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(Dedicated to the memory of Professor J.N. Kapur)

A NOTE ON ALMOST PERIODICITY

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ABSTRACT

The theory of Almost Periodic Functions was created by Bohr in 1924. In 1960 three Russian mathemaicians Antonovskii, Boltyanskii and Sarymsakov introduced a new space known as topological semifield and for the first time Iśeki[4] interpreted some results on almost periodicity on topological semifield.

In the present paper we have generalized some results of Gottschalk, Reddy, and Bagley, R.W on almost periodicity in context with topological semifield.

In this paper, we shall consider some results on almost periodic transformations in a metric space over a topological semi field [1].

Definition. If we consider a set F of uniformly continuous transformations defined on the real line, then a transformation f of F will be called almost periodic if to each positive number ϵ there corresponds a positive integer L, so great that among each L successive positive integers there is an integer N satisfying

$$|f^N(x) - f(x)| < \epsilon, -\infty < x < \infty.$$

Let X be a metric space over a topological semi field R. Let ρ be the metric defined on X. Let T be topological group and f be a mapping defined by

$$f : X \times T \to X \text{ such that } f(x,t) = xt \ (t \varepsilon T) \text{ for } x \varepsilon X.$$

We write $f^t(x)$ in place of $f(x,t)$. We shall also suppose that the group $\{f^t : t \varepsilon T\}$ is a transformation group. We assume that T is generative and let S be a replete semi group [2] in T.

Definition. A subset B of T is said to be S syndetic [3] if there exists a compact subset K of T such that $S \subset BK$.

Definition. A transformation group $\{f^t : t \varepsilon T\}$ is said to be S-almost periodic if for each neighborhood *(nbd)* U of 0 in R and for each s in S, there exists a compact subset K of T such that either of the following conditions are satisfied :

i) $\rho\,(f^k(x), f^s(x))\,\varepsilon\, U$ for each $k\varepsilon K$, $x\varepsilon X$.

ii) $\rho\,(x, f^{ks}(x))\,\varepsilon\, U$ for each $k\varepsilon K$, $x\varepsilon X$.

Definition. The transformation group T is discretely almost periodic if and only if the transitive group $\{f^t : t\varepsilon T\}$ is totally bounded in its space index uniformity.

Theorem . Let X be a metric space over a topological semi field R. Then the following propositions are equivalent :

1) $\{f^t : t\varepsilon T\}$ is S-almost periodic,

2) $\{f^t : t\varepsilon T\}$ is discretely almost periodic,

3) The transformation group $\{f^s : s\varepsilon S\}$ is totally bounded by the natural topology of the metric space X over the topological semi field R.

Proof. (1)→ (2) : To prove $\{f^t : t\varepsilon T\}$ is discretely almost periodic, it is sufficient to show that $\{f^s : s\varepsilon S\}$ is totally bounded.

Let U be a *nbd of 0* in R. Take a nbd V of 0 in R such that $V+V\subset U$. Since $\{f^t : t\varepsilon T\}$ is S-almost periodic, there erists a compact subset K of T such that $\rho\,(f^k(x), f^s(s))\varepsilon\, U$ for some $k\varepsilon K$ and for each $s\varepsilon S$ $(x\varepsilon X)$. Since the mapping $f : XxT \to X$ is uniformly continuous, therefore applying the theorem 5 of [4] for the compact set K, $\{f^k : k\varepsilon K\}$ is equi-continuous and is totally bounded also. Hence there exists a finite set P of K such that $\rho\,(f^p(x), f^k(x))\varepsilon V$ for some $p\varepsilon P$, $k\varepsilon K$ $(x\varepsilon X)$. Thus for each $s\varepsilon S$, $k\varepsilon K$ and $p\varepsilon P$, we have

$\rho\,(f^p(x), f^s(x)) \ll \rho\,(f^p(x), f^k(x)) + \rho\,(f^k(x), f^s(x))$

$\varepsilon\;\; V + V \subset U.$

which shows that $\{f^t : t\varepsilon T\}$ is discretely S-almost periodic.

(2) → (3) : It is obvious by the definition of discretely S-almost periodic function.

(3)→ (4) : For a given *nbd* U of 0 in R, we choose a *nbd* V of 0 in R such *that* $V+V+V\subset U$.

Since $\{f^s : s\varepsilon S\}$ is totally bounded, there exists a finite subset P of T such that

$\rho\,(f^p(x), f^s(x))\varepsilon V$ for $s\varepsilon S$ and for some $p\varepsilon P$ $(x\varepsilon X)$.

We select a nbd W of 0 in R such that $\rho\,(x,y)\,\varepsilon\, W$ implies

$\rho\,(f^p(x), f^s(y))\varepsilon V$ for $x, y\varepsilon X$.

We shall show that $\rho(x,y)\varepsilon W$ implies
$\rho(f^{s}(x), f^{s}(y))\varepsilon U$.
Let $\rho(x,y)\varepsilon W$ and let $s\varepsilon S$. We select $p\varepsilon P$ such that
$\rho(f^{p}(x), f^{s}(x))\varepsilon V$.
Thus for $\rho(x,y)\varepsilon W$ and for some $s\varepsilon S$, we have
$\rho(f^{s}(x), f^{s}(y)) << \rho(f^{s}(x), f^{p}(x)) + \rho(f^{p}(x), f^{p}(y)) + \rho(f^{p}(y), f^{s}(y))$
$\varepsilon V+V+V \subset U$,
which shows that $\{f^{s}: s\varepsilon S\}$ is uniformly equicontinuous.

Theorem [5]. Let X be a metric space over a topological semi field R. If $\{f^{t}: t\varepsilon T\}$ be an abelian transformation group defined on X and is almost periodic, then $\{f^{t}: t\varepsilon T\}$ is equicontinuous.

Remark: $\{f^{t}: t\varepsilon T\}$ is said to be equi continuous at $x\varepsilon X$, if for ever *nbd* U of 0 in R, there is a *nbd* V of 0 in R such that $\rho(x,y)\varepsilon V$ implies
$\rho(f^{t}(x), f^{t}(y))\ \varepsilon\ U$ for all $t\ \varepsilon\ T$.
If $\{f^{t}: t\varepsilon T\}$ is equi continuous at each point of X, then
$\{f^{t}: t\varepsilon T\}$ is said to be equi continuous on X.

Proof of Theorem. Let U be a *nbd of* 0 in R. Select a *nbd* V **of** 0 in R such that $V+V+V\subset U$.
since $\{f^{t}: t\varepsilon T\}$ is almost periodic, there exists a compact subset K of T and a set A of T such that $T = AK$ and
$\rho(x, f^{a}(x))\varepsilon$ V for each aεA and $x\varepsilon X$.
Since K is compact therefore $\{f^{k}: k\varepsilon K\}$ is equi continuous, thus for all $x,y\varepsilon X$ and for each *nbd* V of 0 in R there exists *nbd* W of 0 in R such that
$\rho(x,y)\ \varepsilon\ W$ implies
$\rho(f^{k}(x),f^{k}(y))\ \varepsilon\ V(k\ \varepsilon\ K)$.
Now for $t\varepsilon T$ there are elements $a\varepsilon A$ and $k\varepsilon K$ such that $t = ak$. Hence for $\rho(x,y)\varepsilon W$, we have
$= \rho(f^{t}(x),f^{t}(y) = \rho(f^{ak}(x),f^{ak}(y)$
$\rho(f^{ka}(x),f^{ka}(y))$
$<< \rho(f^{ka}(x),f^{k}(x) + \rho(f^{k}(x),f^{k}(y) + \rho(f^{k}(y),f^{ka}(y))$
$\varepsilon\ V+V+V\subset U$.
Hence the proof is completed.

REFERENCES

[1] M. Y. Antonovskii, V.G. Boltyanskii, and T.A. Sarymsakov, *Topological semifields*, Samarkand State University, Tashkent, USSR. 1960.

[2] W.H, Gottschalk and G.A. Hedlund. Almost periodicity, equi continuity and totally boundedness, *Bull. Amer. Math. Soc.*, 52 (1946), 633-636.

[3] W. Reddy. Almost periodic semi groups in transformation groups, *Illinois Journal of Math.*,12 (1968), 494-509.

[4] K. Iseki, A note on almost periodic transformation., *Proc. Japan. Acad.*, 38 (1962), 10-11.

[5] R.W. Bagley, A note on almost periodic transformation groups, *Proc. Japan. Acad.*, 38 (1962), 10-11.