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(Dedicated to the memory of Professor J.N. Kapur)

NEARLY METACOMPACT SPACES

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ABSTRACT

The notion of near metacompactness is introduced by generalizing the notion of metacompactness as well as that of near paracompactness. A topological space X is said to be nearly metacompact if every regular open cover of X has a point-finite open refinement. The class of nearly metacompact spaces contains the class of metacompact spaces as well as that of nearly paracompact spaces both. In collectionwise normal spaces the notion of near metacompactness is equivalent to that of near paracompactness. In semi-regular spaces the notion of near metacompactness coincide with that of metacompactness. The near metacompactness strengthens countably compactness to near compactness.

Introduction . In the present paper we introduce the notion of near metacompactness by generalizing the notion of metacompactness as well as that of near paracompactness. A topological space X is said to be nearly metacompact if every regular open cover of X has a point-finite open refinement. The class of nearly metacompact spaces contains the class of metacompact spaces as well as that of nearly paracompact spaces both. In collectionwise normal spaces the notion of near metacompactness is equivalent to that of near paracompactness. In semi-regular spaces the notion of near metacompactness is equivalent to that of metacompactness. The near metacompactness strengthens countably compactness to near compactness. The notion of near compactness is due to Singal and Mathur [5]. A topological space X is said to be nearly compact if every regular open cover of X has a finite subcover. The notion of near paracompactness is due to Singal and Arya [4]. A topological space X is said to be nearly paracompact if every regular open cover of X has a locally finite open refinement, the whole paper is divided into five sections. In section 1 the notion of near metacompactness is introduced and studied. In section 2 several counter examples are given which demonstrate the invalidity of several possible conjectures concerning nearly metacompact spaces. In section 3 nearly metacompact spaces and sum and product theorems are given. In section 4 nearly metacompact spaces and subset theorems are given. Lastly in section 3 nearly metacompact spaces and mapping theorems are given.

1. Nearly Metacompact Spaces

1.1 Definition . A topological space X is said to be nearly metacompact if every regular open cover of X has a point-finite open refinement.

1.2 Theorem. A topological space X is nearly metacompact iff every δ-open cover of X has a point-finite open refinement.

Proof. Obvious in view of the fact that regularly open sets form a base of δ-open sets.

1.3 Theorem . Every nearly paracompact space is nearly metacompact.

Proof. Obvious in view of the fact that every locally finite collection is point-finite.

1.4 Theorem. Every metacompact space is nearly metacompact.

Proof. Obvious in view of the fact that every regular open cover is also an open cover.

1.5 Theorem. A semi-regular space X is metacompact iff X is nearly metacompact.

Proof. If X is metacompact then by Theorem 1.4 above, X is nearly metacompact. So assume that X is a semi-regular nearly metacompact space. Let U be an open cover of X. Since in a semiregular space every open set is δ-open. therefore, U is also a δ-open cover of X. Since X is nearly metacompact, therefore, by Theorem 1.2, U has a point-finite open refinement. Hence X is metacompact.

Michael [2] has proved the following theorem:

1.6 Theorem. [2] Every point-finite open covering of a collectionwise normal space has a locally finite open refinement.

Using Theorem 1.6 above we prove the following theorem :

1.7 Theorem . A collectionwise normal space X is nearly paracompact iff

X is nearly metacompact.

Proof. Let X be a collectionwise normal space. If X is nearly paracompact then by Theorem 1.3 X is nearly metacompact. So assume that X is a nearly metacompact space. Let U be a regular open cover of X. Then by near metacompactness of X, U has a point-finite open refinement, say V. Now by Theorem 1.6 above, V has a locally finite open refinement, say W. Then W is a locally finite open refinement of U. Hence X is nearly paracompact.

The following theorem is due to Richard and Dugundji [3]

1.8 Theorem. [3] Let $\{A_\alpha : \alpha \varepsilon \Delta\}$ be a point-finite covering of a topological space X. Then there exists an irreducible subcovering, that is, a subcovering that, when any single set is removed, is no longer a covering of X

Using theorem 1.8 above we prove the following theorem:

1.9 Theorem. Every countably compact nearly metacompact space is *nearly compact.*

Proof. Let X be a countably compact nearly metacompact space. Let $U = \{U_\alpha : \alpha \varepsilon A\}$ be a regular open cover of X. By the near metacompactness of X, U has a point-finite open refinement, say $V: \{V_\beta : \beta \varepsilon B\}$. Then by Theorem 1.8 above, V has an irreducible subcovering, say $\{V_\gamma : \gamma \varepsilon \Gamma\}$. This minimal covering must be finite. For, we can find in each V_γ a point y_γ belonging to no set other than V_γ and if $V = \{V_\gamma : \gamma \varepsilon \Gamma\}$ is not finite then $\{y_\gamma : \gamma \varepsilon \Gamma\}$ is an infinite subset of X. Let Δ be a countably infinite subset of Γ. We may index the element of Δ as follows : $\Delta = \{\gamma_1, \gamma_2 \ldots\}$. Then clearly $\{y_{\gamma_1}, y_{\gamma_2}, \ldots\}$ is a sequence in X. Let $x \varepsilon X$. Then there exists a $\gamma \varepsilon \Gamma$ such that $x \varepsilon V\gamma$. Now if $\gamma \varepsilon \Delta$, then there is some positive integer i such that $y_{\gamma_i} \varepsilon V_\gamma$. Now consider the positive integer $i + 1$, then clearly by the construction of the set $\{y_\gamma : \gamma \varepsilon \Gamma\}$ and the sequence $\{y_{\gamma_1}, y_{\gamma_2}, \ldots\}$ it follows that there is no positive interger $n \geq i+1$ such that $y_{\gamma_n} \varepsilon V_\gamma$. and if $\gamma \notin \Delta$, then clearly V_γ contains no point of the sequence $\{y_{\gamma_1}, y_{\gamma_2} \ldots\}$. Hence x is not an accumulation point of the sequence $\{y_{\gamma_1}, y_{\gamma_2} \ldots\}$. From this it follows that the sequence $\{y_{\gamma_1}, y_{\gamma_2}, \ldots\}$ in X has no accumulation point which contradicts the fact that X is cuntably compact. Therefore, $\{V_\gamma : \gamma \varepsilon \Gamma\}$ is a finite open covering of X. Choose for each $\gamma \varepsilon \Gamma$ an $\alpha(\gamma) \varepsilon A$ such that $V_\gamma \subset U_{\alpha(\gamma)}$. Then $\{U_{\alpha(\gamma)} : \gamma \varepsilon \Gamma\}$ is a finite subcovering of U. Therefore, X is nearly compact.

2. Counter-Examples.

2.1 ***Example.*** Nearly metacompact space which is not metacompact.

Let $X = \{a_{ij}, a_i, a : i, j = 1, 2, \ldots\}$. Let each a_{ij} be an isolated point. Let $\{U^n(a_i):$

$n = 1, 2, \ldots\}$ be the fundamental system of neighborhoods of a_i, where $\{U^n(a_i) = \{a_i, a_{1i}, a_{ij} : j \geq n\}$ for each $n = 1, 2, \ldots$. Let $\{V^n(a) : n = 1, 2, \ldots\}$ be the fundamental system of neighborhoods of a, where $V^n(a) = \{a, a_{11}\, a_{ij} : i \geq n, j \geq n\}$ for each $n = 1, 2, \ldots$. Then $B = \{\{a_{ij}\}\ i, j = 1, 2, \ldots\}\ U\ \{\{U^n(a) : n = 1, 2, \ldots\} : i = 1, 2, \ldots\}\ U\ \{V^n(a) : n = 1, 2, \ldots\}$ is a base for a topology, say τ, on X. Then (X, τ) is a nearly metacompact space but it is not metacompact.

2.2 ***Example.*** Nearly metacompact normal Hausdorff space which is not *nearly paracompact.*

The Michael's modification [2] of the Bing's space [1] is an example of space which is nearly metacompact normal and Hausdorff but not nearly paracompact.

2.3 ***Example.*** A nearly metacompact space with a subset which is not *nearly metacompact.*

The Tychonoff plank $X = [0, \Omega] \times [0, \omega]$ is a nearly metacompact space but the deleted Tychonoff plank $Y = X - \{\Omega, \omega)\}$ is not a nearly metacompact space.

2.4 ***Example.*** A nearly metacompact space X such that $X \times X$ is not *nearly metacompact.*

The Sorgenfrey line S is nearly metacompact but the Sorgenfrey plank $S \times S$ is not nearly metacompact.

2.5 Note . Frolm the results of section 2.1 and 2.2 above together with other known results we have the following implication diagram:

$$
\begin{array}{ccc}
\textit{paracompact} & \overset{\not\Leftarrow}{\Rightarrow} & \textit{metacompact} \\
\not\Uparrow\Downarrow & & \Downarrow\not\Uparrow \\
\textit{nearly paracompact} & \overset{\not\Leftarrow}{\Rightarrow} & \textit{nearly metacompact.}
\end{array}
$$

3. Nearly Metacompact Spaces and Sum and Product Theorem

3.1 Theorem. The disjoint topological sum of nearly metacompact space is nearly metacompact.

Proof. Lef $\{X_\alpha : \alpha \varepsilon A\}$ be a disjoint family of nearly metacompact spaces. Let X denote the disjoint topological sum of this family. Let $U = \{U_\beta :: \beta \varepsilon B\}$ be a regularly open cover of X. Then for each $\alpha \varepsilon A$, $\{U_\beta \cap X_\alpha : \beta \varepsilon B\}$ is an open cover of X_α and so $\{Int_{X\alpha}\, Cl_{X\alpha}(U_\beta \cap X_\alpha) : \beta \varepsilon B\}$ is a regularly open cover of X_α Since X_α is nearly metacompact, therefore $\{Int_{X\alpha}\, Cl_{X\alpha}(U_\beta \cap X_\alpha) : \beta \varepsilon B\}$ has a point-finite open $(in\ X_\alpha)$, refinement, say V_α .Then surely $U = \cup_{\alpha \varepsilon A} V_\alpha$ is a point-finite open refinement of U . Hence X is nearly metacompact.

3.2 Theorem. Let $\{G_\alpha : \alpha \varepsilon A\}$ be a family of subsets of topological space X

such that $\{Int_X G_\alpha : \alpha\varepsilon A\}$ forms a point-finite open covering of X. If each G_α is nearly metacompact, then X is nearly metacompact.

Proof. Let $\{U_\beta : \beta\varepsilon B\}$ be any regular open covering of X. Then for each $\alpha\varepsilon A$, $\{U_\beta : \beta\varepsilon B\}$ is a relatively open covering of G_α. Then $\{Int_{G\alpha}\, Cl_{G\alpha}(U_\beta \cap G_\alpha)$ $: \beta\varepsilon B\}$. is a regularly open (in G_α) cover of G_α for each $\alpha\varepsilon A$. Since G_α is nearly metacompact, therefore $\{Int_{G\alpha}\, Cl_{G\alpha}(U_\beta \cap G_\alpha) : \beta\varepsilon B\}$ has a point-finite (in G_α) open (in G_α) refinement, say $\{V_\beta \cap G_\alpha : \gamma\varepsilon\Gamma_\alpha\}$ where each V_α is open in X. Then surely $\{V_\gamma \cap Int_x G_\alpha : \alpha\varepsilon A\}$ is a point-finite open refinement of $\{U_\beta : \beta\varepsilon B\}$. Hence X is nearly metacompact.

3.3 Corollary. Let $\{G_\alpha : \alpha\varepsilon A\}$ be a point - finite open covering of a topological space X. If each G_α is nearly metacompact, then X is nearly metacompact.

3.4 Theorem. The product of a nearly metacompact space with a nearly compact space is nearly metacompact.

Proof. Let X be a nearly metacompact and Y a nearly compact space. Let $\boldsymbol{U}$ be a regularly open cover of $X \times Y$. Let $(x,y)\ \varepsilon\ X \times Y$. Then there exists regularly open subset V_{xy} and W_{xy} of X and Y respectively, such that $(x,y)\ \varepsilon\ V_{xy}\ W_{xy} \subset U$ for some $U\varepsilon\ \boldsymbol{U}$. Let $I^x = \{x\} \times Y$ for each $x\ \varepsilon\ X$. Then $\{W_{xy}$ $: (x,y)\ \varepsilon\ I^x\}$ is a regularly open covering of the nearly compact space Y and therefore there exists a finite subset J^x of I^x such that $\{W_{xy} : (x,y)\ \varepsilon\ J^x\}$ is a covering of Y. For each $x\varepsilon X$ let $V_x = \cap\ \{V_{xy}\ (x,y)\ \varepsilon\ J^x\}$. Then V_x is a regularly open subset of X containing x. Let $\boldsymbol{V}\ \{V_x : x\varepsilon\ X\}$. Then $\boldsymbol{V}$ is a regularly open covering of X. Since X is nearly metacompact, therefore, $\boldsymbol{V}$ has a point -finite open *(in X)* refinement, say $\boldsymbol{G}$. Now for each $G\varepsilon\ \boldsymbol{G}$ there exists $x_G\varepsilon\ X$ such that $G \subset V_{x_G}$. Now let $\boldsymbol{H} = \{G \cap W_{xy}: G\varepsilon\boldsymbol{G}, (x,y)\varepsilon J^x,$ $x\ \varepsilon\ X\}$. Then surely $\boldsymbol{H}$ is a point finite open refinement of $\boldsymbol{U}$. Hence $X \times Y$ is nearly metacompact.

3.5 Corollary. The product of a nearly metacompact space with a compact space is nearly metacompact.

4. Nearly Metacompact Spaces and Subset Theorem.

4.1 Theorem. Every clo-open subset of nearly metacompact space is nearly metacompact.

Proof. Let X be a nearly metacompact space and Y be a clo-open subset of X. Let $\boldsymbol{U} = \{U_\alpha : \alpha\varepsilon A\}$ be a cover of Y by regularly open subsets of Y. Since regularly open subsets of a clo-open subset Y of X is a regularly open subset of X, therefore, each U_α is a regularly open subset of X. Also clearly $X-Y$ is a regularly open subset of X. Therefore, $\boldsymbol{U} \cup \{X-Y\}$ is a regularly open cover of X. Since X is nearly metacompact, therefore, there exists a point-finite open (in X) refinement, say $V = \{V_\beta : \beta\varepsilon B\}$ of $U \cup \{X - Y\}$. Then surely $V \cap Y = \{V_\beta \cap Y : \beta\varepsilon B\}$ is a point-finite (in Y) open (in Y) refinement of U. Therefore, Y is nearly metacompact.

5. Nearly Metacompact Space and Mapping Theorem.

5.1 Theorem . Let $f : X \to Y$ be an almost conitinuous open surjection such that $f^{-1}(y)$ is a finite subset of X for each $y\varepsilon Y$. Then Y is nearly metacompact, if X is nearly metacompact.

Proof. Let X be a nearly metacompact space. Let $U = \{U_\alpha : \alpha\varepsilon A\}$ be a regularly open cover of Y. Since f is almost continuous open surjection and hence is almost continuous almost open surjection, therefore, $f^{-1}(U) = \{f^{-1}(U_\alpha) : \alpha\varepsilon A\}$ is a regularly open cover of X. Therefore, there exists a point-finite open refinement, say $V = \{V_\beta : \beta\varepsilon B\}$ of $f^{-1}(U)$. Then surely $f(V) = \{f(V_\beta) : \beta\varepsilon B\}$ is a point - finite open refinement of U. Hence Y is nearly metacompact.

5.2 Theorem. Let $f : X \to Y$ be a continuous δ - closed surjection such that $f^{-1}(y)$ is N-closed for each $y\varepsilon Y$. Then X is nearly metacompact, if Y is nearly metacompact.

Proof. Let Y be a nearly metacompact space. Let $U = \{U_\alpha : \alpha\varepsilon A\}$ be a regularly open cover of X. Let Γ be that collection of all finite subsets of A. Let $V_\gamma = Y - f(X - \cup_{\alpha\varepsilon\gamma} U_\alpha)$ for each $\gamma\varepsilon\Gamma$. Then surely $V = \{V_\gamma : \gamma\varepsilon\Gamma\}$ is a δ-open cover of Y such that $f^{-1}(V_\gamma) \subset \cup_{\alpha\varepsilon\gamma} U_\alpha)$ for each $\gamma\varepsilon\Gamma$. Since Y is nearly metacompact. therefore, V has a point-finite open refinement, say $W = \{W\delta : \delta\varepsilon A\}$. Since for each $\delta\varepsilon A$ there is $\gamma_\delta\varepsilon\Gamma$ such that $W_\delta \subset V_{\gamma\delta}$ therefore, for each $\delta\varepsilon\Delta$, there exists a $y_\delta\varepsilon\Gamma$ such that $f^{-1}(W_\delta) \subset f^{1}(V_{\gamma\delta}) \subset \cup_{\alpha\varepsilon\gamma\delta} U_\alpha$ Then surely $\{f^{-1}(W_\delta) \cap U_\alpha : \alpha\varepsilon\gamma_\delta, \delta\varepsilon\Delta$ is a point- finite open refinement of U. Hence X is nearly metacompat.

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