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ON DECOMPOSITION CURVATURE TENSOR FIELDS IN A TACHIBANA RECURRENT SPACE

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ABSTRACT

In the present paper, we prove some theorems on decomposition curvature tensor fields in a Tachibana recurrent space.

1. Introduction. A Tachibana space T_n is a Riemannian space, which admits a structure tensor field F_i^h satisying the conditions [4]

$$F_j^h F_h^i = \delta_j^i, \quad ...(1.1)$$

$$F_{ij} = -F_{ji}\left(F_{ij} = F_i^a g_{aj}\right) \quad ...(1.2)$$

and

$$F_{ij}^n = 0, \quad ...(1.3)$$

where the comma (,) followed by an index denotes the operation of covariant differentiation with respect to the Riemannian metric tensor g_{ij}.

The Riemannian curvature tensor field R_{ijk}^h is defined by

$$R_{ijk}^h = \partial_i \begin{pmatrix} h \\ jk \end{pmatrix} - \partial_j \begin{pmatrix} h \\ ik \end{pmatrix} \begin{pmatrix} h \\ il \end{pmatrix} \begin{pmatrix} l \\ jk \end{pmatrix} - \begin{pmatrix} h \\ jl \end{pmatrix} \begin{pmatrix} l \\ jk \end{pmatrix}, \quad ...(1.4)$$

where $\partial_i \equiv \partial / \partial x^i, \{x^i\}$ denotes real local coordinates.

The Ricci tensor and scalar curvature are given by $R_{ij} = R_{aij}^a$ and $R = R_{ij} g^{ij}$ respectively.

It is well known that these tensors satisfy the following identities [3] :

$$R_{ijk,a}^h = R_{jk,i} - R_{ikj} , \quad ...(1.5)$$

$$R_i = 2R_{i,a}^a , \quad ...(1.6)$$

$$F_i^a R_{aj} = -R_{ia} F_j^a , \quad ...(1.7)$$

and

$$F_i^a R_a^j = R_i^a F_a^j . \quad ...(1.8)$$

The Bianchi identities in T_n are given by

$$R_{ijk}^h + R_{jki}^h + R_{kij}^h = 0, \quad ...(1.9)$$

and

$$R^h_{ijk,a} + R^h_{ika,j} + R^h_{iaj,k} = 0 \qquad ...(1.10)$$

A Tachibana space is said to be Tachibana recurrent if its curvature tensor field satisfies the condition [1]

$$R^h_{ijk,a} = \lambda_a R^h_{ijk}, \qquad ...(1.11)$$

where λ_a is a non-zero recurrence vector field.

The following relations follow immediately from equation (1.11):

$$R_{ik,a} = \lambda_a R_{ik} \qquad ...(1.12)$$

and

$$R_{,a} = \lambda_a R. \qquad ...(1.13)$$

2. Decomposition of Curvature Tensor Field R^h_{ijk}. Consider the docomposition of recurrent curvature tensor field R^h_{ijk} in the following form:

$$R^h_{ijk} = A_{jk} v^h_{,i}, \qquad ...(2.1)$$

where A_{jk} and $v^h_{,i}$ are two non-zero tensor fields such that

$$\lambda_h v^h_{,i} = p_i, \qquad ...(2.2)$$

$$\lambda_h v^h = 1 \qquad ...(2.3)$$

and

$$A_{jk} = \lambda_{j,k} - \lambda_{k,j} \neq 0. \qquad ...(2.4)$$

The vector field p_i is also a non-zero vector field and is called decomposed vector field.

We shall prove the following:

Theorem 1. Under the docomposition (2.1), the Bianchi identities take the forms:

$$P_i A_{jk} + p_j A_{ki} + p_k A_{ij} = 0 \qquad ...(2.5)$$

and

$$\lambda_a A_{jk} + \lambda_j A_{ka} + \lambda_k A_{aj} = 0 \qquad ...(2.6)$$

Proof. In view of equation (1.9) and (2.1), we have

$$A_{jh}, v^h_{,i} + A_{ki} v^h_{,j} + A_{ij} v^h_{,k} = 0. \qquad ...(2.7)$$

Multiplying the above equation by λ_h and using (2.2), we obtain the required result (2.5). Making use of equation (1.10), (1.11) and (2.1), we get

$$v^h_{,i}\left(\lambda_a A_{jk} + \lambda_j A_{ka} + \lambda_k A_{aj}\right) = 0 \qquad ...(2.8)$$

Multiplying the above equation by λ_h and using (2.2), we have

$$P_i\left(\lambda_a A_{jk} + \lambda_j A_{ka} + \lambda_k A_{aj}\right) = 0 \qquad ...(2.9)$$

Since p_i is a non zero recurrence vector field, we obtain

$$\lambda_a A_{jk} + \lambda_j A_{ka} + \lambda_k A_{aj} = 0.$$

This completes the proof of the theorem.

Theorem 2. Under the decomposition (2.1), the tensor field R^h_{ijk}, R_{ij} and A_{jk} satisfy the relations

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} = p_i A_{jk} \qquad \text{...(2.10)}$$

Proof. With the help of equations (1.5), (1.11) and (1.12), we obtain

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} \qquad \text{...(2.11)}$$

Multiplying equations (2.1) by λ_h and using the relation (2.2), we have

$$\lambda_a R^a_{ijk} = p_i A_{jk}. \qquad \text{...(2.12)}$$

In view of equations (2.11) and (2.12), we get the required result (2.10).

Theorem 3. Under the decomposition (2.1), the quantities λ_a and $v^h_{,i}$ behave like recurrent vector and tensor fields respectively. The recurrent forms of these quantities are given by

$$\lambda_{a,m} = \mu_m \lambda_a, \qquad \text{...(2.13)}$$

and

$$v^h_{,i,m} \underset{def}{=} v_m v^h_{,i}, \qquad \text{...(2.14)}$$

where $v^h_{,i,m} = v^h_{,im}$ and μ_m, v_m are non-zero recurrence vector fields.

Proof. Defferentiating (2.11), convariently with respect to x^m and using equations (2.1) and (2.10), we have

$$\lambda_{a,m} A_{jk} v^a_{,i} = \lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}. \qquad \text{...(2.15)}$$

Multiplying the above equation by λ_a and using (2.11), we get

$$\lambda_{a,m}\left(\lambda_i R_{jk} - \lambda_j R_{ik}\right) = \lambda_a\left(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}\right). \qquad \text{...(2.16)}$$

Now multipling equation (2.16) by λ_h, we get

$$\lambda_{a,m}\left(\lambda_i R_{jk} - \lambda_j R_{ik}\right)\lambda_h = \lambda_a \lambda_h \left(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik}\right). \qquad \text{...(2.17)}$$

Since the expression on the right hand side of the above equation is a symmetric in a and h, therefore we get

$$\lambda_{a,m} \lambda_h = \lambda_{h,m} \lambda_a, \qquad \text{...(2.18)}$$

provided that $\lambda_i R_{jk} - \lambda_j R_{ik} \neq 0$.

The vector λ_a being non-zero, we can obtain a proportional vector field μ_m such that $\lambda_{a,m} = \mu_m \lambda_a$.

Further, differentiating (2.2) covariently with respect to x^m and using (2.13), we get

$$\lambda_h v^h_{,im} = p_{i,m} - \mu_m p_i \qquad \text{...(2.19)}$$

From the above equation it is obvious that

$$\lambda_h v^h_{,im} = \lambda_a v^a_{,im} \qquad \text{...(2.20)}$$

Since λ_a is a non-zero recurrence vector field, we can get a proportional vector field v_m such that $V^h_{,im} = v_m v^h_{,i}$.

This completes the proof of the theorem.

Theorem 4. Under the decomposition (2.1), the vector field p_i and tensor field A_{jk} behave like recurrent vector and recurrent tensor fields and their recurrent forms are given respectively by

$$P_{i,m} = (\mu_m + v_m)p_i, \quad ...(2.21)$$

and

$$A_{jk,m} = (\lambda_m - v_m)A_{jk}. \quad ...(2.22)$$

Proof. Differentiating (2.2) covariantly with respect to x^m and using equations (2.2), (2.13) and (2.14), we obtain the required result (2.21).

Further, differentiating equation (2.1) covariently with respect to x^m and using equations (1.11), (2.1) and (2.14), we get the required recurrent form (2.22).

Theorem 5. Under decomposition (2.1), the curvature tensor and holomorphically projective curvature tensor are equal iff

$$A_{km}\{(p_i S_j^h - p_j S_i^h) + p_a(F_j^h F_i^a - F_i^h F_j^a)\} + 2p_a A_{ji} F_k^h F_i^a = 0. \quad ...(2.23)$$

Proof. The holomorphically projective curvature tensor field p_{ijk}^h in T_n is defined by

$$p_{ijk}^h = R_{ijk}^h + (R_{ik}\delta_j^h - R_{jk}\delta_i^h + S_{ik}F_j^h - S_{jk}F_i^h + 2S_{ij}F_k^h)/(n+2). \quad ...(2.24)$$

which may be expressed briefly as

$$p_{ijk}^h = R_{ijk}^h + D_{ijk}^h, \quad ...(2.25)$$

where

$$nD_{ijk}^h = (R_{ik}\delta_j^h - R_{jk}\delta_i^h + S_{ik}F_j^h - S_{jk}F_i^h + 2S_{ij}F_k^h)/(n+2). \quad ...(2.26)$$

Contracting indices h and k in (2.1), we get

$$R_{ij} = A_{jl}v_{,i}^l. \quad ...(2.27)$$

In view of the above equation, we have

$$S_{ij} = F_i^a R_{aj} = F_i^a A_{jl} v_{,a}^l \quad ... (2.28)$$

Making use of (2.27) and (2.28) in (2.26), we have

$$D_{ijk}^h = [A_{kl}\{(S_j^h v_{,i}^l - S_i^h v_{,j}^l) + v_{,a}^D(F_j^h F_i^a - F_i^h F_j^a)\} + 2A_{jl}F_i^a F_k^h v_{,a}^l]/(n+2). \quad ...(2.29)$$

From (2.25) it is clear that

$P_{ijk}^h = R_{ijk}^h$ iff $D_{ijk}^h = 0$, which in view of equation (2.29), gives

$$A_{ki}\{(\delta_j^h v_{,i}^l - \delta_i^h v_{,j}^l) + v_{,a}^l(F_j^h F_i^a - F_i^h F_j^a)\} + 2A_{jl}F_i^a F_k^g v_{,a}^l = 0. \quad ...(2.30)$$

Multiplying the above equation by λ_a and using (2.2), we obtain the required condition (2.23).

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