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(*Dedicated to the memory of Professor J.N. Kapur*)

# *Bd*-EXPANDABILITY PROPERTY AND BOUNDEDLY PARACOMPACT SPACES

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## ABSTRACT

The notions of *Bd*-expandability and bounded *Bd*-expandability properties are introduced and studied. It is proved that *Bd*-expandability property is a topological property other than finite dimensionality coupled with Hausdorffness which strengthens paracompactness to bounded paracompactness. Not only this, *Bd*-expandability property strengthens each of the following properties to bounded paracompactness : θ-refinability, metacompactness, bounded metacompactness, subparacompactness. screenability and Lindelöness.

**Introduction.** In 1970 Fletcher, McCoy and Slover [3] introduced the notion of boundedly paracompact spaces. A topological space $X$ is said to be boundedly paracompact if every open cover of $X$ has a bounded locally finite open refinement. The class of boundedly paracompact spaces contains the class of compact spaces and is contained in the class of paracompact spaces. But a paracompact space need not be boundedly paracompact and a boundedly paracompact space need not be compact. However, Fletcher, McCoy and Slover [3] have proved that finite dimensionality together with Hausdorffness strengthens paracompactness to bounded paracompactness. Also, Fletcher, McCoy and Slover [4] have proved that bounded countable paracompactness together with Hausdorffness strengthens hereditary paracompactness to bounded paracompactness. But it is not known, whether any other topological property strengthens paracompactness to bounded paracompactness. Therefore, the following question arises:

Is there any topological property other than finite dimensionality coupled with Hausdorffness which strengthens paracompactness to bounded paracompactness?

All the known expandability properties being weaker than paracompactness cannot strengthen paracompactness to bounded paracompactness. In the present paper, we introduce the concept of *Bd*-expandability as a stronger form of expandability and answer the above question in the affirmative. The whole paper is divided into two sections. In section 1 the concept of *Bd*-expandability is introduced, discussed and the above question is answered in the affirmative by proving that every *Bd*-expandable paracompact space is boundedly paracompact. The notion of bounded *Bd*-expandability is also introduced by generalizing the notion of *Bd*-expandability. Various inclusion relations are obtained and several counter examples are given which demonstrate the invalidity of a number of plausible conjectures concerning these properties. In section 2, open problems are given.

## 1. *Bd*-Expandability Property

**1.1 Definitions.** A topological space $X$ is said to be *Bd*-expandable (respectively, boundedly *Bd*-expandable) if for each locally finite (respectively, bounded locally finite) collection $\{F_\alpha: \alpha\varepsilon A\}$ of subsets of $X$, there exists a bounded locally finite collection $\{G_\alpha: \alpha\varepsilon A\}$ of open subsets of $X$ such that $F_\alpha \subset G_\alpha$ for each $\alpha\varepsilon A$.

**1 .2 Theorem.** A topological space $X$ is Bd-expandable (respectively, boundedly *Bd*-expandable) iff for each locally finite (respectively, bounded locally finite) collection $\{F_\alpha: \alpha\varepsilon A\}$ of closed subsets of $X$, there exists a bounded locally finite collection $\{G_\alpha: \alpha\varepsilon A\}$ of open subsets of $X$ such that $F_\alpha \subset G_\alpha$ for each $\alpha\varepsilon A$.

**Proof.** Obvious.

**1.3 Theorem.** A topological space $X$ is boundedly *Bd*-expandable iff for each discrete collection $\{F_\alpha: \alpha\varepsilon A\}$ of subsets of $X$, there exists a bounded locally finite collection $\{G_\alpha: \alpha\varepsilon A\}$ of open subsets of $X$ such that $F_\alpha \subset G_\alpha$ for each of $\alpha\varepsilon A$.

**Proof.** The proof follows from a simple modification of the proof of the Theorem 2.7 in [10].

1 .4 ***Lemma.*** Let $\{F_\alpha: \alpha\varepsilon A\}$ be a bounded locally finite collection of subsets of a topological space $X$. Let $\Gamma = \{\gamma \subset A : \gamma \text{ is finite}\}$. Then $\{\cap_{\alpha\varepsilon\gamma} F_\alpha: \gamma\varepsilon\Gamma\}$ is bounded locally finite.

**Proof.** Obvious.

Using Lemma 1.4 above we give a characterization of $Bd$-expandable spaces as follows :

**1.5 Theorem.** A topological space $X$ is $Bd$-expandable iff for each locally finite collection $F$ of (closed) subsets of $X$, there exists a bounded locally finite open cover $U$ of $X$ and a positive integer $n$ such that each member of $U$ intersects atmost $n$ members of $F$.

**Proof.** Let $X$ be a $Bd$-expandable space. Let $\boldsymbol{F} = \{F_\alpha: \alpha\varepsilon A\}$ be a locally finite collection of (closed) subsets of $X$. Then $Cl\,F = \{ClF_\alpha: \alpha\varepsilon A\}$is locally finite. Therefore, there exists a bounded locally finite collection $\{G_\alpha: \alpha\varepsilon A\}$ of open subsets of $X$ such that $Cl\,F_\alpha \subset G_\alpha$ for each $\alpha\varepsilon A$. Then there exists a positive integer $n$ such that each point $x\varepsilon X$ belongs to at most $n$ members of $\{G_\alpha: \alpha\varepsilon A\}$.

Let. $\Gamma = \{\gamma\subset A : |\gamma| \leq n\}$. For each $\gamma\in\Gamma$ ,define $V_\gamma = \cap_{\alpha\varepsilon\gamma} G_\alpha - \cap_{\alpha\notin\gamma} ClF_\alpha$. It can be proved easily that $\boldsymbol{V} = \{V_\gamma: \gamma\varepsilon\Gamma\}$ is an open cover of $X$ such that each member of $\boldsymbol{V}$ intersects atmost $n$ members of $\boldsymbol{F}$ . Now by Lemma 1.4 above and the fact that $V_\gamma \subset \cap_{\alpha\varepsilon\gamma} G_\alpha$, for each $\gamma\varepsilon\Gamma$. it follows that $V$ is bounded locally finite.

Conversely suppose that $\boldsymbol{F} = \{F_\alpha: \alpha\varepsilon A\}$ is a locally finite collection of (closed) subsets of $X$. There exists then a bounded locally finite open cover $\boldsymbol{U} = \{U_\beta: \beta\varepsilon B\}$ of $X$ and a positive integer $n$ such that each member of $U$ intersects atmost $n$ members of $\boldsymbol{F}$. For each $\alpha\varepsilon A$, define $B_\alpha = \{\beta\varepsilon B : F_\alpha \cap U_\beta \neq \phi\}$ and $G_\alpha = \cup_{\beta\varepsilon B\alpha} U_\beta$ . Then clearly $\boldsymbol{G} = \{G_\alpha: \alpha\varepsilon A\}$ is a collection of open subsets of $X$ such that $F_\alpha \subset G_\alpha$ for each $\alpha\varepsilon A$. Since $\boldsymbol{U}$ is bounded locally finite, therefore, there exists a positive integer $m$ such that each point $x\in X$ has a neighbourhood which intersects atmost $m$ members of $U$. Let $m.n=k$. It can be shown that $\boldsymbol{G}$ is bounded locally finite of order $k$.

**1 .6 Theorem.** A topological space $X$ is boundedly $Bd$-expandable if and only if for each discrete collection $\boldsymbol{F}$ of (closed) subsets of $X$, there exists a bounded locally finite open cover $U$ of $X$ such that each member of $U$ intersects atmost one member of $F$.

**Proof.** The proof follows from the similar arguments used in the proof of theorem 1.5 above.

**1.7 Theorem.** Every countably compact space is *Bd*-expandable.

**Proof.** Obvious in view of Theorem 2.7 in [5]

The converse of Theorem 1.7 above is not known.

**1.8 Theorem.** (i) Every *Bd*-expandable space is boundedly *Bd*-expandable.

(ii) Every *Bd*-expandable space is expandable.

(iii) Every boundedly *Bd*-expandable space is boundedly expandable.

(iv) Every collectionwise normal space is boundedly *Bd*-expandable.

**Proof.** Obvious.

The converse of Theorem 1.8(i) is not true. Following is an example of a boundedly *Bd*-expandable space which is not *Bd*-expandable:

**1.9 Example.** A boundedly *Bd*-expandable space which is not *Bd*-expandable. The topological space X defined in [8, page 179-180] being collectionwise normal is boundedly *Bd*-expandable but it is not *Bd*-expandable because it is not countably paracompact.

The converse of Theorem 1.8 (ii) is not true. Following is an example of an expandable space which is not *Bd*-expandable:

**1.10 Example .** An expandable space which is not *Bd* -expandable. Hilbert space being metrizable is expandable but it is not *Bd*-expandable.

The converse of Theorem 1.8 (iii) is not known.

The converse of Theorem 1.8 (iv) is not true. Following is an example of a boundedly *Bd*-expandable space which is not collectionwise normal:

**1.11 Example.** A boundedly *Bd*-exapnadable space which is not collectionwise normal :

Let $(\boldsymbol{R},\tau)$ be the real line with the usual topology. We define a new topology by defining $\tau^* = \{X \subset \boldsymbol{R} : X = \phi \text{ or } \boldsymbol{R}\text{-}X \text{ is compact in } (\boldsymbol{R},\tau)\}$. Surely $\tau^*$ is a topology on $\boldsymbol{R}$. Then $(\boldsymbol{R}, \tau^*)$ being compact is boundedly *Bd*-expandable but it is not normal and hence not collectionwise normal.

**1.12 Note.** The example 1.10 also shows that a paracompat space need not be *Bd*-expandable. In fact, the notion of *Bd*-expandability is independent of that of paracompactness. Following is an example of *Bd*-expandable space which is not paracompact.

**1.13 Example.** A normal Hausdorff *Bd*-expandable space which is not paracompact. Let $\Omega$ be the first uncountable ordinal. Then $X=[0, \Omega[$ with the usual order topology is a normal Hausdorff Bd-expandable space which is not paracompact.

**1.14 Theorem.** Every *Bd*-expandable space is boundedly countably paracompact.

**Proof.** Let $X$ be a *Bd*-expandable space. Let $\boldsymbol{R} = \{R_i : i = 1,2,...\}$ be a countable open cover of $X$. Put $S_i = \cup_{p=1}^{i} R_p$ for each i. Let $A_1 = S_1$ and $A_i = S_i - S_{i-1}$ for i=2,3,.... Surely $\boldsymbol{A} = \{A_i : i=1,2,...\}$ is a locally finite refinement of $R$ such that $A_i \subset R_i$ for each $i$. Since $X$ is *Bd*-expandable, therefore, there exists a bounded locally finite collection $\{G_i : i=1,2,...\}$ of open subsets of $X$ such that $A_i \subset G_i$ for each $i$. Let $U_i = R_i \cap G_i$ for each i. Then $U = \{U_i = 1,2,...\}$ is a bounded locally finite open refinement of $\boldsymbol{R}$. Therefore $X$ is boundedly countably paracompact.

The converse of Theorem 1.14 above is not known.

**1.15 Theorem.** Every boundedly paracompact space is boundedly *Bd*-expandable.

**Proof.** Let $X$ be a boundedly paracompact space, Let $\boldsymbol{F} = \{F_\alpha : \alpha \varepsilon A\}$ be a bounded locally finite collection of closed subsets of $X$. Then there exists a positive integer $n$ such that each point $x \varepsilon X$ belongs to atmost $n$ members of $F$. Let $\Gamma = \{\gamma \subset A : |\gamma| \leq n\}$. For each $\gamma \varepsilon \Gamma$, define $V_\gamma = X - \cup_{\alpha \notin \gamma} F_\alpha$. Then $V = \{V_\gamma : \gamma \varepsilon \Gamma\}$ is an open cover of $X$ such that each member of $V$ intersects atmost n members of $F$. Since $X$ is boundedly paracompact, therefore, $V$ has a bounded locally finite open refinement say $\boldsymbol{W} = \{W_\delta : \delta \varepsilon \Delta\}$.

Now for each $\alpha \in A$, define $U_\alpha = St(F_\alpha, \boldsymbol{W}) = \cup\{W_\delta \varepsilon\ \boldsymbol{W} : W_\delta \cap F_\alpha \neq \phi\}$. Then surely $\boldsymbol{U} = \{U_\alpha : \alpha \varepsilon A\}$ is bounded locally finite collection of open subsets of $X$ such that $F_\alpha \subset U_\alpha$ for each $\alpha \varepsilon A$. Hence $X$ is boundedly *Bd*-expandable.

The converse of Theorem 1.15 above is not true (cf Example 1.1).

Before proving a Theorem which is the crux of this paper. we give two examples which show that the notion of *Bd*-expandability is different from the notions of finite dimensionality coupled with Hausdorffness and hereditarily paracompactness, respectively.

**1.16 Example.** A *Bd*-expandable papracompact Hausdorff space which is not finite dimensional.

The Hilbert cube being compact Hausdorff is *Bd*-expandable paracompact Hausdorff space but it is not finite dimensional.

**1.17 Example.** A *Bd*-expandable paracompact Hausdorff space which is not hereditarily paracompact.

Let $\Omega$ be the first uncountable ordinal and $\omega$ be the first infinite ordinal. Let $X=[0,\Omega]\times[0,\omega]$ where both ordinal spaces $[0,\Omega]$ and $[0,\omega]$ are given the usual order topology. $X$ is known as the Tychonoff plank. $X$ being compact Hausdorff is *Bd*-expandable paracompact Hausdorff space but it is not hereditarily paracompact because the subspace $Y=X-\{(\Omega,\omega)\}$ is not paracompact.

Now we prove the Theorem which is the crux of this paper.

**1.18 Theorem.** Every *Bd*-expandable paracompact space is boundedly paracompact.

**Proof.** Let $X$ be a *Bd*-expandable paracompact space. Let $\boldsymbol{U}$ be an open cover of $X$. Then $U$ has a locally finite open refinement, say $\boldsymbol{V}=\{V_\beta:\beta\varepsilon B\}$. Then by Theorem 1.5 above, there exists a bounded locally finite open cover $\boldsymbol{W}=\{W\delta:\delta\varepsilon\Delta\}$ of $X$ and a positive integer $n$ such that each member of $\boldsymbol{W}$ intersects atmost n members of $\boldsymbol{V}$. Let $\boldsymbol{G}=\{W_\delta\cup V_\beta:\delta\varepsilon\Delta,\beta\varepsilon B\}$. Then surely $\boldsymbol{G}$ is a bounded locally finite open refinement of $\boldsymbol{V}$ and hence a bounded locally finite open refinement of $U$. Therefore $X$ is boundedly paracompact.

**1.19 Corollary.** Every *Bd*-expandable θ-refinable (respectively, metacompact boundedly metacompact, subparacompact) space is boundedly paracompact.

Note that the converse of Theorem 1.18 is not known.

**1.20 Theorem.** The following are equivalent for a normal space $X$:

a) $X$ is collectionwise normal.
b) $X$ is boundedly *Bd*-expandable.
c) $X$ is boundedly expandable.
d) $X$ is discretely expandable.
e) $X$ is discretely *H.C.* expandable.

**Proof.** (a) ⇒ (b) and (b) ⇒ (c) follows from Theorems 1.8 (iv), and 1.8 (iii), respectively. (c) ⇒ (d) is proved in Theorem 2.7 in [10]. (d) ⇒ (e) follows immediately by definitions and (e) ⇒ (a) is (d) ⇒ (a) in Theorem 2.6 in [10].

**1.21 Corollary.** Every *Bd*-expandable normal space is collectionwise normal.

**1.22 Theorem.** Every *Bd*-expandable screenable space is boundedly paracompact.

**Proof.** The proof follows from Theorems 1.8 (ii), 1.18 and Theorem 2.2 in [9].

**1.23 Corollary.** Every *Bd*-expandable strongly screenable space is boundedly paracompact.

**1.24 Theorem.** If $X$ is a *Bd*-expandable space,then every $A_0$ - cover of $X$ has a bounded locally finite open refinement.

**Proof .** Obvious.

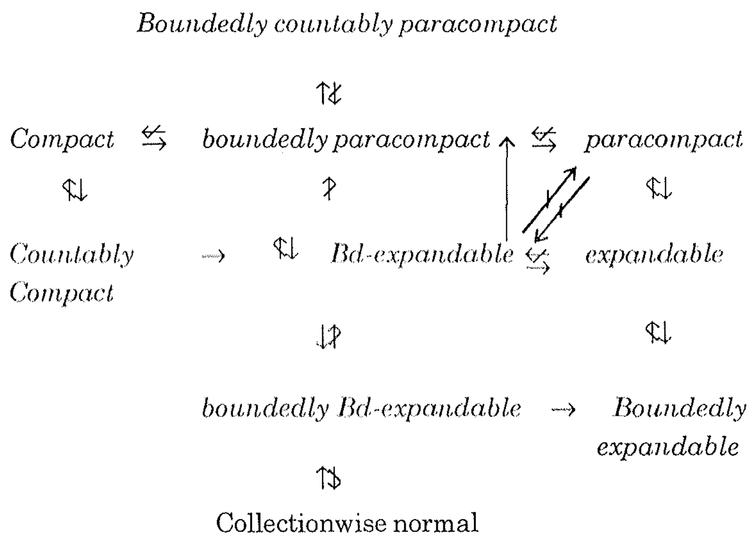
**1.25 Corollary.** If $X$ is a *Bd*-expandable space, then every $A$-cover of $X$ has a bounded locally finite open refinement.

**1.26 Corollary.** Every Bd-expandable Lindelöf space is boundedly paracompact.

**1.27 Theorem.** A topological space $X$ is boundedly *Bd*-expandable iff every $B$-cover of $X$ has a bounded locally finite open refinement.

**Proof.** The proof follows from a simple modification of the proof of Theorem 3.7 in [10].

From the above results together with other known results we have the following implication diagram :




## 2. Open Problems. The following questions remain open :

2.1 Is every boundedly paracompact space *Bd*-expendable?

2.2 Is every *Bd*-expandable space countably compact'?

2.3 Is every boundedly countably paracompact space *Bd*-expandahle?

2.4 Is every boundedly expandable space boundedly *Bd*-expandable ?

If

not,whether these spaces are equivalent in Hausdorff spaces.

Our feeling is such that the answers to problems 2.1 to 2.3 above are in negation, however, we do not know the required counter examples.

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