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DIFFERENTIAL RECURRENCE RELATIONS FOR BIORTHOGONAL TYPE POLYNOMIALS AND THEIR APPLICATIONS

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ABSTRACT

In the present paper, we define biorthogonal type polynomials and establish their differential recurrence relations through special generating functions and then, making an appeal to these relations, we obtain some more generating functions through Lie group theory.

1. Introduction. The biorthogonal polynomials play an important role in the theory of Approximation, Queueing, Coding and other branches of Applied Mathematics and Physics. They are used as weight function. Konhauser [6] has defined rather first the biorthogonal polynomials of degree n such that

$$Z_n^\alpha(x,k)=\frac{\Gamma(kn+\alpha+1)}{n!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{x^{kr}}{\Gamma(kr+\alpha+1)}.$$

Again, Carlitz [3], Prabhakar [7,8], Srivastava [10], Srivastava and Singhal [12] and Thakre [13] studied various biorthogonal polynomials along with generating functions, recurrence relations, orthogonal property and other relevant properties.

Motivating by above work in the present paper we define biorthogonal type polynomials such that

$$Y_n^\alpha(x,k)=\frac{\Gamma(1+\alpha+nk)}{n!}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(1+r)}{\Gamma(1+kr)}\frac{\Gamma(1+n-r)}{\Gamma(1+\alpha+kr)}\frac{\left(-x^k\right)^r}{\Gamma(1+nk-kr)}. \tag{1.1}$$

It is evident that for $k=1$

$$Z_n^\alpha(x,1)=Y_n^\alpha(x,1)=L_n^\alpha(x). \tag{1.2}$$

We also take the generating function concept due to Bateman [1,2] and Rainville [9] to establish the differential recurrence relations for the biorthogonal polynomials and make their applications to find further generating functions of these polynomials with the help of Lie group theory (see Srivastava and Monocha [11]).

2. Theorems. Before evaluation of recurrence relations for the biorthogonal polynomials defined by (1.1), we prove two theorems here, in this section

Theorem 1. From $e^t\psi(x,t)=\sum_{n=0}^{\infty}\sigma_n(x,k)t^{nk}$,

it follows that $\sigma'_0(x,k)=0$ and for $n\geq 1, k>0;$

$$x\sigma'_n(x,k)=nk\sigma_n(x,k)-\sigma_{n-1/k}(x,k). \tag{2.1}$$

Proof. Consider the generating function

$$e^t\psi(x,t)=\sum_{n=0}^{\infty}\sigma_n(x,k)t^{nk}=F \tag{2.2}$$

Then,

$$\partial F/\partial x=te^t\psi'(x,t) \tag{2.3}$$

and

$$\partial F/\partial t=e^t\psi(x,t)+xe^t\psi'(x,t). \tag{2.4}$$

On eliminating $\psi(x,t),\psi'(x,t)$ due to (2.2), (2.3) and (2.4), we get

$$x(\partial F/\partial x)-t(\partial F/\partial t)+tF=0 \tag{2.5}$$

Alternatively, (2.5) may be written as

$$\sum_{n=0}^{\infty}x\sigma'_n(x,k)t^{nk}-\sum_{n=0}^{\infty}nk\sigma_n(x,k)t^{nk}+\sum_{n=0}^{\infty}\sigma_n(x,k)t^{nk+1}=0. \tag{2.6}$$

Further (2.6) may be put in the form

$$\sum_{n=0}^{\infty}x\sigma'_n(x,k)t^{nk}-\sum_{n=0}^{\infty}nk\sigma_n(x,k)t^{nk}+\sum_{n=1/k}^{\infty}\sigma_{n-1/k}(x,k)t^{nk}=0. \tag{2.7}$$

Finally, from the relation (2.7), we get the required result

$$x\sigma'_n(x,k)=nk\sigma_n(x,k)-\sigma_{n-1/k}(x,k).$$

Theorem 2. From $A(t)exp(-xt/(1-t))=\sum_{n=0}^{\infty}\rho_n(x,k)t^{nk}$,

it follows that $\rho'_0(x,k)=0, n\geq 1$ and $k>0$,

$$\rho'_n(x,k)=\rho'_{n-1/k}(x,k)-\rho_{n-1/k}(x,k) \tag{2.8}$$

and

$$\rho'_n(x,k) = -\sum_{m=0}^{n-1/k} \rho_m(x,k). \tag{2.9}$$

Proof. Consider the generating function

$$A(t)exp(-xt/(1-t)) = \sum_{n=0}^{\infty} \rho_n(x,k) t^{nk} = F, \tag{2.10}$$

we have

$$(1-t)(\partial F/\partial x) = -tF. \tag{2.11}$$

Hence, we get

$$\sum_{n=0}^{\infty} \rho'_n(x,k) t^{nk} - \sum_{n=0}^{\infty} \rho'_n(x,k) t^{nk+1} = -\sum_{n=0}^{\infty} \rho_n(x,k) t^{nk+1}, \tag{2.12}$$

which yields $\rho'_n(x,k) = 0$ and for $n \geq 1,\ k>0$,

$$\rho'_n(x,k) = \rho'_{n-1/k}(x,k) - \rho_{n-1/k}(x,k). \tag{2.13}$$

Again, rearranging (2.11) in the form

$$\partial F/\partial x = -(t/(1-t))F, \tag{2.14}$$

so that we obtain

$$\sum_{n=0}^{\infty} \rho'_n(x,k) t^{nk} = -\left(\sum_{n=0}^{\infty} t^{n+1}\right)\left(\sum_{n=0}^{\infty} \rho_n(x,k) t^{nk}\right) = -\sum_{n=0}^{\infty}\sum_{m=0}^{n} \rho_m(x,k) t^{nk+1}$$

$$= -\sum_{n=0}^{\infty}\sum_{m=0}^{n-1/k} \rho_m(x,k) t^{nk}$$

Hence, for $n \geq 1,\ k>0$, we get

$$\rho'_n(x,k) = -\sum_{m=0}^{n-1/k} \rho_m(x,k). \tag{2.15}$$

3. Applications of the theorems to obtain the differential recurrence relations. Consider the generating relation

$$e^t {}_0F_1(-;1+\alpha;-xt) = \sum_{n=0}^{\infty} \frac{Y_n^\alpha(x,k) t^{nk}}{(1+\alpha)_{nk}} \tag{3.1}$$

and theorem 1, we get the relation

$$\frac{x}{(1+\alpha)_{nk}} \frac{d}{dx} Y_n^\alpha(x,k) = \frac{nk}{(1+\alpha)_{nk}} Y_n^\alpha(x,k) - \frac{nk}{(1+\alpha)_{nk-1}} Y_{n-1/k}^\alpha(x,k), \tag{3.2}$$

which readily yields the relation

$$x(d/dx)Y_n^{\alpha}(x,k) = nkY_n^{\alpha}(x,k) - (\alpha+nk)Y_{n-1/k}^{\alpha}(x,k). \tag{3.3}$$

On replacing n by $n+1/k$, we find

$$x(d/dx)Y_{n+1/k}^{\alpha}(x,k) = (nk+1)Y_{n+1/k}^{\alpha}(x,k) - (1+\alpha+nk)Y_n^{\alpha}(x,k). \tag{3.4}$$

Further, consider the generating relation

$$(1-t)^{-1-\alpha} exp(-xt/(1-t)) = \sum_{n=0}^{\infty} Y_n^{\alpha}(x,k)t^{nk}, \tag{3.5}$$

and theorem 2, we find

$$\frac{d}{dx}Y_n^{\alpha}(x,k) = \frac{d}{dx}Y_{n-1/k}^{\alpha}(x,k) - Y_{n-1/k}^{\alpha}(x,k). \tag{3.6}$$

Now, replacing n by $n+1/k$, we get

$$\frac{d}{dx}Y_{n+1/k}^{\alpha}(x,k) = \frac{d}{dx}Y_n^{\alpha}(x,k) - Y_n^{\alpha}(x,k) \tag{3.7}$$

Again, making an appeal to (3.4) and (3.7), we obtain the relation

$$x\frac{d}{dx}Y_n^{\alpha}(x,k) = (nk+1)Y_{n+1/k}^{\alpha}(x,k) - (1+\alpha+nk-x)Y_n^{\alpha}(x,k). \tag{3.8}$$

4. Applications of the differential recurrence relations to obtain further generating functions by Lie thoery. To find the further generating functions due to Lie theory, we choose following relations for the eigen function $Y_n^{\alpha}(x,k)y^{kn}$ as

$$A\left[Y_n^{\alpha}(x,k)y^{kn}\right] = a_{n,k}Y_n^{\alpha}(x,k)y^{kn}, \tag{4.1}$$

$$B\left[Y_n^{\alpha}(x,k)y^{kn}\right] = b_{n,k}Y_{n-1/k}^{\alpha}(x,k)y^{kn-1}, \tag{4.2}$$

$$C\left[Y_n^{\alpha}(x,k)y^{kn}\right] = C_{n,k}Y_{n+1/k}^{\alpha}(x,k)y^{kn+1}, \tag{4.3}$$

provided that A,B,C are first order differential operators and $a_{n,k}$, $b_{n,k}$ and $c_{n,k}$ are the expressions in n and k, and are independent of x and y but not necessarily of α.

Now, making an appeal to the differential recurrence relations (3.3) and (3.8) and Weisner's technique [14-16], we get the operators

$$A = y(\partial/\partial y) + (\alpha+1)/2, \tag{4.4}$$

$$B = xy^{-1}(\partial/\partial x) - \partial/\partial y, \tag{4.5}$$

and $$C = xy(\partial/\partial x) + y^2(\partial/\partial y) + (1+\alpha-x)y. \tag{4.6}$$

The commutation relations due to above operators follow

$[A,B] = AB-BA = -B,$ (4.7)

$[A,C] = AC-CA = C,$ (4.8)

and $[C,B] = CB-BC=2A.$ (4.9)

Again, we consider, here, an operator

$$D=CB+A^2-A$$

$$= x[x(\partial^2/\partial x^2)+(1+\alpha-x)(\partial/\partial x)+y(\partial/\partial y)]+(\alpha^2-1)/4 \quad (4.10)$$

which commutes with A,B and C.

Again, suppose $f(x,y)$ is a common eigen function of the operators D and B and that satisfy the equations

$Df(x,y)=(\alpha^2-1)f(x,y)/4,$ (4.11)

and

$Bf(x,y)=-f(x,y),$ (4.12)

which gives us the partial differential equations

$$\left[x\frac{\partial^2}{\partial x^2}+(1+\alpha-x)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right]f(x,y)=0 \quad (4.13)$$

and

$$\left(xy^{-1}\frac{\partial}{\partial x}-\frac{\partial}{\partial y}-1\right)f(x,y)=0 \quad (4.14)$$

Setting $f(x,y)=e^yR(x.y)$ in (4.13) and with the use of (4.14), we get (4.15)

$$\left[u\frac{d^2}{du^2}+(1+\alpha)\frac{d}{du}+1\right]R(u)=0, \text{ where } u=xy. \quad (4.16)$$

Due to Srivastava and Manocha [§ 1.4 (11)], (4.16) gives the solution

$$R(u)=\Gamma(\alpha+1)u^{-\alpha/2}J_\alpha(2\sqrt{u})={}_0F_1\left[\begin{matrix}-\,;\\1+\alpha\,;\end{matrix}-u\right]. \quad (4.17)$$

Therefore, from (4.15) and (4.17), we have

$$f(x,y)=e^y{}_0F_1\left[\begin{matrix}-\,;\\1+\alpha\,;\end{matrix}-xy\right] \quad (4.18)$$

Now if $\mathbb{G}$ and $\mathbb{G}'$ are the Lie algebras of local Lie groups G and G' respectively and then G and G' are isomorphic (see Srivastava and Manocha [11] thorem 5 p. 319).

Let T denotes the local isomorphism of G onto a neighbourhood of the identity element in G'. Then T is a multiplier representation of G on the representation space $A_{x\beta}$. Particularly, if $L_\beta \varepsilon G'$ is an isomorphic image of $\beta \varepsilon G$, we have

$$\left[T\left(e^{\beta t}\right)f\right](x) = exp\left(tL_\beta\right)f(x) = \sum_{n=1}^{\infty}\frac{t^n}{n!}\left[L_\beta^n f\right](x), \tag{4.19}$$

where t lies in a sufficiently small neighbourhood of $0 \ \varepsilon \ \mathbb{C}$.
Now, to determine the multiplier representation

$$[T(g)f](x,y) \text{ with the aid (4.19), where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon SL_2. \tag{4.20}$$

corresponding to A,B and C operators (See Srivastava and Manocha [§6.4 (18)]). we have

$$[T(g)f](x,y) = (by+d)^{-\alpha-1} exp\left(\frac{bxy}{by+d}\right).f\left(\frac{xy}{(ay+c)(by+d)}, \frac{ay+c}{by+d}\right)$$

$$\text{where } |by/d| < 1, |c/ay| < 1 \tag{4.21}$$

and g given by (4.20) with $ad-bc=1$, lies in a sufficiently small neighbourhood

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \varepsilon SL_2. \tag{4.22}$$

Now, making an appeal to (4.18), (4.19), (4.20) and (4.21) with the condition (4.22), we find

$$[T(g)f](x,y) = (by+d)^{-\alpha-1} exp\left(\frac{ay+bxy+c}{by+d}\right) {}_0F_1\left[\begin{matrix} - \ ; & -xy \\ 1+\alpha\,; & (by+d)^2 \end{matrix}\right] \tag{4.23}$$

which satisfies

$$D[T(g)f](x,y) = (1/4)(\alpha^2-1)[T(g)f](x,y) \tag{4.24}$$

Since $[T(g)f](x,y)$ is analytic at $y=0$, hence we have

$$[T(g)f](x,y) = \sum_{n=0}^{\infty} p_{n,k}(g) L_n^{\alpha}(x,k) y^{nk} \tag{4.25}$$

To compute the coefficient $p_{n,k}(g)$, we put $x=0$ in (4.25) and with the aid of (4.23) we have

$$(by+d)^{-\alpha-1} exp\left(\frac{ay+c}{by+d}\right) = \sum_{n=0}^{\infty} \frac{(1+\alpha)_{nk}}{(nk)!} p_{n,k}(g) y^{nk}. \tag{4.26}$$

from which, on making an appeal to (3.5), we derive

$$(by+d)^{-\alpha-1} exp\left(\frac{ay+c}{by+d}\right) = e^{c/d} d^{-1-\alpha} (-b/d)^{nk} Y_n^{\alpha}(1/(bd),k) y^{nk}. \tag{4.27}$$

Now from (4.26) and (4.27), we obtain

$$p_{n,k}(g) = \frac{\Gamma(1+\alpha)\Gamma(1+nk)}{\Gamma(1+\alpha+nk)} e^{c/d} d^{-1-\alpha} (-b/d)^{nk} . Y_n^{\alpha}(1/(bd),k). \tag{4.28}$$

Thus, making an appeal to (4.23), (4.25) and (4.28), we get

$$\left(1+\frac{by}{d}\right)^{-\alpha-1} exp\left[\frac{y(1+bdx)}{d(by+d)}\right] {}_0F_1\left[\begin{matrix} - ; & \frac{-xy}{(by+d)^2} \\ 1+\alpha ; & \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(1+nk)\Gamma(1+\alpha)}{\Gamma(1+\alpha+nk)} Y_n^{\alpha}(x,k) Y_n^{\alpha}\left(\frac{1}{bd},k\right)\left(\frac{-by}{d}\right)^{nk}; \left|\frac{by}{d}\right| < 1. \tag{4.29}$$

Again, setting $b=i/\sqrt{w}$ and $d=-i/\sqrt{w}$, $i=\sqrt{(-1)}$ in (4.29), we find

$$(1-y)^{-\alpha-1} exp\left[\frac{-(x+w)y}{(1-y)}\right] {}_0F_1\left[\begin{matrix} - ; & \frac{wxy}{(1-y)^2} \\ 1+\alpha ; & \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(1+\alpha)\Gamma(1+nk)}{\Gamma(1+\alpha+nk)} Y_n^{\alpha}(x,k) Y_n^{\alpha}(w,k) y^{nk}; |y| < 1. \tag{4.30}$$

5. Special Case. For $k=1$, (4.30) with the appeal to (1.1) and (1.2), gives the formula due to Hardy [4] and Hille [5]

$$(1-y)^{-\alpha-1} exp\left[\frac{-(x+w)y}{(1-y)}\right] {}_0F_1\left[\begin{matrix} - ; & \frac{wxy}{(1-y)^2} \\ 1+\alpha ; & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n} L_n^{\alpha}(x) L_n^{\alpha}(w) y^n; |y| < 1 \tag{5.1}$$

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