Jñānābha, Vol. 34, 2004

# COMPLETE LIFT OF $F_\lambda(2v+3,1)$ STRUCTURE IN TANGENT BUNDLE

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*(Received : July 1, 2003; Revised : January 15, 2004)*

## ABSTRACT

Andreou [1] studied the structure defined by the (1,1) tensor field $F$ Satisfying $F^5+F+r=0$. Ishihara and Yano [2], defined integrability conditions of $F$-structure. Srivastava and Niwas [7], considered the integrability condition of $F_\lambda(2v+3,1)$ structure. The horizontal and complete lift from a differentiable manifold $M^n$ of class $C^\infty$ to its cotangent bundle ${}^CT(M^n)$ have been studied by Yano and Patterson [5]. Yano and Ishihara [9]. studied lift of $F$-structure in the tangent and cotangent bundle. Kim [3] dealt with the properties of $F$-structure. Srivastava [8], considered lift of $(1,1)$ tensor field $F$ satisfying

$$F^{v+1}=\lambda^2F^{v-1} \text{ and } F^v=(-1)^{v+3}F,$$

Niwas, Ali and Srivastava [4], studied some properties of lifts (horizontal and complete) for the structure satisfying

$$a^nF^n+a_{n-1}F^{n-1}+\ldots+a_1F=0.$$

Here we give some results for the (1,1) tensor field $F$ satisfying $F^{2v+3}=-\lambda^2F$, taking horizontal, complete and diagonal lift.

**1. Preliminaries.** Let $M^n$ be $n$-dimensional differentiable manifold of class $C^\infty$. If $F\neq 0$ be a (1,1) tensor field of class $C^\infty$, rank $r$ satisfying [4]

(1.1) $$F^{2v+3}=-\lambda^2F$$

Here tensors $\ell$ and $m$ are defined as follows :

(1.2) $$\ell=-F^{2v+2}/\lambda^2,\ m=1+F^{2v+2}/\lambda^2.$$

From (1.2), we have

(1.3) $$\ell+m=1,\ \ell^2=\ell,\ m^2=m,\ \ell m=m\ell=0,\ F\ell=F,\ Fm=0.$$

Thus there exits in $M^n$ two complementary distributions $D_\ell$ *and* $D_m$ corresponding to the projection tensors $\ell$ and $m$ respectively. If rank of $f$ is $r$ then $D_\ell$ is $r$ dimensional and $D_m$ is $(n-r)$ dimensional. $D_mM^n=n$.

**2. The Complete Lift of $F_\lambda$ Structure in the Tangent Bundle $T(M^n)$.** Let $(M^n)$ be an $n$-dimensional differentiable manifold of class $C^\infty$ and $T_p(M^n)$ is the tangent space at a point $p$ of $M^n$ then

$$T(M^n)=\bigcup_{p\in M^n}T_p(M^n)$$

is the tangent bundle cover the manifold $M^n$. The tangent bundle $T(M^n)$ of $M^n$ is a differentiable manifold of dimension $2n$.

Let $T_s^r$ denote the set of tensor field of class $C^\infty$ and type $(r,s)$ in $M^n$ and let $T_s^r\,[T(M^n)]$ denotes the corresponding set of tensor fields in $T(M^n)$. The complete lift $F^c$ of an element $F$ of $T_1^1(M^n)$ with local components $F_1^h$ has components of the form Yano and Ishihara [9]

(2.1) $$F^c = \begin{pmatrix} F_i^h & 0 \\ \partial F_i^h & F_i^h \end{pmatrix}.$$

Let $F, G \in T_1^1(M^n)$ then we have [9]

(2.2) $(FG)^c = F^c G^c$

which yields on putting $F=G$

(2.3) $(F)^c = (F^c)^2$.

Similarly on putting $G=F^2$ in (4.3.2) we get

(2.4) $(F^3)^c = (F^c)^3$.

Continuing the above process of replacing $G$ in equation (2.2) by some higher power of $F$, we get

(2.5) $(F^K)^c = (F^c)^K$, where $K$ is any positive integer.

If $G$ and $H$ are tensors of the same type then

(2.6) $(G+H)^c = G^c + H^c$.

Taking complete lifts on both sides of (1.1), we get

(2.7) $(F^{2v+3} + \lambda^2 F)^c = 0$

$\Rightarrow (F^{2v+3})^c + (\lambda^2 F)^c = 0$ or $(F^c)^{2v+3} + \lambda^2 F^c = 0$.

The rank of $F^c$ in $2r$ iff rank of $F$ is $r$. Thus we have

**Theorem** 1. Let $F \in T_1^1(M^n)$, the $F$ satisfies this structure equation (1.1), if $F^c$ satisfies (1.4), further more $F$ is of rank $r$ iff $F^c$ is of rank $r$. Let $F$ be an $F_\lambda(2v+3,1)$ – structure of rank $r$ in $M^n$. Then complete lift $\ell^c$ of $\ell$ and $m^c$ of $m$ are complementary projection tensors in $T(M^n)$. Thus there exist in $T(M^n)$ two complementary distributions $D_\ell^c$ and $D_m^c$ are respectively complete lifts $D_\ell^c$ and $D_m^c$ of $D_\ell$ and $D_m$.

**3. Integrability Conditions of $F_\lambda(2v+3,1)$ Structure in Tangent Bundle.** Let $F \in T_1^1(M^n)$ and $F$ satisfies (1.1) then the Nijenhuis tensor $N_F$ of $F$ is a tensor field of type (1,2) given by Yano and Ishihara [9]

(3.1) $N_F(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y]$.

Let $N^c$ be the Nijenhuis tensor of $F^c$ in $T(M^n)$, where $F^c$ is the complete lift of $F$ in $M^n$, then we have

(3.2) $N^c(X^c,Y^c)=[F^cX^c,F^cY^c]-F^c[F^cX^c,Y^c]-F^c[X^c,F^cY^c]+(F^2)^c[X^c,Y^c].$

$\forall\, X,Y\in I_0^1(M^n)$ and $F\in I_1^1(M^n)$, we have

(3.3) (a) $[X^c,Y^c]=[X,Y]^c,$ (b) $[X,Y]^c=X^c+Y^c,$ (c) $F^cX^c=(FX)^c.$

From (1.3) and (3.3) we have

(3.4) $F^c\ell^c=(F\ell)^c=F^c$ and $F^cm^c=(Fm)^c=0,$

$(\ell^c)^2=(\ell^2)^c=\ell^c,\qquad (m^c)^2=(m^2)^c=m^c.$

**Theorem 2.** In the $F^{2\upsilon+3}=-\lambda^2F$ structure, we have

(3.5) (a) $\left(N^c(m^cX^c,m^cY^c)\right)^{(2\upsilon+3)/2}\neq\lambda^2F^c[m^cX^c,m^cY^c],$

(b) $m^cN^c(X^c,Y^c)=m^c[F^cX^c,F^cY^c],$

(c) $m^cN^c(\ell^cX^c,\ell^cY^c)=m^c[F^cX^c,F^cY^c],$

(d) $m^cN^c\left[(F^{2\upsilon+3}+\lambda^2F)^cX^c,(F^{2\upsilon+3}+\lambda^2F)^cY^c\right]=\lambda^2m^2N^c(\ell^cX^c,\ell^cY^c).$

**Proof.** Since $N^c(m^cX^c,m^cY^c)=[F^cm^cX^c,F^cm^cY^c]-F^c[F^cm^cX^c,m^cY^c]$

$-F^c[F^cm^cX^c,m^cY^c]-F^c[m^cX^c,F^cm^cY^c]+(F^2)^c[m^cX^c,m^cY^c]$

$=(F^c)^2[m^cX^c,m^cY^c].$

Thus we get $N^c(m^cX^c,m^cY^c)=(F^c)^2[m^cX^c,m^cY^c].$

Since $(F^c)^{2\upsilon+3}=-\lambda^2F^c$, therefore above equation reduces to

$[N^c(m^cX^c,m^cY^c)]^{(2\upsilon+3)/2}=-\lambda^2F^c[m^cX^c,m^cY^c]$

Thus we get (3.5)(a).

Similarly other results can easily be derived.

**Theorem 3.** For any $X,\,Y\in T_1^1(M^n)$ the following conditions are equivalent

(3.6) (a) $m^cN^c(X^c,Y^c)=0$ , (b) $m^cN^c(\ell^cX^c,\ell^cY^c)=0,$

(c) $m^cN^c\left[(F^{2\upsilon+3}+\lambda^2F)^cX^c,(F^{2\upsilon+3}+\lambda^2F)^cY^c\right]=0.$

If we put $m^cN^c\left[(F^{2\upsilon+3}+\lambda^2F)^cX^c,(F^{2\upsilon+3}+\lambda^2F)^cY^c\right]=0$

in (3.5)(d), then we get $m^cN^c(\ell^cX^c,\ell^cY^c)=0$ i.e. we get (3.6)(b) and also from

(3.5)(c) and (3.6)(b) we see that if (3.6)(c) is satisfied then (3.6)(a)and (3.6)(b) are also satisfied.

**Theorem 4.** The complete lift $D_m^c$ in $T(M^n)$ of a distribution $D_m$ in $M^n$ is integrable if $D_m$ is integrable in $M^n$.

**Proof.** The distribution $D_m$ is integrable iff Srivastava [7]

(3.7) $\ell[mX,mY]=0,\ \forall X,Y \in T_0^1(M^n)$ and $\ell = 1-m$.

Taking complete lift of both sides, we get

(3.8) $\ell^c[m^c X^c, m^c Y^c]=0,\ \forall X,Y \in T_0^1(M^n)$,

where $\ell^c = (1\text{-}m)^c = 1 - m^c$ is the projection tensor complementary to $m^c$. Thus the condition (3.7) implises (3.8).

**Theorem 5.** The complete lift $D_m^c$ in $T(M^n)$ of a distribution $D_m$ in $M^n$ is integrable if $\ell^c N^c(m^c X^c, m^c Y^c)=0$, or equivalently $N^c(m^c X^c, m^c Y^c)=0\quad \forall X,Y \in T_0^1(M^n)$.

**Proof.** The distribution $D_m$ is integrable in $M^n$ iff $N(mX,mY)=0\ \ \forall X,Y \in T_0^1(M^n)$.

Since $N^c(m^c X^c, m^c Y^c) = (F^c)^2[m^c X^c, m^c Y^c]$,

Therefore, by multiplying throughout by $\ell^c$, we get

$\ell^c N^c(m^c X^c, m^c Y^c) = (F^c)^2 \ell^c[m^c X^c, m^c Y^c]$,

but since $\ell^c[m^c X^c, m^c Y^c]=0$, Therefore

(3.9) $\ell^c m^c(m^c X^c, m^c Y^c)=0$ and also

(3.10) $m^c N^c(m^c X^c, m^c Y^c)=0$.

Adding (3.9) and (3.10) we get

$$(\ell^c + m^c)N^c(m^c X^c, m^c Y^c)=0 .$$

Since $\ell^c + m^c = I^c = 1$, therefore we prove $N^c(m^c X^c, m^c Y^c)=0$.

**Theorem 6.** Let the distribution $D_\ell$ be integrable in $M^n$, that is $mN(X,Y)=0$ $\forall X,Y \in T_0^1(M^n)$. Then the distribution $D_\ell^c$ is integrable in $T(M^n)$ iff any one of the conditions of theorem (2.2) is satisfied $\forall\ X,Y \in T_0^1(M^n)$.

**Proof.** Since the distribution $D_\ell$ is integrable in $M^n$ iff $mN(\ell X, \ell Y)=0$.

Thus distribution $D_\ell^c$ is integrable is $T(M^n)$ iff

$m^c N^c(\ell^c X^c, \ell^c Y^c)=0$. Thus theorem follows.

**Theorem 7.** The complete lift $F^c$ of an $F^{2v+3}+\lambda^2 F=0$ structure in $M^n$ is partially integrable in $T(M^n)$ iff $F$ is partially integrable in $M^n$.

**Proof.** $F^{2v+3}+\lambda^2F=0$ structure in $M^n$ is partially integrable iff

$N(\ell X,\ell Y)=0,\ \forall X,Y\in T_0^1(M^n)$.

Keeping in view of (1.1) and (3.1) we obtain

(3.11) $N^c[\ell^c X^c,\ell^c Y^c]=[N(\ell X,\ell Y)]^c$

which implies

$$N^c(\ell^c X^c,\ell^c Y^c)=0 \text{ iff } N(\ell X,\ell Y)=0.$$

Also from theorem (3.5) $N^c(\ell^c X^c,\ell^c Y^c)=0 \Rightarrow N(\ell X,\ell Y)=0$.

Hence the theorem follows.

**4. The Horizontal Lift of Structure.** The horizontal lift $S^H$ of a tensor field $S$ of arbitrary type in $M^n$ to $T(M^n)$ is defined by Yano and Ishihara [9].

(4.1) $S^H=S^C-\nabla_\gamma S,$

where $S$ is a tensor field defined by

$S=S_{k\ldots j}^{i\ldots h}\frac{\partial}{\partial x^i}\otimes\ldots\otimes\frac{\partial}{\partial x^h}\partial x^k\otimes\ldots\otimes\partial x^j$ in $M^n$ with affine connection $\nabla$ and $\nabla_\gamma S$ is a tensor field in $T(M^n)$ given by

(4.2) $\nabla_\gamma S=\left(\gamma^\ell\nabla_\ell S_{k\ldots j}^{i\ldots h}\right)\frac{\partial}{\partial y^i}\otimes\ldots\otimes\frac{\partial}{\partial y^h}\partial x^k\otimes\ldots\otimes\partial x^j$

with respect to induced coordinates $(xh,yh)$ in $\pi^{-1}(U)$.

The horizontal lift $F^H$ of a tensor field $F$ of type (1,1) in $M^n$ with local components $F_i^h$ in $M^n$ has components

(4.3) $F^H:\begin{pmatrix}F_i^h & 0\\ 0 & F_i^h\end{pmatrix}$.

**Theorem 8.** Let $F\in I_1^1(M^n)$ be an $F_\lambda$-structure in $M^n$ then its horizontal lift $F^H$ is also a $F_\lambda$-structure in $T(M^n)$.

**Proof.** If $P(t)$ is a polynomial in one variable $t$, then we have

(4.4) $(P(F))^H=(P(F^H))$

Also $I$ be the identity tensor field of type (1,1) in $M^n$ then

(4.5) $I^H=I.$

Thus

(4.6) $\left(F^{2v+3}\right)^H+\lambda^2(F)^H=0\Rightarrow\left(F^H\right)^{2v+3}+\lambda^2F^H=0$.

Now from the local components of $F^H$, we see that if $F$ is of rank $r$, then $F^H$ is of rank $2r$. Thus we have

**Theorem 9.** If $F$ is structure of rank $r$ in $M^n$ then its horizontal lift $F^H$ is also $F_\lambda$-structure of rank $2r$ in $T(M^n)$

**Proof.** Let $m$ be a projection tensor field of type (1,1) in $M^n$ defined by (1.2) and $m^2=m$ then $(m^H)^2=m^H$.

Thus $m^H$ is also a projection tensor in $T(M^n)$. Hence there exists in $T(M^n)$ a distribution $D^H$ corresponding to $m^H$, which is called the horizontal lift of the distribution $D$.

**5. Diogonal Lift in Cotangent Bundle.** If the diagonal lift of the tensor field $F$ of type (1,1) in $M^n$ is $F^D$ in ${}^cT(M^n)$ then Yano and Ishihara [9]

(5.1) $$F^D : \widetilde{F}_\beta^\alpha = \begin{pmatrix} F_i^h & 0 \\ 0 & -F_h^i \end{pmatrix},$$

(5.2) $$F^D G^D + G^D F^D = (FG + GF)^H,$$

(5.3) $$F^D G^H + G^D F^H = F^H G^D + G^H F^D = (FG + GF)^D,$$

(5.4) $$F^H F^D = F^D F^H = (F^2)^D,$$

(5.5) $$(F^D)^{2p} = (F^{2p})^H, (F^D)^{2p+1} = (F^{2p+1})^D \quad \forall F \in I_1^1(M^n)$$

**Theorem 10.** The diagonal lift of an almost $F_\lambda(2v+3, 1)$ structure in $M^n$ with symmetric affine connection is also an almost $F_\lambda$ in ${}^cT(M^n)$.

**Proof.** Thus from (1.1) keeping in view of (5.4) and (5.5), we obtain

$$(F^{2v+3})^D = \lambda^2 (F)^D \Rightarrow (F^D)^{2v+3} = \lambda^2 F^D.$$

**Theorem 11.** If $F$ is $F_\lambda$-structure of rank $r$ in $M^n$ then its diagonal lift $F^D$ is also $F_\lambda$-structure of rank $2r$ in ${}^cT(M^n)$.

**Proof.** Since $m$ be a projection tensor field of type (1,1) in $M^n$ defined by (1.2) and $m^2=m$ then

$(m^2)^D=(m^D)^2=m^D$.

Thus $m^D$ is also a projection tensor in ${}^cT(M^n)$. Hence there exists in ${}^cT(M^n)$ a distrubtion $D$ corresponding to $m^D$ which is called diagonal lift of the distribution $D^D$.

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