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(Dedicated to the memory of Professor J.N. Kapur)

COMMON FIXED POINT THEOREM IN REFLEXIVE BANACH SPACES

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ABSTRACT

In this paper few common fixed point theorems of mappings which map a reflexive Banach Space into itself are proved. The results of this paper extend the results which have been proved in {[1-3],[5],[10]}.

Introduction. A large number of literatures are available which deal with common fixed points of two mappings or a family of mappings in a complete metric space or in a Hilbert Space etc. The conditions considered in the theorems of these literatures are taken in such a way that the method of iterations provides the way to get fixed points easily. But if the condition is extended or multiplier is extended it becomes very difficult to obtain fixed points of mappings by iteration method. In this case either some conditions require to impose on the space or on the mappings or both. Also the method is different. One can see { [1- 3],[5],[10]}.

In this paper we have proved a common fixed point for two mappings in the *Ist* theorem and in the *2nd* theorem for a sequence of mappings in a reflexive Banach space with some additional condition on the space as well as on the operators. The following is quoted from [5].

Definition [5]. Let A be a bounded subset of a Banach Space X. A point $a\in X$ is said to be a non-diametrial point of A if $\sup\{\|x-a\|, x\in A\}<\delta)(A)$.

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A bounded convex subset K of X is said to have normal structure if for each convex subset H of K which contains more than one point there exists an $x \in H$ which is a non-diametral point of H.

We now proved the following theorems.

Theorem 1. Let X be a reflexive Banach Space and K be a non-empty bounded closed convex subset of X. Let $T_1, T_2 : K \to K$ be such that

(A) $\|T_1 x\ T_2 y\| \leq max\{\|x-y\|, \|x-T_1 x\|, \|y-T_2 y\|$ for all $x, y \in K$

(B) K has a Normal Structure,

(C) $T_1 C \subset C$ if and only if $T_2(C) \subset C$ for each closed subset C of K, and

(D) Either $\sup_{x \in F} \|x - T_1 x\| < \delta(F$ or $\sup_{y \in F} \|x - T_2 x\| < \delta(F)$,

for every non-empty bounded closed convex subset of K which is mapped into itself by T_1 or T_2. Then T_1 and T_2 have a common fixed point in X.

Proof. Let Y denotes the family of all non-empty bounded closed convex subsets of K ordered by set inclusion which are mapped into itself by T. By Smulian's result [9] i.e. X is reflexive if and only if every decreasing sequence of non-empty bounded closed convex subsets of X has a non-empty intersection and by Zorn's lemma, it follows that X possesses a minimal element F, say. If F contains only one element then that element becomes a fixed point of T. We shall show that F contains only one element.

We suppose, on the contrary, that F contains more than one point, which we shall show implies a contradiction.

Let $A = \sup_{y \in F} \|T_2 y - y\|$. By the condition, $A < \delta(F)$ we now define the following terms.

For $x \in F$, let,

$$\upsilon_x(F) = max\{\sup_{y \in F}\|x - y\|, A\},$$

$$\upsilon_x(F) = inf\{V_x(F), x \in F\},$$

$$F_c = \{x \in F\, ; \upsilon_x(F) = \upsilon(F)\}.$$

We now show that F_c is nonempty, closed and convex.

For a positive integer n and for $x \in F$, let

$$F(x, n) = \{y \in F\, ; \|x - y\| \leq \upsilon(F) + 1/n\} \text{ and}$$

$$C_n = \bigcap_{y \in F} F(x, n).$$

We show first that C_n is non-empty. If possible let $C_n = \phi$, then there exist $x_1\ x_2 \in F$ such that $F(x_i, n) \cap F(x_2, n) = \phi$. By construction

$F(x_1, n) = \{y \in F : \|x_1 - y\| < \upsilon(F) + 1/n\}$ and similarly for $F(x_2, n)$. We obtain from the disjointness of $F(x_1, n)$ and $F(x_2, n)$

$$\|x_2 - x_2\| \leq 2\upsilon(F) + 2/n$$

Now for $x \in F$, $\sup_{y \in F} \|x-y\| \frac{\delta(F)}{2}(F)$ and so $\upsilon_x(F) \geq \frac{\delta(F)}{2}$ and this implies.

$$\frac{\delta(F)}{2} \leq \upsilon(F)$$

Therefore $\delta(F) < 2\upsilon(F) + 2/n$. So from(1)

$\|x_2 - x_2\| \delta(F)$ which is a contradiction because $x_1, x_2 \in F$.

C_n is non-empty.

It may further be verified that C_n is closed bounded convex and that $C_{n+1} \subset C_n$.

We wish to show that $F_c = \bigcap_{n=1}^{\infty} C_n$.

For this, let $y \in F_c$ then $\upsilon \in (F) = \upsilon(F)$.

So $\max\{\sup_{x \in F} \|y-x\|, A\} = \upsilon(F)$ and so

$$\sup_{x \in F} \|y - x\| \leq \upsilon(F). \qquad \text{...(2)}$$

We verify that $y \in F(x,n)$ for all $x \in F$ and for all n.

If possible, let $y \notin F(x,n)$ for some x and for some n. Then

$$\|x - y\| > \upsilon(F) + 1/n \qquad \text{... (3)}$$

From (2) we see that $\|x - y\| \leq \upsilon(F)$ which is a contradiction to (3).

Therefore $\quad y \in \bigcap_{n=1}^{\infty} C_n$ *and so* $F_C \subset \bigcap_{n=1}^{\infty} C_n$.

Next let $y \in \bigcap_{n=1}^{\infty} C_n$. Then $y \in F(x,n)$ for all x and for all n and this implies that $\quad \sup_{x \in F} \|y - x\| \leq \upsilon(F)$.

Also $A \leq v(F)$. These two together give $\upsilon_y(F) \leq \upsilon(F)$. But $\upsilon(F) \leq \upsilon_y(F)$ always, and therefore $\upsilon_y(F) = \upsilon(F)$ and this gives $y \in F_c$. Thus $\bigcap_{n=1}^{\infty} C_n \subset F_c \therefore F_c = \bigcap_{n=1}^{\infty} C_n$.

This equality further gives that F_c is closed and convex by Smulians result [9] non empty.

Next we show that $\delta(Fc) < \delta(F)$.

Since K has normal structure and $A < \delta(F)$. There exists a point $x \in F$ such that $\upsilon_x(F) < \delta(F)$. If $x_1, x_2 \in Fc$ then $\|x_1 - x_2\| < \upsilon x_1(F) = \upsilon(F)$.

So $\quad \delta(F_c) = \operatorname{Sup}\{\|x_1 - x_2\| : x_1, x_2 \in F_c\}$

$$< \upsilon(F) \leq \upsilon_x(F) < \delta(F) \qquad \text{...(4)}$$

If $x \in F_c$ and y is an arbitrary element of F we obtain

$$\|T_1x - T_2y\| < \max\{\|x-y\|, \|x - T_1x\|\}$$

$$\leq \max\{\operatorname{Sup}_{y \in F}\|x-y\|, \operatorname{Sup}_{y \in F}\|y - T_2y\|, \operatorname{Sup}_{y \in F}\|x - T_1x\|\}$$

$$\leq \max\{\operatorname{Sup}_{y \in F}\|x - y\|, A\} = \upsilon_x(F) = \upsilon(F).$$

So the set $T_2(F)$ is contained in a closed sphere with centre at T_1x and radius $\upsilon(F)$. We denote this sphere by $\bar{U}$.
Clearly $T_2(F \cap \bar{U}) \subset F \cap \bar{U}$ and because F is minimal, $F \subset \bar{U}$ and so,

$$\sup_{y \in F} \|T_1x - y\| \leq \upsilon(F) \qquad \ldots(5)$$

Now

$$\upsilon_{T_1x}(F) = \max\{\sup_{y \in F} \|T_1x - y\|, A\}$$
$$\leq \max\{\upsilon(F), A\}, \text{ from (5)},$$
$$= \upsilon(F), \text{ because } \upsilon(F) \geq A.$$

Hence $\upsilon_{T_1x}(F) \leq \upsilon(F)$. But we always have $\upsilon(F) < \upsilon_{T_1x}(F)$. So $\upsilon_{T_1x}(F) = \upsilon(F)$. This implies that $T_1(x) \in F_c$. Similarly $T_2(x) \in F_c$. Therefore F_c is a non empty closed convex subset of F which is mapped into itself by T_1 and T_2 and because of (4), $\delta(F_c) < \delta(F)$. Therefore F_c is a proper subset of F. This contradicts the fact that F is minimal. Thus F can not contain more than one element. But F is non-empty. Hence F contains only one element which is clearly fixed point of T_1 and T_2.

Note 1. If $T_1 = T_2$ the theorem proved in Achari and Lahiri [1] and Tiwary and Lahiri [10] follows.

Note 2. $T_1 = T_2$ and if the condition (C) is withdrawn the theorem proved in Kirk[5] follows.

Theorem 2. Let X be a reflexive Banach space and K be a non-empty bounded closed convex subset of X. Let $\{T_n\}$ be a sequence of mappings which map K into itself by $\{T_n\}$ and satisfy

(A) $\|T_ix - T_jy\| \leq \max\{\|x - y\|, \|x - T_ix\|, \|y - T_jy\|\}$ *for all* $x, y \in K$

(B) K has a normal structure,

(C) $T_i(C) \subset C$ if and only if $T_j(C) \subset C$ for each closed subset C of K and $T_i, T_j \in \{T_n\}$ and

(D) Either $\sup_{x \in F} \|x - T_ix\| < \delta(F)$

or

$$\sup_{x \in F} \|x - T\| < \delta(F),$$

for every non empty bounded closed convex subset of K which is mapped into itself by either T_i or T_j.

Then $\{T_n\}$ has a common fixed point in X.

Proof. Let F denote the family of all non empty bounded closed convex subsets of K which are mapped into itself by $\{T_n\}$ and ordered by set inclusion. The family F is non empty and because X is reflexive by Smulian's result [9] every decreasing sequence of non empty bounded closed convex subsets of X has a non empty intersection and by Zorn's lemma, it follows that X possesses a minimal element F, say.

Picking any two mapping T_i and T_j from $\{T_n\}$ and following the proof of the Theorem 1 it follows that, T_i and T_j have a common fixed point in X.

Since T_i and T_j are any two mappings, it follows that $\{T_n\}$ has a common fixed point in X.

This completes the proof of the theorem.

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