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SOME COMMON FIXED POINT THEOREMS IN COMPLETE METRIC SPACES

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ABSTRACT

In the present paper, some comman fixed point theorems have been proved which generalize the results recently proved in [3], [7] and [10].

1. Introduction. After the works of Khan et. al [3], Pathak and Sharma [6], Sastri and Babu [8], Sastri et. al. [10] etc. have obtained several common fixed point theorems in metric space. Sastri et. al. [10] have shown that the inequality used in Khan et. al. [3] does not hold for $x=y$ because a(0) is not defined there in while the unicity condition fails in Pathak and Sharma [6]. In their support they have provided examples [10].

Let R^+ denote the set of all non negative reals, N the set of all natural numbers and ϕ, the set continuous of ψ such that ψ is monotonically increasing ant $\psi(t)=0$ if and only if $t=0$.

Using this function ψ and altering the distance between points, Khan et. al. [3], Park [4,5], Pathak and Sharma [6], Sastri and Babu [8] Sastri et. al [10], etc. have obtained fixed point for one mapping and common fixed point for two mappings. Sastri et. al. [8] have proved the following

Theorem A. Let (X,d) be a bounced metric space and $T:X\to X$ such that

$$\psi(d(Tx,Ty)) \leq k \max\{\psi(d(x,y)),\psi(d(x,Tx)),\psi(d(y,Ty))\}$$

and $0\leq k\leq 1$ for all x, y in X. Then T has a unique fixed point in X.

They have also proved the following

Theorem B. Let (X,d) be a bounded complete metric space and S and T be self maps of X such that $ST=TS$. Further, assume that S and T satisfy the following inequality

there exists $k\in(0,1)$ and $\psi\in\Phi$ such that

$$\psi(d(Sx,Ty)) \leq k \max\{\psi(d(x,y)),\psi(d(x,Sx)),\psi(d(y,Ty))\}$$

for all x, y in X. Then one of S and T (and hence both) have a unique common fixed point in X.

Sessa [11] introduced the notion of weakly commutativity which is weaker than commutativity. Every commutative pair is weakly commutative but the converse is not always true [10].

Definition 1. Two mappings $f,g : X \to X$, where X is a metric space are said to be weakly commutative if and only if $d(fgx,gfx) \leq d(fx,gx)$ for all $x \in X$.

Later on Jungck [1] extended the notion of weakly commutative to compatibility and showed that every weakly commutative maps are compatible but the converse is not always ture.

Definition 2. Two mappings $f,g:X \to X$, where X is a complete metric space, are said to be compatible if and only if whenever $\{x_n\}$ is a sequence in X such that $fx_n, gx_n \to t$ then $d(gfx_n, fgx_n) \to 0$.

Jungck and Rhoades [2] established the following

Proposition A. Let f,g be compatible self maps of a metric space (X,d)

1. If $f(t)=g(t)$ then $fg(t)=gf(t)$,
2. Suppose that $\lim fx_n = \lim gx_n = t$ for some $t \in X$ and $x_n \in X$,

Then

(a) If f is a continuous at t, $\lim gfx_n = f(t)$

(b) If f and g are continuous at t then $f(t)=g(t)$ and $fg(t)=gf(t)$.

In this paper, we obtain a necessary and sufficient condition for a common fixed point of these compatible mappings which extend the results proved in [3],[7] and [10].

We now first prove the following

Theorem 1. Let (X,d) be a complete metric space and let f, g be continuous mappings of X into itself then a mapping $h:X \to f(X) \cap g(X)$ has a fixed point in X if and only if

$\phi(d(hx,hy)) \leq k \max\{\phi(d(hx,fx)), \phi(d(hy,gy)), \phi(d(fx,gy)), \phi((d(hx,gy)/2 + d(hy,fx))\}$...(A)

for all $x,y \in X$, $0 \leq k \leq 1$ and $\phi \in \Phi$ and f, g are compatible with h. Further f,g,h have a unique common fixed point in X.

Proof. We shall first show that the condition is sufficient. Let $x_0 \in X$ such that $hx_0 = fx$. Since $x_1 \in X$ and $h(X) \subset g(X)$, there exists a point $x_2 \in X$ such that $hx_1 = gx_2$. In this way a sequence $\{x_n\}$ is constructed. So that $hx_{2n} = fx_{2n+1}$ and $hx_{2n+1} = gx_{2n+2}$, $n=0,1,2...$.

Define $d_n = d(hx_n, hx_{n+1})$. From (a)

$$\begin{aligned}\phi(d_{2n}) &= \phi(d(hx_{2n}, hx_{2n+1})) = \phi(d(hx_{2n+1}, hx_{2n})) \\ &\leq k \max\{\phi(d(hx_{2n+1}, fx_{2n+1})), \phi(d(hx_{2n}, gx_{2n})), \phi(d(hx_{2n+1}, gx_{2n})),\end{aligned}$$

$$\phi\{d(hx_{2n+1},gx_{2n})+d(hx_{2n},fx_{2n+1})\}/2\}$$
$$=k \max\{\phi(d(hx_{2n+1},hx_{2n})),\phi(d(hx_{2n},hx_{2n-1})),0,$$
$$\phi\{(d(hx_{2n+1},hx_{2n-1})/2\}\}$$
$$\leq k \max\{\phi(d_{2n}),\phi(d_{2n-1}),0,\ \phi\{d(hx_{2n-1},hx_{2n}),(hx_{2n},hx_{2n+1})\}/2\}$$
$$=k \max\{\phi(d_{2n}),\phi(d_{2n-1}),0,\ \phi\{(d_{2n-1}+d_{2n})/2\}$$

If $d_{2n}>d_{2n-1}$, for any n. Then

$$\phi(d_{2n}) \leq k\phi(d_{2n}) <\phi(d_{2n}) \text{ a contradiction.}$$

Therefore

$$\phi(d_{2n}) \leq k\phi(d_{2n-1}) <\phi(d_{2n-1}) \qquad ...(1)$$

and because ϕ is non decreasing, $d_{2n}<d_{2n-1}$.

Similarly we can show that $d_{2n+1}<d_{2n}$, so that for each n, $d_{n+1}<d_n$ and $\{d_n\}$ strictly decreasing sequence of reals. Further as

$$\phi(d_n) \leq k\phi(d_{n-1})\leq k^2\phi(d_{n-2})\leq ... \leq k^{n-1}\phi(d_1)\to 0 \text{ as } n\to\infty.$$

Since $\{d_n\}$ is a decreasing sequence of reals, it converges. Call the limit p. Suppose $p>0$, then since ϕ is continuous $\lim \phi(d_n)=\phi(p)=0$ therefore $p=0$. We now wish to show that $\{hx_n\}$ is a Cauchy sequence. Assume that it is not Cauchy.

Then for every positive number ε and for every poritive integer k there exit two positive integers $2m(k)$ and $2n(k)$ such that $2m(k)>2n(k)>k$ and $d(hx_{2m(k)},hx_{2n(k)})>\varepsilon$. Further let $2m(k)$ denote the smallest even integer for which $2m(k)>2n(k)>k$, $d(hx_{2m(k)},hx_{2n(k)})>\varepsilon$ and $d(hx_{2m(k)-2},hx_{2n(k)})\leq \varepsilon$.

Then $\phi(\varepsilon) <\phi(d(hx_{2n(k)},hx_{2m(k)}))$

$$\leq\phi(d(hx_{2n(k)},hx_{2m(k)-2})+d_{2m(k)-2}+d_{2m(k)-1})).$$

Taking limit as $n\to\infty$ we have

$$\phi(\varepsilon) = \lim \phi(d(hx_{2n(k)},hx_{2m(k)})) \ . \qquad ... (2)$$

Using triangle inequality

$$|d(hx_{2m(k)},hx_{2n(k)+1})-d(hx_{2m(k)},hx_{2n(k)})|\leq d_{2n(k)};$$
$$|d(hx_{2m(k)+1},hx_{2n(k)+1})-d(hx_{2m(k)},hx_{2n(k)+1})|\leq d_{2m(k)} \text{ and}$$
$$|d(hx_{2m(k)+1},hx_{2n(k)+2})-d(hx_{2m(k)+1},hx_{2n(k)+1})|\leq d_{2n(k)+1}.$$

These imply $\lim d(hx_{2m(k)},hx_{2n(k)+1})$

$$= \lim d(hx_{2m(k)},hx_{2n(k)}) = \lim d(hx_{2m(k)},hx_{2n(k)}) = \lim d(hx_{2m(k)+1},hx_{2n(k)+1})$$
$$= \lim\ d(hx_{2m(k)+1},hx_{2n(k)+2}).$$

Since ϕ is continuous, $\lim \phi\ (d(hx_{2m(k)},hx_{2n(k)+1})) = \lim d(hx_{2m(k)+1},hx_{2n(k)+2})$

$$= \lim d(hx_{2m(k)+1},hx_{2n(k)+2}) = \phi(\varepsilon)\ .$$

From (A),

$$\phi(d(hx_{2m(k)+1},hx_{2n(k)+2})) \leq k \max\{\phi(d(hx_{2m(k)+1},fx_{2m(k)+1}),$$
$$\phi(d(fx_{2m(k)+1},gx_{2n(k)+2})),\ \phi(d(hx_{2m(k)+2},gx_{2n(k)+2})),$$
$$\phi\{(d(hx_{2m(k)+1},gx_{2n(k)+2})+d(hx_{2n(k)+2},fx_{2m(k)+1}))/2\}$$
$$= k \max\{\phi(d_{2m(k)},\ \phi(d(hx_{2m(k)},hx_{2n(k)+1})),\ \phi(d_{2n(k)+1}),$$
$$\phi\{(d(hx_{2m(k)+1},hx_{2n(k)+1}))/2\}.$$

Taking limit as $k \to \infty$, we get

$$\phi(\varepsilon) \leq k\phi(\varepsilon) < \phi(\varepsilon), \text{ a contradiction.}$$

So $\{hx_n\}$ is a Cauchy sequence and because. X is complete, it is convergent. Call the limit u. then

$\lim hx_{2n+1} = \lim fx_{2n+1} = u$. Since f and h are compatible,

$$\lim d(fhx_{2n+1}, hfx_{2n+1}) = 0 \quad \ldots (3)$$

Also $\lim hx_{2n+2} = \lim gx_{2n+2} = u$. Since g and h are compatible,

$$\lim d(hgx_{2n+2}, ghx_{2n+2}) = 0. \quad \ldots (4)$$

We now show that the continuity of f will imply

$$fu = gu = hu.$$

From the triangle inequality

$$d(fhx_{2n+1}, ghx_{2n+2}) \leq d(fhx_{2n+1}, hfx_{2n+1}) + d(hfx_{2n+1}, hgx_{2n+2}) + d(hgx_{2n+2}, ghx_{2n+2}).$$

Taking limit as $n \to \infty$ using (3) and (4) and the continuity of f and g we have

$$d(fu, gu) \leq \lim (hfx_{2n+1}, hgx_{2n+2}).$$

From (1)

$$\phi(d(hfx_{2n+1}, hgx_{2n+2}) \leq k \max \{\phi(d(hfx_{2n+2}, ffx_{2n+1})), \phi(d(hgx_{2n+2}, ggx_{2n+2}), \phi(d(ffx_{2n+1}, ggx_{2n+2})), \phi(d(hgx_{2n+1}, ffx_{2n+2}))/2\} \quad \ldots (5)$$

From (3) and continuity of f,

$$\lim d(hfx_{2n+1}, ffx_{2n+1}) \leq \lim d(hfx_{2n+1}, fhx_{2n+1}) + \lim d(fhx_{2n+1}, ffx_{2n+1}) = 0.$$

From (4) and continuity of g,

$$\lim d(hgx_{2n+2}, ggx_{2n+2}) \leq \lim d(hgx_{2n+2}, ghx_{2n+2}) + \lim d(ghx_{2n+2}, ggx_{2n+2}) = 0.$$

From the continuity of f and g,

$$\lim (d(ffx_{2n+1}, ggx_{2n+2}) = d(fu, gu).$$

From (3) and (4) and continuity of f and g,

$$\lim \{d(hfx_{2n+1}, ggx_{2n+2})\} + d(hgx_{2n+2}, ffx_{2n+1})\}/2$$
$$\leq \lim \{d(hfx_{2n+1}, fhx_{2n+1}) + d(fhx_{2n+1}, ggx_{2n+2})\} + d(hgx_{2n+2}, ghx_{2n+2})$$
$$+ d(hgx_{2n+2}, ghx_{2n+2}) + d(ghx_{2n+2}, ffx_{2n+1})/2$$
$$= d(fu, gu).$$

Taking limit in (5) as $n \to \infty$ and using the continuity of ϕ,

$$\phi(d(fu, gu) \leq k\phi(d(fu, gu))$$

which implies that

$$fu = gu.$$

In the similar way it can be shown that $fu = hu$.

So $fu = hu = gu$.

Now

$$\phi(d(hfx_{2n+1}, hx_{2n+2})) \leq k \max \{\phi(d(hfx_{2n+1}, ffx_{2n+1})), \phi(d(hx_{2n+2}, gx_{2n+2})), \phi\{d(hfx_{2n+1}, gx_{2n+2}) + d(hx_{2n+2}, ffx_{2n+1}))/2\}.$$

Taking limit as $n \to \infty$,

$$\lim \phi\{d(hfx_{2n+1},hx_{2n+2})\} \le k\ \phi(d(fu,u))$$

i.e $\phi(d(hu,u)) \le k\ \phi(d(fu,u)) = k\ \phi(d(hu,u))$,

which implies $u=hu=fu=gu$. Let v be the another common fixed point of f, g,h. Then from (A)

$$\phi(d(u,v))=\phi(d(hu,hv)) \le k \max\{\phi(d(hu,fu));\phi(d(hv,gv)), \phi(d(fu,gv)), \phi\{d(hu,gv)+d(hv,fu)/2\}$$

$$= k \max\{\phi(d(u,v)), \phi(d(u,v)), \phi(d(u,v)),\phi(d(u,v))\}$$

Therefore, $\phi(d(u,v)) \le k \max \phi(d(u,v))$

which implies $u=v$. This completes the necessary part of the proof.

The condition is sufficient. Let $fz=gz=z$ for some $z\in X$ and define h by $hx=z \forall x\in X$. Then h is continuous from X to $g(X) \cap f(X)$. Moreover for $x\in X$, $hfx=z$, $fhx=fz=z$ and $hgx=z$, $ghx=gz=z$. So h commutes with f and g hence the mapping h and f and h and g are compatible.

further

$$\phi(d(hx,hy))=\phi(d(z,z))=\phi(0)=0.$$

Therefore

$$\phi(d(hx,hy)) \le \mathrm{k} \max\{\phi(d(hx,fx)), \phi(d(hy,gy)), \phi(d(fx,gy)), \phi(d(hx,gy)+d(hy,fx))/2\} \ \forall\ x,y\in X.$$

So, h satisfies the condition (A). This completes the proof of the theorem.

Theorem 2. Let (X,d) be complete metric space and let f,g and h be self maps on X such that for some positive integers m,n,p,

$$\phi(d(h^p x,h^p y)) \le k \max\{\phi(d(h^p x,f^m x)), \phi(d(h^p y,g^n y)), \phi(d(f^m x,g^n y)/2), \phi\{d(h^p x,g^n y)+d(h^p y,f^m y))\}$$

for all $x,y\in X$, $0 \le k<1$ and $\phi \in \Phi$.

Let $fh=hf$, $gh=hg$ and f, g be continuous. Then f, g, h have a unique common fixed point in X.

Proof. Let $H=h^p$, $F=f^m$ and $G=g^m$. Since $fh=hf$, $gh=hg$ it follows that $FH=HF$; $GH=HG$ and so F, H and G, H are compatible. Also F, G, H satisfy the inequality of Theorem 3, hence it follows that F,G and H have a unique common fixed point z, say in X.

It now remains to show that z is also unique common fixed point of f,g and h. Now

$$\phi(d(z,hz))=\phi(d(Hz,H(hz)))$$

$$\le k \max\{\phi(d(Hz,Fz)), \phi(d(H(hz),G(hz)),\phi(d(Fz,G(hz)), \phi\{d(Hz,G(hz)/2)+d(Hhz,Fz))\}$$

$$= k \max\{\phi(d(z,z)), \phi(d(hz,hz)),\phi(d(z,hz), \phi((z,hz)+d(hz,z))/2\},$$

i.e. $\phi(d(z,hz))=k\phi(d(z,hz))$

implies $\phi(d(z,hz))=0$ *implies* $hz=z$.

Next we show that $hz=fz=gz$. Now,

$$\phi(d(hz,fz))=\phi(d(Hhz,Hfz))$$
$$\leq k \max\{\phi(d(Fhz,Hhz)),\ \phi(d(Gfz,Hfz))\phi(d(Fhz,Gfz)),$$
$$\phi\{\ d(Hz,G(hz)+d(Hhz,Fz))/2\}$$
$$=k \max\{\phi(d(z,z)),\ \phi(d(hz,hz))\phi(d(z,hz)),\ \phi\{(d(z,hz))/2+d(hz,z))\}$$

i.e. $\phi(d(z,hz))\leq\phi(d(z,hz))$ implies $\phi(d(z,hz))=0$ implies $hz=z$.

Next we show that $hz=fz=gz$. Now,

$$\phi(d(hz,fz))=\phi(d(Hhz,Hfz))$$
$$\leq k \max\{\phi(d(Fhz,Hhz)),\ \phi(d(Gfz,Hfz))\phi(d(Gfz,hfz)),$$
$$\phi\{(d(Fhz,Hfz)+d(Gfz,Hhz))/2\}$$
$$=k \max\{\phi(d(FZ,HZ)),\ \phi(d(fz,fz))\phi(d(hz,fz)),\ \phi\{(d(hz,fz)/2+d(fz,hz))\}$$
$$=k\ \phi\ (d(fz,hz))$$

which implies

$$\phi(d(gz,hz))=0 \text{ implies } d(fz,\ hz)=0$$

implies

$$fz=hz.$$

Similarly we can show that $gz=hz$ and $gz=hz=fz=z$.

The unicity part of the theorem follows easily.

Fixed point for a sequence of mappings.

Theorem 3. Let (X,d) be a complete metric space and let $\{T_i\}$ be a sequence of continuous mappings on X into itself. Then a mapping $h:X\to X$ has a unique fixed point in X if and only if

$$\phi(d(hx,hy)) \leq k \max\{\phi(d(hx,T_ix)),\phi(d(hy,T_jy)),\phi(d(T_ix,T_jy)),$$
$$\phi(d(hx,T_jy)/2+d(hy,T_ix))\}$$

for all $x,y\in X$, $0 \leq k<1$; $i, j\in N$ and h is compatible with each $T_i\in\{T_i\}$ and h have a unique common fixed point in X.

Proof. Result follows from the Theorem 1.

REFERENCES

[1] G. Jungck, Compatible mappings and common fixed points, *International J. Math. Math Sci.*, **9** (1986), 771.

[2] G. Jungck and B.E. Rhoades, Fixed point theorems for Compatible mappings, *International J. Math and Math Sci.*, **16** (1993), 417.

[3] M.S. Khan, M. Swaleh and S. Sessa, Fixed point theorems altering distance between the points, *Bull. Anstr. Math. Soc.* **30** (1984), 1.

[4] S. Park, A unified approach to fixed points of contractive maps. *J. Korean Math Soc.*, **16** (1980), 95.

[5] S. Park, Fixed points and periodic points of contractive pairs of maps, *Proc. College Nat. Sci, Seol Nat. Univ.* **5** (1980), 9.

[6] H.K. Pathak and R. Sharma, A note on fixed point theorems of Khan, Swaleh and Sessa. *The Mathematics Education,* **28** (1994), 151.

[7] B.E. Rhoades, Kalishankar Tiwary and G.N. Singh, A common fixed point Theorems for compatible mappings, *Indian J. Pure Appl. Math.* **26** (1995), 403.

[8] K.P.R. Sastri and G.V.R. Babu, Fixed point theorems in metric space by altering distances, *Bull. Cal. Math. Soc.* **90** (1998), 175.

[9] K.P.R. Sastri and G.V.R. Babu, Some fixed point theorems by altering distance between the points, *Indian J. Pure Appl. Math.* **30**, (1999), 6.

[10] K.P.R. Sastri, G.V.R. Babu and D. Narayana Rao, Fixed point Theorems in Complete metricspace by using a continuous control functions, *Bull. Cal. Math. Soc.*, **9** (1999), 493.

[11] S. Sessa, On a weak Commutativity Condition of mappings in fixed point consideration, *Publ. Inst. Math.*, **32**, (1982), 149.