Jñānābha, Vol. 33, 2003
(*Dedicated to the memory of Professor J.N. Kapur*)

A COMMON FIXED POINT THEOREM INVOLVING BEST APPROXIMATION

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(*Received : February 25, 2003*)

Key words and phrases : Compact metric space, fixed-point and normed space, best C-approximant, starshaped subset compatible pair, coincidently commuting pairs.

ABSTRACT

A fixed-point theorem due to Jungck is utilized to derive a common fixed-point theorem for six mappings in compact metric spaces, which is also used to prove another common fixed-point theorem involving best approximation. In process results due to Brosowski, Singh Hicks-Humphris and Sahab et al. are generalized and improved.

1. Introduction . A self-mapping T of a normed space X is said to be non-expansive (resp. I-non expansive) if $\|Tx-Ty\|\leq\|x-y\|$ { resp. $\|Tx-Ty\| \leq \|Ix-Iy\|$} for all x,y in X. If $\bar{x}$ is a point of X and C a subset of X, then set $B_c(\bar{x})$ of best C-approximant to $\bar{x}$ consist of the points y in C such that $\|y-\bar{x}\| = inf \| z-\bar{x}\| : z \in C$ }. A subset C of X is said to be starshaped (cf[2]) with respect to a point $q \in C$ if for all x in C and all $0 \leq \lambda \leq 1$,. $\lambda x + (1-\lambda)q$ is in C. Clearly, a convex set is starshaped with respect to each of its points but the converse is not always true.

Brosowski[1] proved that if T is non-expansive with $\bar{x} \in F(T)$, $T(C) \subset C$ and $B_c(\bar{y})$ is nonempty, compect convex, then T has a fixed point in $B_c(\bar{x})$. Subrahmanyam[13] substituted the nonempty requirement of $B_c(\bar{x})$ with the finite dimensionality of C (as a subspace of X) where as

Singh [10,11] noted that Brosowski's result remains true if $B_c(\bar{x})$ is only starshped, but soon noticed that non-expansive property of T on $B_c(\bar{x}) \cup \{\bar{x}\}$ is enough for his earlier result. In this continuation Hicks-Humphries[4] observed that Singh's first result remains true, if one replaces $T(C) \subset C$ by $T(\delta C) \subset C$, where δC denotes the boundary of C in X. Smoluk[12] substituted 'finite dimensionality of C' in Subrahmanyam's
result by 'linearity of T and compactness of $T(D)$ for every bounded subset D of C which was later improved by Habiniak[3], by relaxing the linearity of T.

In what follows, $F(I, T)$ denotes the set common fixed points of I and T where as $F(A,B,S,T,I,J)$ denotes the set common fixed point of the mapping $A,B,S,T,I,$ and J.

In an attempt to unify and generalize the results due to Hicks-Humphries[4] and Singh[188] Sahab et al [9] proved the following :

Theorem 1.1 : (Sahab et al.[9]) Let X be a normed space, I and T self-maps of X with $\bar{x} \in f(T,I)$, $C \subset X$ with $T(\delta C)$ and $q \in F(I)$. If $D = B_c(\bar{x})$ is compact and q-star shaped $I(D)$-D, I is continuous and linear on D, I, and T are commuting on D and T is I-non expansive on $D \cup (\bar{x})$, then I and T have a common fixed point in D.

We essentially require the following definitions :

Definition 1.1. (Jungck[6]) A pair of self-mappings $(B.I)$ of a normed *space* X is said to be compatible if

$$\lim_{n\to\infty} \|BIx_n - IBx_n\| = 0,$$

whenever $\{x_n\}$ is a sequence in X such that, $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Ix_n = t \in X$

Definition 1.2. (Jungck and Rhoades[8]) A pair of self-mappings (B,I) on X is said to be coincidently commuting if (B,I) commute at the coincidence points of B and I.

In this note using a variant of a fixed point theorem due to Jungck[7], we first derive a common fixed point theorem in compact metric spaces involving six mappings, which is then used to prove yet another extension of Theorem1.1. In process relevent results due to Brosowski[1], Singh[10, 11], Hick-Humphries[4] and Sahab et al [9] are generalized and improved.

2. Main Results. Motivated from the observations explained in Jungck and Rhoades[8], one can state the following variant of a fixed point theorem, which is due to Jungck[7].

Theorem 2.1. (Jungck [7]) Let A, S, I and J be continuous self-mappings of a compact metric space (X,d) with $A(X) \subset J(X)$ and $S(X) \subset I(X)$. If either (A,I) are compatible and (S,J) coincidently commuting or (S,J) are compatible and (A,I) coincidently commuting and

$$d(Ax, Sy) < M(x, y) \text{ where}$$

$$M(x, y) = max\ \{ d(Ix, Jy), d(Ix, Ax), d\ (Jy,Sy), 1/2\ [d\ (Ix, Sy)+d(Jy, Ax)]\},$$

for all $x, y \in X$ with $M\ (x, y) > 0$ then A,S,I and J have a unique common fixed point.

Proof . The proof is almost the same as that of Jungck's[7] theorem hence it is omitted

Remark 2.1. Theorem 2.1 was originally proved with compatibility of both the pairs in Jungck[7].

As an application of Theorem 2.1 we derive a common fixed point theorem for six mappings, as follows :

Theprem 2.2. Let A, B, S, T, I and J be self-mappings of a compact metric space (X, d) such that $A(X) \subset TJ(X)$, $S(X)$ with A, S, T, J and BI being continuous. If either (A, BI) are compatible and (S, TJ) coincidently commuting or (ST,J) are compatible and (A, BI) coincidently commuting and

$$d(Ax, Sy) < M(x, y)\ ,$$

where $M(x, y) = max\ \{ d(BIx, TJy,)\ d\ (BIx,Ax),\ d\ (TJy, Sy),$

$$½[d\ (BIx, Sy) + d\ (TJy, Ax)]\},$$

for all $x,y \in X$ with $M\ (x, y) > 0$, then A, S, BI and TJ have a unique common fixed point z in X . Moreover, if the pairs (B, I) (T, J) (A, B), (A, I), (S, T) and (S, J) commute at the fixed point z then z remains the unique common fixed point of A, B, S, T, I and J separately.

Proof. We begin by noting that the coutinuity of BI (resp. TJ) does not demand the continuity of B or I or both *(resp. T or J or both)*. But for maps A, S, BI and TJ all the conditions of Theorem 2.1 are satisfied ensuring the existence of unique common fixed point z of A, S, BI and TJ. Here it is worth noting that z is the common fixed point of borth pairs (A, BI) and (S, TJ) respectively.

Now it remains to show that z is also a common fixed point A, B, S, T, I and J. For this let z is the unique common fixed point of both the pairs (A, BI) and (S, TJ), then

$$Bz = B(BIz) = B\ (IBz) = BI(Bz), \qquad Bz = B\ (Az) = A(Bz),$$

$$Iz = I(BIz) = IB\ (Iz) = BI(Iz), \qquad Iz = I\ (Az) = A(Iz),$$

$$Tz = T(TJz) = T(JTz) = TJ(Tz), \qquad Tz = T(Sz) = S(Tz),$$
$$Jz = J(TJz) = JT(Jz) = TJ(Bz), \qquad Jz = J(Sz) = S(Jz),$$

which shows that Bz and Iz (*resp.* Tz and Jz) are other fixed points of the pair (A, BI) and (*resp.* S, TJ). Now in view of the uniqueness of common fixed point of the pairs (A, BI) and (S, TJ), we get

$$Z = Bz = Iz = Tz = Jz = BIz = TJz = = Az = Sz,$$

which shows that z also remains the common fixed point of A,B,S,T, I and J separately. This completes the proof.

Remark 2.2. By restricting A,B,S,T,I, and J suitably and modifying the remaining hypothesis accordingly, one can derive a multitude of known and unknown fixed point theorems. So far we are not familiar of any fixed point theorem involving five or six mappings in compact metric spaces.

As an application of Theorem 2.2, we prove the following fixed point theorem (employing the notion of best approximation) which generalizes earlier results due to Brosowski[1], Hicks-Humphries[4], Singh[10], Sahab et al.[9] and others.

Theorem 2.3. Let A,B,S,T,I and J be self-mappings of a normed space X and C be a subset of X such that A, S, : $\delta C \to C$ with $\bar{x} \in F(A, B, S, T, I, J)$ if A, B, S, T, I and J satisfy the condition

$\|Ax - Sy\| < M(x, y)$, with A ans S being continuous where,

$$M(x, y) = max\{\|BIx - TJy\|, \|BIx - Ax\|, \|TJy - Sy\|, \tfrac{1}{2}[\|BIx - Sy\| + \|TJy - Ax)\|]\},$$

for all $x, y \in D' = D \cup C\{\bar{x}\}$.

Futher suppose that the pairs (A, BI) and (S, TJ) are compatible with BI and TJ being linear and continuous on D. If D be a nonempty, compact and starshaped with respect to a point $q \in D$ and $BI(D) = D = TJ(D)$ then $\quad D \cap F(A, B, S, T, I, J) \neq \phi$,

provided the pairs (B, I), (T, J), (A, B), (S, TJ), (A, I) and (S, J) commute at the common fixed point of BI, TJ, A and S.

Proof : Let $y \in D$ then $BIy \in D$ as $BI(D) = D$ Also if $y \in \delta C$ then $Ay \in C$ as $A(\delta C) \subset C$

Using condition (2.5.1), we obtain

$$\|Ay - \bar{x}\| = \|Ay - S\bar{x}\| < M(y, \bar{x}),$$

giving thereby $Ay \in D$. Thus A is a self-mapping of D. Similarly S is also a self-mapping of D.

Let $\{t_n\}$ be a sequence of real nunbers such that $0 \leq t < 1$ and converging to 't'. We define sequences $\{A_n\}$ and of $\{S_n\}$ mapping by

$$A_n x = t_n\, Ax + (1 - t_n)\, q$$
$$0S_n x = t_n\, Sx + (1 - t_n)\, q$$

for all $x \in D$ and for each n.

Since D is starshaped with respect to q hence $\{A_n\}$ maps D into itself and so does $\{S_n\}$ also. Since I is linear, one can have

$$(A_n)\, Ix_n = t_n\, (AIx_n) + (1 - t_n)\, Iq,$$

and

$$I(A_n)\, x_n = t_n\, (IAx_n) + I\,(1 - t_n)\, q.$$

Since (A, BI) are compatible

$$0 \leq \lim_{n \to \infty} \|(BI)\, A_n x_n - A_n (BI) x_n\|$$
$$\leq \lim_{n \to \infty} \|(BI)\, Ax_n - A(BI)x_n\| + \lim_{n \to \infty} (1 - t_n)\, \|q - Aq\| = 0$$

whenever, $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} BIx_n = t \in D$ for all n

Hence (BI, A_n) on D. Similarly it can be shown that (TJ, S_n) are compatible on D.

Futher from

$$\|A_n x - S_n y\| = t_n \|Ax - Sy\| < t_n M(x, y) < M\,(x, y),$$

for all $x, y \in D$. Since I and J are continuous ans D is compact, therefore by Theorem 2.2.

$$F(A_n) \cap F\,(BI) \cap F\,(S_n) \cap F\,(TJ) = \{X_n\},$$

for each n Also since D is compact so $\{x_n\}$ has a convergent subseqence $\{x_{n_i}\}$ converging to z in D.

Now $x_{n_i} = A_{n_i} x_{n_i} = t_{n_i}\, Ax_{n_i} + (1 + t_{n_i})\, q,$

which on letting $n \to \infty$ reduces to $Az = z$ giving thereby $z \in D \cap F\,(A)$. Similarly, it can be shown that $z \in D \cap F(S)$. Since BI and Tj are continuous, we have

$$BIz = BI \lim_{i \to \infty} x_{n_i} = \lim_{i \to \infty} BIx_{n_i} = \lim_{i \to \infty} x_{n_i} = z$$
$$TJz = TJ \lim_{i \to \infty} x_{n_i} = \lim_{i \to \infty} TJ\, x_{n_i} = \lim_{i \to \infty} x_{n_i} = z$$

yielding thereby $BIz + TJz = Az = Sz = z$.

Let the pairs (A, BI) and (S, TJ) have different fixed point u and v respectively, then $\|u - v\| = \|Au - Sv\|$

$$< max\ \{\|BIu - TJv\|, \|Blu - Au\|\, TJv - Sv\|,$$
$$½\ [\|BIu - Sv\| + \|TJv - Au\|]\}$$

which is a contradiction, implying thereby $u = v$, thus both the pairs have same common unique fixed point $u = v = z$.

Now on the lines of the proof of Theorem 2.2, it can be easily shown that

z remains the unique common fixed point of A, B, S, T, I and J

Hence, we conclude that

$$D \cap F(A, B, S, T, I, J) \neq \phi .$$

This completes the proof.

Remarks 2.3

(i) Theorem 2.3 extends the result of Sahab et al[9] as we generalized contractions along with compatibility (cf [6]) instead of commutativity. Also Theorem 2.3 involves six mappings instead of two mappings. In Process related results due to Hicks-Humphries[4], Singh [10], Brosowski[1] and others are modified and improved either partially or completely.

(ii) If we use a fixed point theorem in complete spaces corresponding to Theorem 2.2 then the continuity requirement of any one of the maps A,S,BI or TJ can serve the purpose which is possible due to the fact that compact metric spaces are always complete. But due to a shorter proof we opt to use Theorem.

REFERENCES

[1] B. Brosowski, Fixpunktsatze in der oppoximations theorie, Mathematica *(Cluj)*, **11** (1969), 195-200.

[2] W. J. Dotson, Fixed point theorem for non-expansive mappings on starshaped subsets of Banach spaces. *J. London Math. Soc.*, (2) 4 (1972), 408-410.

[3] L. Habiniak, Fixed point theorems and invariant appoximations, *J. Aprox. Theory*, **56** (1989), 241-244

[4] T. L. Hicks, and M. D. Humphries, A note on fixed point theorem, *J. Approx. Theory*, **34** (1982), 221-225.

[5] G. Jungck, An iff fixed point criterion, *Math Magazine*, **49** (1976), 32-34.

[6] G. Jungck, compatible mappings and common fixed points, *Internat. J. Math. and Math. Sci.*, **9 (4)** (1986), 771-779.

[7] G. Jungck, Common fixed points for commuting and compatible maps on compacta, *Proc. Math. Soc.*, **103** (1988), 977-993.

[8] G. Jungck and B.E. Rhoades, Fixed points for set-valued mappings without continuity, Prepreint.

[9] S. A. Sahab, M. S. Khan and S. Sessa, Aresult in best approximation theory, *J. Approx. Theory*, **55** (1988), 349-351.

[10] S.P. Singh, An application of fixed point theorem to approximation theory, *J. Approx. Theory*, **25** (1979). 89-91.

[11] S.P. Singh, *Application of fixed point theorems in approximation theory in "Applied Nonlinear Analysis"* (V. Lakshmikantham, Ed), *PP*. 389-397, Academic Press, New York, 1979.

[12] A. Smoluk Invariant approximations, *Mathematyka Stosowana*, **17** (1981) 17-22.

[13] P.V. Subrahmanyam, An application of a fixed point theorem to best approximation, *J. Approx Theory*, **20** (1977), 165-172