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COMMON FIXED POINT OF COMPATIBLE MAPPINGS OF TYPE (α)

By
Sushil Sharma
Department of Mathematics, Madhav Science College
Ujjain-456010, Madhya Pradesh, India

Kamal Choubey and Abha Tenguria
Government Narmada Postgraduate College of Excellence
Hoshangabad-461001, Madhya Pradesh, India
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ABSTRACT

In this paper we prove a common fixed point theorem for six mappings in a complete metric space. Our theorem generalizes the entire results of Banach [1], Kannan [10], Fisher [6], Chatterjee [2] and Cho [3].

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1. Introduction. Sessa [14] defined a generalization of commutativity, which is called weak commutativity. Further, Jungck [7] introduced more generalized commutativity, so called compatibility, which is more general than that of weak commutativity del. The utility of compatibility in the context of fixed point theory was demonstrated by extending a theorem of Park and Bae [13]. Also, Jungck [8] extended the results of Khan and Imdad [11].

Kang et. al. [12] extended the results of Ding [5], Diviccaro and Sessa [4] and Jungck [8] by using one of the mappings continuous and employing compatible mappings.

Recently, Jungck-Murthy-Cho [9] introduced the concept of compatible mappings of type (α) in metric spaces.

In this paper, we prove a common fixed point theorem for six mappings under the condition of compatible mappings of type (α), in a complete metric space.

2. Compatible mappings of type (α). In this section, we give some definitions and the concept of compatible mappings of type (α) in metric spaces and some properties of these mappings for our main result.

Definition 1. Let (X,d) be a metric space and $\{x_n\}$ be a sequence in X.

(1) $\{x_n\}$ is said to be convergent to a point x in X, if

$$\lim_{n\to\infty} d(x_n,x) = 0$$

(2) $\{x_n\}$ in X is said to be Cauchy sequence, if $\lim_{m,n\to\infty} d(x_m, x_n) = 0$

(3) A metric spame (X,d) is said to be complete if every Cauchy sequence in X is convergent.

Definition 2 [7]. Let A and B be mappings from a metric space (X,d) into itself. Then A and B are said to be compatible, if

$$\lim_{n\to\infty} d(ABx_n, BAx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \text{ for some } z \in X.$$

Definition 3 [9]. Let A and B be mappings from a metric space (X,d) into itself. Then A and B are said to be compatible of type (α) if

$$\lim_{n\to\infty} d(ABx_n, BBx_n) = 0 \text{ and } \lim_{n\to\infty} d(BAx_n, AAx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Ax_n = z \text{ for some } z \in X.$$

Proposition 1 [9]. Let (X,d) be a metric space. Let A and B be continuous mappings from X into itself. Then A and B are compatible if and only if they are compatible of type (α).

Proposition 2 [9]. Let (X,d) be a metric space and A and B are mappings from X into itself. If A and B are compatible of type (α) and $Az = Bz$ for some $z \in X$, then

$$ABz = BBz = BAz = AAz.$$

Proposition 3 [9]. Let (X,d) be a metric space and A and B are mappings from X into itself. If A and B compatible of type (α) and $\{x_n\}$ is sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \text{ for some } z \text{ in } X, \text{ then}$$

(i) $\lim_{n\to\infty} BAx_n = Az$, if A is continuous at z,

(ii) $ABz = BAz$ and $Az = Bz$, if A and B are continuous at z.

3. A Common Fixed Point Theorem. In this section, we prove a common fixed point theorem for six mappings satisfying some conditions.

Theorem. Let (X,d) be a complete metric space. Let A,B,S,T,P and Q be mappings from X into itself such that

(3.1) $P(X) \subset AB(X), Q(X) \subset ST(X)$,

(3.2) $AB=BA$, $ST = TS$, $PA = AP$, $PB=BP$, $QS=SQ$, $QT=TQ$,

(3.3) A,B,S and T are continuous,

(3.4) the pairs $\{P, AB\}$ and $\{Q, ST\}$ are compatible of type (α),

$$(3.5)\quad d(Px,Qy)\le \alpha_1\left[\frac{d(STy,Qy)d(ABx,ABy)+d(ABy,Qy)d(ABx,Px)}{d(ABx,Qy)+d(ABx,STy)}\right]$$

$$+\alpha_2[d(STy,Px)+d(STy,Qy)+d(ABx,STy)]$$

$$+\alpha_3[d(ABx,Qy)+d(ABy,Px)+d(STy,ABy)]$$

$$+\alpha_4[d(ABx,Px)+d(ABy,Qy)]$$

for all x, $y \in X$, where each $\alpha_i \ge 0$ and $2\alpha_1+2\alpha_2+2\alpha_3+\alpha_4<1$.

Then A, B, S,T, P and Q have a common fixed point in X.

Proof . By (3.1), Since $P(X) \subset AB(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $Px_0 = ABx_1$. Since $Q(X)\ ST \subset (X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Qx_1 = STx_2$. Inductively, we can define a sequence $\{y_n\}$ in X as follows.

$y_{2n} = Px_{2n} = ABx_{2n+1}$

$y_{2n+1} = Qx_{2n+1} = STx_{2n+2}$ for $n = 0,1,2, \ldots$

By Putting $x=x_{2n+1}$ and $y=x_{2n+2}$ in (3.5) we write

$d(Px_{2n+1}, Qx_{2n+2})$

$$\le \alpha_1\left[\frac{d(STx_{2n+2},Qx_{2n+1})d(ABx_{2n+1},ABx_{2n+2})+d(ABx_{2n+2},Qx_{2n+2})d(ABx_{2n+1},Px_{2n+1})}{d(ABx_{2n+1},Qx_{2n+2})+d(ABx_{2n+1},STx_{2n+2})}\right]$$

$$+\alpha_2[d(STx_{2n+2},Px_{2n+1})+d(STx_{2n+2},Qx_{2n+2})+d(ABx_{2n+1},STx_{2n+2})]$$

$$+\alpha_3[d(ABx_{2n+1},Qx_{2n+2})+d(ABx_{2n+2},Px_{2n+1})+d(STx_{2n+2},ABx_{2n+2})]$$

$$+\alpha_4[d(ABx_{2n+1},Px_{2n+1})+d(ABx_{2n+2},Qx_{2n+2})]$$

$$\le \alpha_1\left[\frac{d(y_{2n+1},y_{2n+2})d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})d(y_{2n},y_{2n+1})}{d(y_{2n},y_{2n+2})+d(y_{2n},y_{2n+1})}\right]$$

$$+\alpha_2[d(y_{2n+1},y_{2n+1})+d(y_{2n+1},y_{2n+2})+d(y_{2n},y_{2n+1})]$$

$$+\alpha_3[d(y_{2n},y_{2n+2})+d(y_{2n+1},y_{2n+1})+d(y_{2n+1},y_{2n+1})]+\alpha_4[d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})].$$

$$d(y_{2n+1},y_{2n+2})\le \frac{(2\alpha_1+\alpha_2+\alpha_3+\alpha_4)}{(1-\alpha_2-\alpha_3-\alpha_4)}d(y_{2n},y_{2n+1}).$$

Putting $h=\dfrac{(2\alpha_1+\alpha_2+\alpha_3+\alpha_4)}{(1-\alpha_2-\alpha_3-\alpha_4)}$, we find $h<1$, since $2\alpha_1+2\alpha_2+2\alpha_3+\alpha_4<1$.

Hence

$$d(y_{2n+1},y_{2n+2})\le h d(y_{2n},y_{2n+1}).$$

Similarly we can see that

$$d(y_{2n}, y_{2n+1}) \leq hd(y_{2n-1}, y_{2n}).$$

Proceeding in this way, we have

$$d(y_{2n}, y_{2n+1}) \leq h^{2n-1} d(y_0, y_1).$$

By routine calculations the following inequalities hold for $k > n$

$$d(y_n, y_{n+k}) \leq \sum_{i=1}^{k} (dy_{n+i-1}, y_{n+1})$$

$$\leq \sum_{i=1}^{k} h^{n+i-1} d(y_0, y_1)$$

$$\leq \frac{h^n}{1-h} d(y_0, y_1) \to 0 \text{ as } n \to \infty.$$

Here $h < 1$. Hence $\{y_n\}$ is a Cauchy sequence and by completeness of X we see that $\{y_n\}$ is convergent to a point z in X.

Since Px_{2n}, Qx_{2n+1}, ABx_{2n+1} and STx_{2n+2} are subsequences of $\{y_n\}$, they also converge to a point z, that is as $n \to \infty$,

$$Px_{2n}, Qx_{2n+1}, STx_{2n+1} \to z.$$

Since A and B are continuous and the pair $\{P, AB\}$ is compatible of type α, by proposition 3 we have, as $n \to \infty$,

$$P(AB)x_{2n+1} \to ABz, \quad (AB)^2 x_{2n+1} \to ABz.$$

Now putting $x = ABX_{2n+1}$ and $y = x_{2n+2}$ in (3.5), we write

$d\,(P(AB)x_{2n+1}, Qx_{2n+2})$

$$\leq \alpha_1 \left[\frac{d(STx_{2n+2}, Qx_{2n+2}) d((AB)^2 x_{2n+2}, ABx_{2n+2}) + d(ABx_{2n+2}, Qx_{2n+2}) d((AB)^2 x_{2n+1}, P(AB)x_{2n+1})}{d((AB)^2 x_{2n+1}, Qx_{2n+2}) + d((AB)^2 x_{2n+1}, STx_{2n+2})} \right]$$

$$+ \alpha_2 \left[d(STx_{2n+2}, P(AB)x_{2n+1}) + d(STx_{2n+2}, Qx_{2n+2}) + d((AB)^2 x_{2n+1}, STx_{2n+2}) \right]$$

$$+ \alpha_3 \left[d((AB)^2 x_{2n+1}, Qx_{2n+2}) + d(ABx_{2n+2}, P(AB)x_{2n+1}) + d(STx_{2n+2}, ABx_{2n+2}) \right]$$

$$+ \alpha_4 \left[d((AB)^2 x_{2n+1}, P(AB)x_{2n+1}) + d(ABx_{2n+2}, Qx_{2n+2}) \right]$$

which implies that, as $n \to \infty$

$$d(ABz, z) \leq (2\alpha_2 + 2\alpha_3) d(ABz, z),$$

which is a contradiction since $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$.

Therefore, we have $ABz = z$.

By putting $x = Px_{2n}$ and $y = x_{2n+1}$ in (3.5), we write

$d(P(Px_{2n}), Qx_{2n+1})$

$$\leq \alpha_1 \left[\frac{d(STx_{2n+1}, Qx_{2n+1})d(AB(Px_{2n}), ABx_{2n+1}) + d(ABx_{2n+1}, Qx_{2n+1})d(AB(Px_{2n}), P(Px_{2n}))}{d(AB(Px_{2n}), Qx_{2n+1}) + d(AB(Px_{2n}), STx_{2n+1})} \right]$$

$$+ \alpha_2 [d(STx_{2n+1}, P(Px_{2n})) + d(STx_{2n+1}, Qx_{2n+1}) + d(AB(Px_{2n}), STx_{2n+1})]$$

$$+ \alpha_3 [d(AB(Px_{2n}), Qx_{2n+1}) + d(ABx_{2n+1}, P(Px_{2n})) + d(STx_{2n+1}, ABx_{2n+1})]$$

$$+ \alpha_4 [d(AB(Px_{2n}), P(Px_{2n}) + d(ABx_{2n+1}, Qx_{2n+1})].$$

Taking the limit $n \to \infty$, we have

$$d(Pz, z) \leq (2\alpha_2 + 2\alpha_3 + \alpha_4) d(Pz, z),$$

which is a contradiction since $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$.

Therefore, we have $Pz=z=Abz$.

Now, we show that $Bz=z$. By putting $x=Bz$ and $y=x_{2n+1}$ in (3.5), we write

$d(p(Bz), Qx_{2n+1})$

$$\leq \alpha_1 \left[\frac{d(STx_{2n+1}, QX_{2n+1})d(AB(Bz), ABx_{2n+1}) + d(ABx_{2n+1}, Qx_{2n+1})d(AB(Bz), P(Bz))}{d(AB(Bz), Qx_{2n+1}) + d(AB(Bz), STx_{2n+1})} \right]$$

$$+ \alpha_2 [d(STx_{2n+1}, P(Bz)) + d(STx_{2n+1}, Qx_{2n+1}) + d(AB(Bz), STx_{2n+1})]$$

$$+ \alpha_3 [d(AB(Bz), Qx_{2n+1}) + d(ABx_{2n+1}, P(Bz)) + d(STx_{2n+1}, ABx_{2n+1})]$$

$$+ \alpha_4 [d(AB(Bz), P(Bz)) + d(ABx_{2n+1}, Qx_{2n+1})].$$

Taking the limit $n \to \infty$, we have

$$d(Bz, z) \leq (2\alpha_2 + 2\alpha_3) d(Bz, z),$$

which is a contradiction since $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$ and

which implies that $Bz=z$. Since $ABz=z$, $Az=z$.

Now by putting $x=z$ and $y=STx_{2n+2}$ in (3.5), we write

$d(Pz, Q(STX_{2n+2}))$

$$\leq \alpha_1 \left[\frac{d((ST)^2 x_{2n+2}, Q(ST)x_{2n+2})d(ABz, AB(ST)x_{2n+2}) + d(AB(ST)x_{2n+2}, Q(ST)x_{2n+2})d(ABz, Pz)}{d(ABz, Q(ST)x_{2n+2}) + d(ABz, (ST)^2 x_{2n+2})} \right]$$

$$+ \alpha_2 \left[d((ST)^2 x_{2n+2}, Pz) + d((ST)^2 x_{2n+2}, Q(ST)x_{2n+2}) + d(ABz, (ST)^2 x_{2n+2}) \right]$$

$$+ \alpha_3 \left[d(ABz, Q(ST)x_{2n+2}) + d(AB(ST)x_{2n+2}, Pz) + d((ST)^2 x_{2n+2}, AB(ST)x_{2n+2}) \right]$$

$$+ \alpha_4 [d(ABz, Pz) + d(AB(ST)x_{2n+2}, Q(ST)x_{2n+2})].$$

Taking the limit $n \to \infty$, we have

$$d(z, STz) \leq (2\alpha_2 + 2\alpha_3 + \alpha_4) d(z, STz),$$

which is a contradiction since $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$.

Therefore we have $STz=z$.

Now by putting $x=z$ and $y=Qx_{2n+1}$ in (3.5), we write

$$d(Pz,Q(Qx_{2n+1}))$$

$$\leq \alpha_1\left[\frac{d(ST(Qxx_{2n+1}),Q(Qx_{2n+1}))d(ABz,AB(Qx_{2n+1}))+d(AB(Qx_{2n+1}),Q(Qx_{2n+1}))d(ABz,Pz)}{d(ABz,Q(Qxx_{2n+1}))+d(ABz,ST(Qx_{2n+1}))}\right]$$

$$+\alpha_2[d(ST(Qx_{2n+1}),Pz)+d(ST(Qx_{2n+1}),Q(Qx_{2n+1}))+d(ABz,ST(Qx_{2n+1}))]$$

$$+\alpha_3[d(ABz,Q(Qx_{2n+1}))+d(AB(Qx_{2n+1}),Pz)+d(ST(Qx_{2n+1}),AB(Qx_{2n+1}))]$$

$$+\alpha_4[d(ABz,Pz)+d(AB(Qx_{2n+1}),Q(Qx_{2n+2}))].$$

Taking the limit $n\to\infty$, we have

$$d(z,Qz)\leq(\alpha_2+\alpha_3+\alpha_4)d(z,Qz),$$

which is a contradiction since $2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$.

Therefore we have $Qz=z$ and hence $STz=z=Qz$.

Finally we show that $Tz=z$. By putting $x=z$ and $y=Tz$ in (3.5), we write

$$d(Pz,Q(Tz)) \leq \alpha_1\left[\frac{d(ST(Tz),Q(Tz))d(ABz,AB(Tz))+d(AB(Tz),Q(Tz))d(ABz,Pz)}{d(ABz,Q(Tz))+d(ABz,ST(Tz))}\right]$$

$$+\alpha_2[d(ST(Tz),Pz)+d(ST(Tz),Q(Tz))+d(ABz,ST(Tz))]$$

$$+\alpha_3[d(ABz,Q(Tz))+d(AB(Tz),Pz)+d(ST(Tz),AB(Tz))]$$

$$+\alpha_4[d(ABz,Pz)+d(AB(Tz),Q(Tz))],$$

which implies that

$$d(z,Tz)\leq(2\alpha_2+2\alpha_3)d(z,Tz).$$

Therefore we have $Tz=z$. Since $STz=z$, we have $z=STz=Sz$.

Therefore by combining the above results, we have

$$Az=Bz=Sz=Tz=Pz=Qz=z,$$

that is z is common fixed point of A, B, S, T, $P$and Q.

For uniqueness let w ($w\neq z$) be another fixed point of A, B, S, T, P and Q, then by (3.5), we write

$$d(Pz,Qw) \leq \alpha_1\left[\frac{d(STw,Qw)d(ABz,ABw)+d(ABw,Qw)d(ABz,Pz)}{d(ABz,Qw)+d(ABz,STw)}\right]$$

$$+\alpha_2[d(STw,Pz)+d(STw,Qw)+d(ABz,STw)]$$

$$+\alpha_3[d(ABz,Qw)+d(ABz,Pz)+d(STw,ABw)]$$
$$+\alpha_4[d(ABz,Pz)+d(ABw,Qw)],$$

which implies that

$$d(z,w)\leq(2\alpha_2+2\alpha_3)d(z,w),$$

which is a conradiction since $2\alpha_1+2\alpha_2+2\alpha_3+\alpha_4<1$.

Therefore we have $z = w$.

This completes the proof of the theorem.

Remark 1. Taking $AB=ST=I$ (I is the identify map on X) and $P=Q$ and $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0$ in the Theorem, we obtain the result due to Fisher [6].

Remark 2. Taking $AB=I$, $P=Q$ and $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0$ in our Theorem, we obtain the result due to Kannan [10].

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