{z=0} = (dW_1/dz)_{z=0} + \epsilon e^{int} (dW_2/dz)_{z=0} = \tau_p + i\tau_s \quad \dots(4.1)$$

Using (4.1) primary skin-friction ( $\tau_p$ ) due to primary velocity and secondary skin-friction ( $\tau_s$ ) due to secondary velocity are :

$$\tau_p = \tau_{0p} + \epsilon (A_{15} \cos nt - B_{15} \sin nt) \quad \dots(4.2)$$

$$\tau_s = \tau_{0s} + \epsilon (B_{15} \cos nt + A_{15} \sin nt) \quad \dots(4.3)$$

Also, the rate of heat transfer ( $N_u$ ) and the rate of mass transfer ( $S_h$ ) at the plate at  $z=0$  are :

$$N_u = (dT/dz)_{z=0} = (dT_1/dz)_{z=0} + \epsilon e^{int} (dT_2/dz)_{z=0}, \quad \dots(4.4)$$

$$S_h = (dC/dz)_{z=0} = (dC_1/dz)_{z=0} + \epsilon e^{int} (dC_2/dz)_{z=0}. \quad \dots(4.5)$$

Hence, considering that the real part only is of significance, we get

$$N_u = -H_1 + \epsilon [A_{16} \cos nt - B_{16} \sin nt], \quad \dots(4.6)$$

$$S_h = H_1 F_0 - S_c (1 + F_0) + \epsilon [A_{17} \cos nt - B_{17} \sin nt]. \quad \dots(4.7)$$

In terms of amplitude and phase we write

$$N_u = -H_1 + \epsilon |L_1| \cos(nt + \alpha), \quad \dots(4.8)$$

$$S_h = H_1 F_0 - S_c (1 + F_0) + \epsilon |L_2| \cos(nt + \beta). \quad \dots(4.9)$$

**Table 1**

**Skin-friction due to primary and secondary velocity**

( $P_r=0.71$ ,  $k_0=10.0$ ,  $\alpha_0=1.0$ ,  $n=5.0$ ,  $G_m=10.0$ ,  $t=1.0$  and  $\epsilon=0.002$ )

$S_c$	$M$	$G_r$	$S_0$	$E$	$\tau_p$	$\tau_s$
0.22	1.0	10.0	1.0	1.0	10.30405	-5.49032
0.78	1.0	10.0	1.0	1.0	9.035954	-3.49691
0.22	1.5	10.0	1.0	1.0	9.607512	-3.23245
0.22	1.0	20.0	1.0	1.0	14.27663	-6.73981
0.22	1.0	10.0	2.0	1.0	10.11975	-5.63872
0.22	1.0	10.0	1.0	2.0	7.311972	-5.03227

**Table 2**  
**Amplitude  $|K_1|$ , phase  $\tan\alpha$  and rate of heat transfer  $N_u$**   
 ( $n=5.0, t=1.0$  and  $\varepsilon=0.002$ )

S.No.	$P_r$	$\alpha_0$	$ K_1 $	$\tan\alpha$	$N_u$
1	0.71	1.0	1.6957	0.4207	0.6879
2	0.025	1.0	0.2138	0.4714	0.0214
3	7.00	1.0	11.5739	0.0286	5.9864
4	11.4	1.0	20.2873	0.0095	9.8197
5	0.71	2.0	1.9871	0.4998	0.6792

**Table 3**  
**Amplitude  $|K_2|$ , phase  $\tan\beta$  and rate of heat transfer  $S_h$**   
 ( $n=5.0, t=1.0$  and  $\varepsilon=0.002$ )

S.No.	$S_c$	$S_0$	$ K_2 $	$\tan\beta$	$S_h$
1	0.22	1.0	0.8271	0.7014	0.1987
2.	0.30	1.0	1.0092	0.6813	0.2632
3	0.60	1.0	1.5482	0.6192	0.5389
4	0.66	1.0	1.6412	0.6138	0.6218
5	0.78	1.0	1.9173	0.6085	0.7485
6	0.22	2.0	0.9915	0.7754	0.1991

**5. Discussion and Conclusions.** The effects of Schmidt number ( $S_c$ ), magnetic parameter ( $M$ ), Grashof number ( $G_r$ ), Soret number ( $S_0$ ) and rotation parameter ( $E$ ) on primary velocity ( $u$ ) and secondary velocity ( $v$ ) at  $P_r=0.71$ ,  $k_0=10.0$ ,  $n=5.0$ ,  $\alpha_0=1.0$ ,  $t=1.0$ ,  $G_m=10.0$  and  $\varepsilon=0.002$  are numerically observed and are shown in Fig.-1 and Fig.-2. The above stated effects on skin-friction ( $\tau_p$ ) due to primary velocity and skin-friction ( $\tau_s$ ) due to secondary velocity at the plate are observed numerically and are represented in Table 1. To be realistic the values of Prandtl number ( $P_r$ ) are chosen for mercury ( $P_r=0.025$ ), air ( $P_r=0.71$ ), water ( $P_r=7.0$ ) and water at  $4^{\circ}\text{C}$  ( $P_r=11.4$ ). The values of Schmidt number ( $S_c$ ) are chosen for Hydrogen ( $S_c=0.22$ ), Helium ( $S_c=0.30$ ), Water-vapour ( $S_c=0.60$ ), Oxygen ( $S_c=0.66$ ) and Ammonia ( $S_c=0.78$ ). The values of the remaining parameters are chosen arbitrarily. The conclusions of the study are as follows:

- (i) An increase in  $G_r$  or  $S_0$  increases primary velocity while an increase in  $S_c$ ,  $M$  or  $E$  decreases the primary velocity.
- (ii) An increase in  $G_r$  and  $M$  both decreases the primary velocity.
- (iii) An increase in  $S_0$  and  $E$  both decreases the primary velocity.
- (iv) The primary velocity increases near the plate and after attaining a maximum value it decreases as  $z$  increases.

- (v) An increase in  $S_c M$  or  $E$  increases secondary velocity while an increase in  $G_r$  or  $S_0$  decreases the secondary velocity.
- (vi) An increase in  $G_r$  and  $M$  both, the secondary velocity increases.
- (vii) An increase in  $G_m$  and  $E$  both, the secondary velocity increases.
- (viii) The secondary velocity decreases near the plate and after attaining a minimum value it increases as  $z$  increases.
- (ix) An increase in  $S_c$ ,  $M$ ,  $S_0$  or  $E$  decreases the skin-friction due to primary velocity while an increase in  $G_r$ , increases the skin-friction due to primary velocity.
- (x) An increase in  $S_c$ ,  $M$  or  $E$  increases the skin-friction due to secondary velocity while an increase in  $G_r$  or  $S_0$  decreases the skin-friction due to secondary velocity.
- (xi) The values of  $|K_1|$  and  $N_u$  are least for mercury and highest for water at  $40^\circ\text{C}$  while reverse effects are noted for  $\tan\alpha$ .
- (xii) An increase in  $\alpha_0$  results in an increase in amplitude  $|K_1|$  and phase  $\tan\alpha$  but decreases the rate of heat transfer  $N_u$ .
- (xiii) The values of  $|K_2|$  and  $S_h$  increase due to the increase in  $S_c$  corresponding to the said gases while reverse effect is noted for the phase  $\tan\beta$ .
- (xiv) An increase in  $S_0$  results in an increase in amplitude  $|K_2|$  and phase  $\tan\beta$  but decreases the rate of mass transfer  $S_h$ .

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### APPENDIX

$$H_1 = \left[ P_r + \sqrt{P_r^2 + 4\alpha_0} \right] / 2, \quad H_2 = P_1 + iQ_1 = \left[ P_r + \sqrt{P_r^2 + 4(inP_r + \alpha_0)} \right] / 2,$$

$$H_3 = P_2 + iQ_2 = \left[ S_c + \sqrt{S_c^2 + 4inS_c} \right] / 2, \quad H_4 = P_3 + iQ_3 = \left[ 1 + \sqrt{(1 + 4M_1 + 8iE)} \right] / 2$$

$$H_5 = P_4 + iQ_4 = \left[ 1 + \sqrt{1 + 4M_1 + 4i(2E + n)} \right] / 2, \quad F_0 = S_c S_0 H_1^2 / [H_1(H_1 - S_c)]$$

$$F_1 = (A_1 + iB_1) = \frac{H_1 P_r}{(H_1^2 - H_1 P_r - \alpha_0) - i(nP_r)}, \quad F_2 = A_2 + iB_2 = iS_c/n$$

$$F_3 = A_3 + iB_3 = \frac{S_0 S_c H_2^2 (1 - F_1)}{H_2^2 - H_2 S_c - inS_c}, \quad F_4 = A_4 + iB_4 = \frac{S_0 S_c F_1 H_1^2}{H_1^2 - H_1 S_c - inS_c},$$

$$F_5 = A_5 + iB_5 = 1 - F_2 + F_3 - F_4, \quad F_6 = A_6 + iB_6 = \frac{G_r + G_m F_0}{H_1^2 - H_1 - M_1 - i2E},$$

$$F_7 = A_7 + iB_7 = \frac{G_m (1 + F_0)}{S_c^2 - S_c - M_1 - i2E}, \quad F_8 = A_8 + iB_8 = 1 + F_6 + F_7,$$

$$F_9 = A_9 + iB_9 = \frac{F_6 (1 + k_0 H_1) + k_0 G_r F_1 - G_m F_4 k_0}{k_0 [H_1^2 - H_1 - M_1 - i(2E + n)]},$$

$$F_{10} = A_{10} + iB_{10} = \frac{G_m F_3 - G_r (1 - F_1)}{[(H_2^2 - H_2 - M_1) - i(2E + n)]},$$

$$F_{11} = A_{11} + iB_{11} = \frac{G_m F_5}{H_3^2 - H_3 - M_1 - i(2E + n)},$$

$$F_{12} = A_{12} + iB_{12} = \frac{F_9 (k_0 H_4 - 1)}{k_0 [H_4^2 - H_4 - M_1 - i(2E + n)]},$$

$$F_{13} = A_{13} + iB_{13} = \frac{k_0 G_m F_2 + F_7 (k_0 S_c - 1)}{k_0 [S_c^2 - S_c - M_1 - i(2E + n)]},$$

$$F_{14} = A_{14} + iB_{14} = 1 - F_9 - F_{10} + F_{11} - F_{12} + F_{13},$$

$$P_1 = P_r / 2 + \left[ \sqrt{(P_r^2 + 4\alpha_0)^2 + 16n^2 P_r^2 + (P_r^2 + 4\alpha_0)} \right]^{1/2} / 2\sqrt{2},$$



$$Q_1 = \left[ \sqrt{(P_r^2 + 4\alpha_0)^2 + 16n^2 P_r^2} + (P_r^2 + 4\alpha_0) \right]^{1/2} / (2\sqrt{2}),$$

$$P_2 = S_c/2 + \left[ S_c \sqrt{S_c^2 + 16n^2} + S_c^2 \right]^{1/2} / 2\sqrt{2}, \quad Q_2 = \left[ S_c \sqrt{S_c^2 + 16n^2} - S_c^2 \right]^{1/2} / (2\sqrt{2}),$$

$$P_3 = 1/2 + \left[ \sqrt{(1 + 4M_1)^2 + 64E^2} + (1 + 4M_1) \right]^{1/2} / (2\sqrt{2}),$$

$$Q_3 = \left[ \sqrt{(1 + 4M_1)^2 + 64E^2} - (1 + 4M_1) \right]^{1/2} / (2\sqrt{2}),$$

$$P_4 = 1/2 + \left[ \sqrt{(1 + 4M_1)^2 + 16(2E + n)^2} + (1 + 4M_1) \right]^{1/2} / (2\sqrt{2}),$$

$$Q_4 = \left[ \sqrt{(1 + 4M_1)^2 + 16(2E + n)^2} - (1 + 4M_1) \right]^{1/2} / (2\sqrt{2}),$$

$$A_1 = \frac{H_1 P_r \alpha_1}{\alpha_1^2 + b_1^2}, \quad B_1 = \frac{H_1 P_r b_1}{\alpha_1^2 + b_1^2}, \quad A_2 = 0, \quad B_2 = \frac{S_c}{n}, \quad A_3 = \frac{\alpha_2 \alpha_3 + b_2 b_3}{\alpha_3^2 + b_3^2}, \quad B_3 = \frac{b_2 \alpha_3 - \alpha_2 b_3}{\alpha_3^2 + b_3^2},$$

$$A_4 = \frac{S_c S_0 H_1^2 [A_1 \alpha_4 - B_1 b_4]}{\alpha_4^2 + b_4^2}, \quad B_4 = \frac{S_c S_0 H_1^2 [A_1 b_4 - B_1 \alpha_4]}{\alpha_4^2 + b_4^2}, \quad A_5 = 1 - A_2 + A_3 - A_4,$$

$$B_5 = B_3 - B_2 - B_4, \quad A_6 = \frac{(G_r + F_0 G_m) \alpha_5}{\alpha_5^2 + b_5^2}, \quad B_6 = \frac{(G_r + F_0 G_m) b_5}{\alpha_5^2 + b_5^2}, \quad A_7 = \frac{\alpha_6 G_m (1 + F_0)}{\alpha_6^2 + b_6^2},$$

$$B_7 = \frac{b_6 G_m (1 + F_0)}{\alpha_6^2 + b_6^2}, \quad A_8 = 1 + A_6 + A_7, \quad B_8 = B_6 + B_7, \quad A_9 = \frac{\alpha_7 \alpha_8 - b_7 b_8}{\alpha_8^2 + b_8^2},$$

$$B_9 = \frac{b_7 \alpha_8 - \alpha_7 b_8}{\alpha_8^2 + b_8^2}, \quad A_{10} = \frac{\alpha_9 \alpha_{10} + b_9 b_{10}}{\alpha_{10}^2 + b_{10}^2}, \quad B_{10} = \frac{b_9 \alpha_{10} + \alpha_9 b_{10}}{\alpha_{10}^2 + b_{10}^2}, \quad A_{11} = \frac{G_r (A_5 \alpha_{11} + B_5 b_{11})}{\alpha_{11}^2 + b_{11}^2},$$

$$B_{11} = \frac{G_r (B_5 \alpha_{11} - A_5 b_{11})}{\alpha_{11}^2 + b_{11}^2}, \quad A_{12} = \frac{\alpha_{12} \alpha_{13} + b_{12} b_{13}}{\alpha_{13}^2 + b_{13}^2}, \quad B_{12} = \frac{b_{12} \alpha_{13} - \alpha_{12} b_{13}}{\alpha_{13}^2 + b_{13}^2},$$

$$A_{13} = \frac{\alpha_{14} \alpha_{15} + b_{14} b_{15}}{\alpha_{15}^2 + b_{15}^2}, \quad B_{13} = \frac{b_{14} \alpha_{15} - \alpha_{14} b_{15}}{\alpha_{15}^2 + b_{15}^2}, \quad A_{14} = A_{13} - A_{12} + A_{11} - A_{10} - A_9 + 1,$$

$$B_{14} = B_{13} - B_{12} + B_{11} - B_{10} - B_9, \quad \alpha_1 = H_1^2 - H_1 P_r - \alpha_0,$$

$$\alpha_2 = S_0 S_c [(1 - A_1)(P_1^2 - Q_1^2) + 2P_1 Q_1 B_1], \quad \alpha_3 = P_1^2 - Q_1^2 - P_1 S_c, \quad \alpha_4 = H_1 (H_1 - S_c),$$

$$\alpha_5 = H_1^2 - H_1 - M_1, \quad \alpha_6 = S_c^2 - S_c - M_1, \quad \alpha_7 = A_6 (1 - k_0 H_1) - k_0 A_1 G_r - k_0 F_0 G_m, \quad \alpha_8 = k_0 \alpha_5,$$

$$\alpha_9 = G_m A_3 - G_r (1 - A_1), \quad \alpha_{10} = P_1^2 - Q_1^2 - P_1 - M_1, \quad \alpha_{11} = P_2^2 - Q_2^2 - P_2 - M_1,$$

$$\alpha_{12} = A_8 (k_0 P_3 - 1) - B_8 k_0 Q_3, \quad \alpha_{13} = k_0 [P_3^2 - Q_3^2 - P_3 - M_1], \quad \alpha_{14} = A_2 k_0 G_m + A_7 (k_0 S_c - 1),$$

$$\alpha_{15} = k_0 [S_c^2 - S_c - M_1], \quad b_1 = n P_r, \quad b_2 = S_c S_0 [2P_1 Q_1 (1 - A_1) - B_1 (P_1^2 - Q_1^2)],$$

$$b_3 = 2P_1 Q_1 - Q_1 S_c - n S_c, \quad b_4 = n S_c, \quad b_5 = b_6 = 2E, \quad b_7 = B_6 (1 - k_0 H_1) - B_1 k_0 G_r,$$

$$b_8 = b_{15} = k_0 (2E + n), \quad b_9 = G_m B_3 + G_r B_1, \quad b_{10} = 2P_1 Q_1 - Q_1 - 2E - n,$$

$$b_{11} = 2P_2 Q_2 - Q_2 - 2E - n, \quad b_{12} = B_8 (P_3 k_0 - 1) + A_8 Q_3 k_0, \quad b_{13} = k_0 [2P_3 Q_3 - Q_3 - 2E - n],$$

$$b_{14} = B_2 k_0 G_m + B_7 (k_0 S_c - 1), \quad \tau_{0p} = A_7 H_1 + A_8 S_c + Q_3 B_9 - P_3 A_9,$$

$$\tau_{0z} = B_7 H_1 + B_8 S_c - Q_3 A_9 - P_3 B_9,$$

$$A_{15} = A_{13} S_c - A_{14} P_4 + B_{14} Q_4 - A_{12} P_3 + B_{12} Q_3 - A_{11} P_2 + B_{12} Q_2 - A_{10} P_1 + B_{10} Q_1 - A_9 H_1$$

$$B_{15} = B_{13} S_c - A_{14} Q_4 - B_{14} P_4 - A_{12} Q_3 - B_{12} P_3 - A_{11} Q_2 - B_{12} P_2 - A_{10} Q_1 - B_{10} P_1 - B_9 H_1,$$

$$A_{16} = P_1(A_1 - 1) - Q_1 B_1 - H_1 A_1, \quad B_{16} = Q_1(A_1 - 1) - P_1 B_1 - H_1 B_1,$$

$$A_{17} = A_3 P_1 - B_3 Q_1 - A_5 P_2 + B_5 Q_2 - A_2 S_c - A_4 H_1,$$

$$B_{17} = A_3 Q_1 + B_3 P_1 - A_5 Q_2 - B_5 P_2 - B_2 S_c - B_4 H_1,$$

$$|L_1| = |A_{16} + iB_{16}|, \quad |L_2| = |A_{17} + iB_{17}|, \quad \tan\alpha = B_{16}/A_{16} \quad \text{and} \quad \tan\beta = B_{17}/A_{17}$$

