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A THEOREM ON LOCALIZATION OF $|N,p_n|$ SUMMABILITY OF A FACTORED FOURIER SERIES

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ABSTRACT

In the present paper, our object is to investigate a suitable type of factor so that the summability $|N,p_n|$ of the factored Fourier series becomes a local property. It may be observed that for $p_n=1/(n+1)$, our theorem reduces to the results of a well known theorem due to Lal (1963) on the absolute harmonic summability of the factored Fourier series.

1. Introduction. Let Σu_n be an infinite series with the sequence $\{S_n\}$ of its partial sums. Let $\{p_n\}$ be sequence of constants with $p_n>0$ and

$$P_n = \sum_{m=0}^{n} p_m.$$

The sequence-to sequence transformation given by

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}S_k \tag{1.1}$$

defines the n^{th} Nörlund mean or the n^{th} (N,p_n) mean of the sequence $\{S_n\}$ of partial sums of the series Σu_n.

If t_n tends to a fixed and finite sum S as $n\to\infty$, then the series Σu_n or the sequence $\{S_n\}$ of its partial sums is said to be summable (N,p_n) to the sum S [Hardy [2]]. The conditions for the regularity of the summability (N,p_n) defind by (1.1) are

$$\lim_{n\to\infty}(p_n/P_n) \text{ and } \sum_{v=0}^{n}|p_v| = o(p_n).$$

For $p_n = 1/(n+1), P_n \sim log n,$

the Nörlund mean (N,p_n) reduces to the harmonic mean $(N+1/(n+1))$ [Riesz [5]].

The series Σu_n is said to be absolutely summable (N,p_n) or simply summable $|N,p_n|$ if the sequence $\{t_n\}$ is of bounded variation. Mears [4].

Let $f(t)$ be a 2π-periodic and Lebesgue integrable function of t in the inerval $(-\pi,\pi)$. Without any loss of generality, we may assume that the constant term in the Fourier series of $f(t)$ is zero, i.e $\int_{-\pi}^{\pi} f(t)dt = 0$,

and then

$$f(t) \sim \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t).$$

We use the following notations :

$\phi(t) = [f(x+t) + f(x-t) - 2f(x)]/2,$

$$\Phi_\alpha(t) = \frac{1}{\Gamma\alpha}\int_0^t (t-u)^{\alpha-1}\phi(u)du, \qquad (\alpha > 0)$$

$\Phi_0(t) = \phi(t),$

$\phi_n(t) = \Gamma(\alpha+1)t^{-\alpha}\Phi_\alpha(t), \qquad (0 \le \alpha \le 1)$

$\Delta p_n = p_n - p_{n+1}.$

Lal [3] has discussed the localization problem for $|N, 1/(n+1)|$ summability of the factored Fourier series by proving the following

Theorem A. If $\{\lambda_n\}$ is a convex sequence such that the serise $\sum n^{-1}\lambda_n$ is convergent, then the summability $|N, 1/(n+1)|$ of the series $\sum (A_n(t) \log(n+1)\lambda_n / n)$ at a point can be ensured by a local property.

Applying the absolute Nörlund summability method, which is more genral than $|N, 1/(n+1)|$ summability, the object of this paper is to investigate a suitable type of factor so that the summability $|N, p_n|$ of the factored Fourier series becomes a local property.

In what follows we establish the following main

Theorem. If $\{p_n\}$ and $\{\Delta p_n\}$ are both non-negative, monotonic, non-increasing and $\{\lambda_n\}$ is a convex sequence such that is $\sum p_n\lambda_n$ is convergent, then $|N, p_n|$ summability of $\Sigma A_n(t)\lambda_n P_n p_n$ depends only on the behaviour of the generating function $f(t)$ in the immediate neighbourhood of the points $t=x$, provided

$$\sum_{k=0}^{m-2}\left|p_{k+1}^2\lambda_{k_{+1}}\right| = o(1) \tag{1.2}$$

and

$$\sum_{k=0}^{m-2} P_{k+1}p_{k+1}\Delta\lambda_{k_{+1}} = o(1). \tag{1.3}$$

2. **Proof of the Main Theorem.** To prove our theorem we require the following

Lemma (Basanquet and Kastelman [1] Theorem 1).

Suppose that $f_n(x)$ is measurable in (a,b) where $b\text{-}a \leq \infty$ for $n=1,2,\ldots$. Then a necessary and sufficient condition that, for every function $\lambda(x)$ integrable (L) over (a,b) the functions $f_n(x)\lambda(x)$ should be integrable (L) in (a,b) and

$$\sum_{n=0}^{\infty}\left|\int_a^b \lambda(x) f_n(x)dx\right| \leq K,$$

is that

$$\sum_{n=1}^{\infty}|f_n(x)| \leq K,$$

where K is an absolute constant, for almost every x in (a,b).

Since

$$t_n = \frac{1}{P_n}\sum_{\upsilon=0}^{n} P_{n-\upsilon}u_\upsilon = \frac{1}{P_n}\sum_{\upsilon=0}^{n} P_\upsilon u_{n-\upsilon}, \tag{2.1}$$

where

$$u_n = A_n(t)p_n P_n \lambda_n,$$

we have

$$\begin{aligned} t_n - t_{n-1} &= \sum_{\upsilon=0}^{n-1}(P_\upsilon/P_n - P_{\upsilon-1}/P_{n-1})u_{n-\upsilon} \\ &= \frac{1}{P_n P_{n-1}}\sum_{\upsilon=0}^{n-1}(P_n p_\upsilon - P_\upsilon p_n)u_{n-\upsilon} \\ &= \frac{1}{P_n P_{n-1}}\sum_{\upsilon=0}^{n-1}(P_n p_{n-\upsilon-1} - P_{n-\upsilon-1}p_n)u_{\upsilon+1}. \end{aligned} \tag{2.2}$$

For the Fourier series of $f(t)$ at $t=x$

$$A_n(t) = \frac{2}{\pi}\int_0^\pi \phi(t) cosntdt,$$

so that

$$\begin{aligned} t_n - t_{n-1} &= \frac{2}{\pi}\int_0^\pi \phi(t)\left\{\frac{1}{P_n P_{n-1}}\sum_{k=0}^{n-1}(P_n p_{n-k-1} - P_{n-k-1}p_n)P_{k+1}p_{k+1}\lambda_{k+1}\cos(k+1)t\right\}dt \\ &= \frac{2}{\pi}\int_0^\pi \phi(t)K(n,t)dt, say \end{aligned} \tag{2.3}$$

Hence

$$\sum_{n=2}^{\infty}|t_n - t_{n-1}| \leq \sum_{n=2}^{\infty}\left|\frac{2}{\pi}\int_0^\delta \phi(t)K(n,t)dt\right| + \sum_{n=2}^{\infty}\left|\frac{2}{\pi}\int_\delta^\pi \phi(t)K(n,t)dt\right|.$$

In order to prove the theorem, it is now required prove that

$$\sum_{n=2}^{\infty}\left|\frac{2}{\pi}\int_\delta^\pi \phi(t)K(n,t)dt\right| < \infty \tag{2.4}$$

But by virtue of the lemma, it is a sufficient for our purpose to show that

$$\sum_{n=2}^{\infty}|K(n,t)|dt \leq A, \tag{2.5}$$

for $0<\delta\leq t\leq\pi$, where A is a positive constant, not necessarily the same at each occurrence.

Now

$$\sum_{n=2}^{m}|K(n,t)|=\sum_{n=2}^{m}\frac{1}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}(P_np_{n-k-1}-P_{n-k-1}p_n)P_{k+1}p_{k+1}\lambda_{k+1}\cos(k+1)t\right|$$

$$=\sum_{n=2}^{m}\frac{1}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}M(n,k,t)\right|,\quad \text{say.} \tag{2.6}$$

Applying Abel's transformation, we get

$$\sum_{k=0}^{n-1}M(n,k,t)=\sum_{k=0}^{n-2}\left[\Delta\{(P_np_{n-k-1}-P_{n-k-1}p_n)P_{k+1}p_{k+1}\lambda_{k+1}\}\sum_{\upsilon=0}^{k}\cos(\upsilon+1)t\right]$$

$$+(P_np_0-P_0p_n)P_np_n\lambda_n\sum_{\upsilon=0}^{n-1}\cos(\upsilon+1)t\ . \tag{2.7}$$

Therefore, for $0<\delta\leq t\leq\pi$, we have

$$\left|\sum_{k=0}^{n-1}M(n,k,t)\right|\leq A\sum_{k=0}^{n-2}[\Delta\{(P_np_{n-k-1}-P_{n-k-1}p_n)P_{k+1}p_{k+1}\lambda_{k+1}\}]+AP_n^2p_n\lambda_n\ . \tag{2.8}$$

Clearly

$$\sum_{n=2}^{m}\frac{1}{P_nP_{n-1}}.AP_n^2p_n\lambda_n\leq A\sum_{n=2}^{m}p_n\lambda_n=o(1) \tag{2.9}$$

as $m\to\infty$, since $\sum p_n\lambda_n$ has been assumed to be convergent, so that the sequence of its partial sums is bounded.

Further

$$\sum_{k=0}^{n-2}|\Delta|\{(P_np_{n-k}-P_{n-k-1}p_n)P_{k+1}p_{k+1}\lambda_{k+1}\}|$$

$$\leq\sum_{k=0}^{n-2}|\Delta(P_np_{n-k-1}-P_{n-k-1}p_n)|P_{k+1}p_{k+1}\lambda_{k+1}+\sum_{k=0}^{n-2}|(P_np_{n-k-1}-P_{n-k-2}p_n)\Delta(P_{k+1}p_{k+1}\lambda_{k+1})| \tag{2.10}$$

Hence

$$\sum_{n=2}^{m}\frac{1}{P_nP_{n-1}}\sum_{k=0}^{n-2}\Delta|(P_np_{n-k-1}-P_{n-k-1}p_n)|P_{k+1}p_{k+1}\lambda_{k+1}$$

$$=\sum_{k=0}^{m-2}(P_{k+1}p_{k+1}\lambda_{k+1})\sum_{n=k+2}^{m}\frac{|\Delta(P_np_{n-k-1}-P_{n-k-1}p_n)|}{P_nP_{n-1}}$$

$$\leq\sum_{k=0}^{m-2}(P_{k+1}p_{k+1}\lambda_{k+1})\sum_{n=k+2}^{m}\left[\frac{|\Delta p_{n-k-1}|}{p_{n-1}}+\frac{p_{n-k-1}p_n}{P_nP_{n-1}}\right]$$

$$= o\left[\sum_{k=0}^{m-2}(p_{k+1}\lambda_{k+1})\sum_{n=k+2}^{m}\left|\Delta p_{n-k-1}\right|\right]+o\left[\sum_{k=0}^{m-2}(P_{k+1}p_{k+1}\lambda_{k+1})\sum_{n=k+2}^{m}\frac{p_n}{p_n p_{n-1}}\right]$$

$$= o\left[\sum_{k=0}^{m-2}p_{k+1}\lambda_{k+1}\right]$$

$$=o(1), \text{ as } m\to\infty \,. \tag{2.11}$$

Again

$$\sum_{k=0}^{n-2}\left|(P_n p_{n-k-2}-P_{n-k-2}p_n)\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|$$

$$\le \sum_{k=0}^{n-2}(P_n-P_{n-k-2})p_n\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|+\sum_{k=0}^{n-2}(p_{n-k-2}-p_n)P_n\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right| \tag{2.12}$$

But

$$\sum_{n=2}^{m}\frac{1}{P_nP_{n-1}}\sum_{k=0}^{n-2}(P_n-P_{n-k-2})\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|$$

$$=\sum_{k=0}^{n-2}\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|\sum_{n=k+2}^{m}(P_n-P_{n-k-2})\Delta(1/p_{n-1})$$

$$\le A\sum_{k=0}^{m-2}\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|P_{k+2}\sum_{n=k+2}^{m}\Delta(1/p_{n-1})$$

$$=o\left[\sum_{k=0}^{m-2}\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|\right]$$

$$=o\left[\sum_{k=0}^{m-2}\left|p_{k+1}^2\lambda_{k+1}\right|\right]+o\left[\sum_{k=0}^{m-2}P_{k+1}p_{k+1}\Delta\lambda_{k+1}\right]+o\left[\sum_{k=0}^{m-2}P_{k+1}\lambda_{k+1}\Delta p_{k+1}\right]$$

$=o(1)+o(1)$, (using (1.2) and (1.3))

$$=o(1) \tag{2.13}$$

as $m\to\infty$, since (P_n-P_{n-k-2}) decreases as n increases.

Also

$$\sum_{n=2}^{m}\frac{1}{P_nP_{n-1}}\sum_{k=0}^{n-2}(p_{n-k-2}-p_n)P_n\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|$$

$$=\sum_{n=2}^{m}\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}(p_{n-k-2}-p_n)\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|$$

$$\le\sum_{n=2}^{m}\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}(p_{n-k-2}-p_{n-k-1})(k+2)\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|$$

$$=\sum_{k=0}^{m-2}(k+2)\left|\Delta(P_{k+1}p_{k+1}\lambda_{k+1})\right|\sum_{n=k+2}^{m}\left(\frac{p_{n-k-2}-p_{n-k-1}}{P_{n-1}}\right)$$

$$\leq \sum_{k=0}^{m-2} \frac{k+2}{P_{k+1}} \left|\Delta\left(P_{k+1} p_{k+1} \lambda_{k+1}\right)\right| \sum_{n=k+2}^{m} \left(p_{n-k-2} - p_{n-k-1}\right)$$

$$= o\left[\sum_{k=0}^{m-2} \frac{(k+2)p_{k+1}}{P_{k+1}} \left(p_{k+1}^{2} \lambda_{k+1}\right)\right] + o\left[\sum_{k=0}^{m-2} P_{k+1} p_{k+1} \Delta\lambda_{k+1}\right] + o\left[\sum_{k=0}^{m-2} P_{k+1} \Delta p_{k+1} \lambda_{k+1}\right]$$

$=o(1)$ as $m \to \infty$ in (2.13); [sinc $np_n \leq P_n$]. (2.14)

Now the required result in (2.5) follows with the help of the results from (2.6) to (2.14) and this completes the proof of our main thorem.

REFERENCES

[1] L.S. Bosanqnet and H. Kastelman, The absolute convergence of a series of integrals, *Proc. Lond. Math. Soc.*, **45** (1939) 88-97.

[2] G.H. Hardy, *Divergent Series*, Oxfrod (1959).

[3] S.N. Lal , On the absolute harmonic summability of the factored Fourier series. *Proc. Amer. Math. Soc.*, **14** (1963), 311-319.

[4] F.M. Mears, Some multiplication theorems for the Nörlund mean. *Bull. Amer. Math. Soc.*, **41** (1935), 875-880.

[5] M. Riesz, Sur l' equivalence de certain methods le sommation, *Proc. Lond. Math. Soc*, **22** (2), (1924), 412-419.