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(*Dedicated to the memory of Professor J.N. Kapur*)

STRONGLY EXPANDABLE SPACE

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ABSTRACT

The notion of strongly expandable spaces is introduced and studied to answer the following question : Is there any topological property other than normality coupled with Lindelöfness which strengthens paracompactness to strong paracompactness? Strong expandability is independent of normality coupled with Lindelöfness. The above question is answered in the affirmative by proving that strong expandability strengthens paracompactness (respectively, θ-refinability, metacompactness) to strong paracompactness. In fact strongly paracompact spaces are decomposed into strongly expandable and paracompact (respectively, θ-refinable, metacompact) spaces. Also, strongly χ_0-expandable spaces are introduced and it is shown that the notion of strong χ_0-expandability is equivalent to that of strong countable paracompactness. Various inclusion relations are obtained and several counter examples are given which demonstrate the invalidity of a number of plausible conjectures concerning strongly expandable spaces.

Introduction : In 1948 Morita [12] introduced the notion of star-finite property. A topological space X is said to have the star-finite property if every open covering of X has a star-finite open refinement . Begle[1] studied these spaces under the name S-spaces and Hu [3] called them hypocompact spaces. Since every hypocompact space is paracompact, hypocompact spaces were also called strongly paracompact spaces. Also, a topological space X is said to be strongly countably paracompact, if every countable open coverng of X has a star-finite open refinement. The class of strongly paracompact spaces contains the class of compact spaces and is contained in the class of paracompact spaces as well as in the class of strongly countably paracompact spaces. The class of strongly countably paracompact spaces is contained in the class of countably paracompact spaces. In normal spaces the notions of strong countable paracompactness, countable paracompactness and countable metacompactness are equivalent. There are examples of paracompact spaces which are not strongly paracompact [13, Example V-6]. In fact even metrizable spaces need not be strongly paracompact. However, every separable metric space is strongly paracompact. Every strongly countably paracompact Lindelöf space is strongly paracompact. Therefore, in a normal Lindelöf space the notion of countable paracompactness and hence that of paracompactness is equivalent to the notion of strong paracompactness. But it is not known, whether there is any other topoligcal property which strengthens paracompactness to strong paracompactness. Henec the following question arises :

(*) Is there any topological property other than normality coupled with Lindelöfness which strengthens paracompactness to strong paracompactness ?

In the present paper we answer the above posed question (*) in the affirmative by introducing the concept of strongly expandable spaces. The notion of strong expandability property is independent of normality coupled with Lindelöfness. The whole paper is divided into two sections. In section 1 preliminary definitions and results needed for this paper are given . In section 2 the notions of strongly expandable and -strongly χ_0-expandable spaces are introduced and the above posed question (*) is answered in the affirmative by proving that in strongly expandable spaces the notion of paracompactness is equivalent to that of strong paracompactness. Moreover, the strong expandability property strengthens θ-refinability and hence metacompactness to strong paracompactness. In fact strongly paracompact spaces are decomposed into strongly expandable and paracompact (respectively, metacompact, θ-refinable) spaces. Various inclusion relations are obtained and several counter examples are given which demonstrate the invalidity of a number of plausible conjectures concerning strongly expandable spaces. Lastly, the relations of strongly expandable spaces with other topological spaces are summarized in a implication diagram.

1. Preliminary Definitions and Results.

1.1 Definitions [13]. A collection A of subsets of a topological space X is said to be star-finite if each member of A intersects atmost finitely many members of A. A collection A of subsets of a topological space X is said to be σ-star-finite if it is a countable union of star-finite collections.

1.2 Theorem [5]. A normal space is strongly countably paracompact if X is countably paracompact.

1.3 Definition[14]. A topological space X is said to be strongly m-paracompact. where m is an infinite cardinal, if each open covering of X of order atmost m has a star-finite open refinement.

1.4 Theorem [14]. Normal space X is strongly m-paracompact iff X is strongly counably paracompact and every open cover of X of power atmost m has a σ-star-finite open refinement.

The above theorem remains true when "strongly m-paracompact" is replaced by srongly paracompact" and "every open cover of X of power atmost m" is replaced by "every open cover of X".

1.5 Theorem [13]. Every regular Lindelöf space is strongly paracompact.

1.6 Theorem[7]. A topological space X is expandable iff for each locally finite collection F of (Closed) subsets of X, there exists a locally finite open cover U of X such that each member of U intersects atmost finitely many members of F.

The above theorem remains true when "expandable" is replaced by χ_0-expandable" and "locally finite collection F" is replaced by "locally finite countable collection F".

1.7 Theorem [15]. Every expandable screenable space is paracompact.

1.8 Theorem [9]. A topological space X is χ_0-expandable iff X is countably paracompact.

1.9 Theorem [16]. A topological space X is almost χ_0-expandable iff X is countably metacompact.

2. Strongly Expandable Space

2.1 Definition. A topological space X is said to be strongly expandable (respectively, strongly χ_0-expandable) if for each locally finite collection $F = \{F\alpha : \alpha \varepsilon A\}$ of subsets of X (respectively, with $|A| \leq \chi_0$), there exists a star filint open cover U of X such that each member of U intersects atmost finitely many members of F.

2.2 Note. Obviously every strongly expandable space is strongly χ_0-expandable but the converse is not true. The Michael's modification of Bing's space [11] is strongly χ_0-expandable space which is not strongly expandable In the definitions 2.1 above the collection F of subsects of X may be taken as a collection of closed subsets of X.

2.3 Theorem. Every countably compact space is strongly expandable.

Proof. Obvious in view of the fact that in a countable compact space every locally finite collection of subsets is finite.

The converse of above theorem is not true. Following is an example of a strongly expandable space which is not countably compact :

2.4 Example. A strongly expandable space which is not countably compact.

Let IN be the set of all natural numbers with the discrete topology. Let X-= $\Pi\{IN_i : i\varepsilon IN\}$ be the countable Cartesian product of copies of IN with the Tychonoff product topology τ. Then (X,τ) is strongly expandable but it is not countably compact.

2.5 Theorem. Every strongly paracompact space is strongly expandable.

Proof. *Let* X be a strongly paracompact space. Let $F = \{F_\alpha : \alpha\varepsilon A\}$ be a locally finite collection of closed subsets of X. Let $\Gamma = \{\gamma \subset A : \gamma \textit{ is finite}\}$. For each $\gamma\varepsilon\Gamma$, define $V_\gamma = X \cup \{F_\alpha : \alpha \notin \gamma\}$ Then $V = \{V_\gamma : \gamma\varepsilon\Gamma\}$ is an open cover of X such that each member of V intersects atmost finitely many members of F. Since X is strongly paracompact,therefore, V has a star-finite open refinement, say W. Then W is a star-finite open cover of X such that each member of W intersects atmost finitely many members of F. Hence X is strongly expandable.

The converse of above theorem is not true. Even a strongly expandable space need not be paracompact. Following is an example of a strongly expandable space which is not paracompact :

2.6 Example. A strongly expandable space which is not paracompact.

Let Ω be the first uncountable ordinal and let $X = [0, \Omega]$ with the usual order topology. Then X is strongly expandable space which is not paracompact and hence also not strongly paracompact.

The example 2.6 above together with the following example shows that the notion of strong expandability is independent of that of paracompaetness.

2.7 Example . A paracompact space which is not strongly expandable.

Let A be an uncountable index set. Let I_α, $\alpha\varepsilon A$ be copies of the unit interval $[0,1]$. In their union $\cup_{\alpha\varepsilon A} I_\alpha$, we identify all zeros to get a star-shaped set $S(A)$. Then we define the distance between two points of $S(A)$ by $\rho(p,q) = p-q$ if $p,q\varepsilon I_\alpha$ and $\rho(p,q) = p+q$ if $p\varepsilon I_\alpha$, $q\varepsilon I_\beta$ where $\alpha\neq\beta$. Then $S(A)$ is a metric space with metric ρ. Consider the metric topology $\tau(\rho)$ on $S(A)$. Then $(S(A), \tau(\rho)$ is a paracompact space which is not strongly expandable. Since every paracompact space is expandable, therefore, this is also an example of an expandable space, which is not strongly expandable.

2.8 Theorem .

i) Every strongly expandable space is expandable.

ii) Every strongly χ_0-expandable space is χ_0-expandable.

Proof . Obvious in view of theorem 1.6 and the fact that every star-finite open cover of a topological space X is also locally finite open cover of X.

The converse of theorem 2.8 (i) above is not true (cf. Example 2.7). Also, the converse of theorem 2.8 (ii) above is not true because of the facts that χ_0-expandability is equivalent to countable paracompactness (cf.Theorem 2.9 below) and that it is well known that a countably paracompact space need not be strongly countably paracompact.

2.9 Theorem. A topological space X is strongly χ_0-expandable iff X is strongly countably paracompuet.

Proof. Let X be a strongly χ_0-expandable space. Let $R = \{Ri : i = 1,2...\}$ be a countable open cover of X. Since X is strongly χ_0-expandable, therefore, by Theorem 2.8 (ii) above X is χ_0-expandable and hence by theorem 1.8, X is countably paracompact. Therefore, there exists a locally finite countable open cover $G = \{G_i : i=1,2...\}$ of X such that $G_i \subset R_i$ for each i. Then there exists a star-finite open cover $U = \{U_\alpha : \alpha\varepsilon A\}$ of X such that each member of U intersects atmost finitely many members of G. Then clearly $V = \{U_\alpha \cap G_i : \alpha\varepsilon A, i=1,2...\}$ is a star-finite open refinement of R . Hence X is strongly countably paracompact.

Conversely suppose that X is strongly countably paracompact. Then proceeding as in the proof of Theorem 2.5, by taking $|A| \leq \chi_0$ it is easy to prove that X is strongly χ_0-expandable.

2.10 Theorem. The following are equivalent for a normal space X

a) X is strongly χ_0-expandable.

b) X is χ_0-expandable.

c) X is almost χ_0-expandable.

d) X is countably metacompact.

e) X is countably paracompact.

f) X is strongly countably paracompact.

Proof. Obvious in view of Theorems 1.2, 1.9 and 2.9 above.

Now before giving the theorem which is the crux of this paper we give the following two examples which demonstrate that the notion of strong expandability is independent of that of normality coupled with Lindelöfness.

2.11 Example. A strongly expandable space which is neither normal nor Lindelöf.

The long line L is constructed from the ordinal space $[0, \Omega]$ (where Ω is the first uncountable ordinal) by placing between each ordinal α and its successor $\alpha + 1$ a copy of the unit integral$]0,1[$. L is then linearly ordered, and we give it the order topology. To the long line L we add a point p. Open sets of $L\cup\{p\}$ are the open sets of L together with those generated by the following neighbourhood of p: $U_\beta(p) = \{p\}\cup\{\cup^{\Omega}_{\alpha=\beta}]\alpha,\alpha+1[\}$ (where $1 \leq \beta \leq \Omega$). $U_\beta(p)$ is then a right hand ray less the ordinal points. We consider p to be the greatest element of $L\cup\{p\}$. Then $L\cup\{p\}$ is a strongly expandable space which is neither normal nor Lindelöf.

2.12 Example. A normal Lindelöf space which is not strongly expandable Let R be the set of all real numbers. For each $r\varepsilon R$, define $S_r = \{x\varepsilon R : x > r\}$. Then $B=\{S_r : r\in R\}$ is a base for some topology say τ, on R. The topology τ is known as right order topology on R. Then (R, τ) is normal Lindelöf space which is not countably paracompact and hence it is not strongly expandable.

2.13 Theorem. A topological space X is strongly paracompact iff X is strongly expandable and paracompact.

Proof. Only one implication needs proof. Let X be a strongly expandable paracompact space. Let U be an open cover of X. Since X is paracompact, therefore, U has a locally finite open refinement, say $V=\{V_\alpha : \alpha\varepsilon A\}$. Since X is strongly expandable and V is locally finite, therefore, there exists a star-finite open cover $W = \{W_\delta : \delta\varepsilon\Delta\}$ of X such that each member of W intersects atmost finitely members of V. Define G $\{W_\delta\cap V_\alpha : \delta\varepsilon\Delta, \alpha\varepsilon A\}$. Then clearly G is a star-finite open refinement of U. Hence X is strongly paracompact.

2.14 Corollary. A topological space X is strongly paracompact iff X is θ-refinable (respectively, metacompact) and strongly expandable.

2.15 Corollary. Every strongly expandable subparacompact space is strongly paracompact.

2.16 Theorem. Every strongly expandable screenable space is strongly paracompact.

Proof. Let X be a strongly expandable screenable space. Then by Theorem 2.8(i), X is an expandable screenable space and hence by Theorem 1.7, X is paracompact. Therefore, by Theorem 2.13, X is strongly paracompact.

2.17 Theorem. A normal almost χ_0-expandable screenable space is strongly paracompact.

Proof. Let X be a normal almost χ_0-expandable screenable space. Let $U=\{U_\alpha : \alpha \varepsilon A\}$ be an open cover of X. Since X is screenable, therefore, U has a σ-disjoint open refinement, say $V = \cup_{i=1}^{\infty} V_i$,where $V_i = \{V_{\alpha,i} : \alpha \varepsilon A_i\}$ for each i. Since each V_i is a pairwise disjoint open collection, therefore, each V_i is a σ-star-finite collection of open subsets of X. Therefore V is a σ-star-finite open refinement of U. Now since X is normal and almost χ_0-expandable, therefore, by theorem 2.10, X is normal strongly countably paracompact space and hence by Theorem 1.4 X is strongly paracompact.

2.18 Corollary. A normal almost expandable (respectively, expandable) screenable space is strongly paracompact.

Like the class of strongly expandable spaces the class of Bd-expandable spaces [6] also lies between the classes of countably compact and expandable spaces, but in the following example we show that these two concepts are different:

2.19 Example. A strongly expandable space which is not Bd-expandable.

For each positive integer n we define the compact subspace Q^n of E^n by $Q^n = \{(x_1, x_2,\ldots,x_n): 0 \leq x_i \leq 1/n$ for $i=1,2,\ldots,n\}$ and we define X to be the discrete union of the topological spaces Q^n for $n=1,2\ldots$ The space X clearly has a countable basis, Therefore, X is second countable and hence Lindelöf. Also a complete metric can be introduced into X which assigns the usual Euclidean distance between two points of the same Q^n and the distance 1 to any two points of two distinct subspaces Q^n. Since X is separable metric space, therefore, X is strongly paracompact and hence X is strongly expandable. Wenner [18, Example 3.1] has proved that X is not boundedly paracompact. Hence X cannot be Bd-expandable, because of the fact that a *Bd*-expandable paracompact space is boundedly paracompact.

Now from the above results together with other known results, we have the following implication diagram :

Compact $\overset{\not\leftarrow}{\rightarrow}$ *strongly paracompact* $\overset{\not\leftarrow}{\rightarrow}$ *paracompact*

$\downarrow\not\uparrow$ $\qquad\qquad$ $\downarrow\not\uparrow$ $\qquad\qquad$ $\downarrow\uparrow$

Countably $\overset{\not\leftarrow}{\rightarrow}$ *strongly expandable* $\overset{\not\leftarrow}{\rightarrow}$ *expandable*
Compact

$\downarrow\not\uparrow$ $\qquad\qquad$ $\downarrow\not\uparrow$

Strongly χ_0-*expandable* $\quad$ $\overset{\not\leftarrow}{\rightarrow}$ χ_0-*expandable*

$\downarrow\uparrow$ $\qquad\qquad$ $\downarrow\uparrow$

Strongly countably $\overset{\not\leftarrow}{\rightarrow}$ *countably paracompact*
Paracomact

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