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(Dedicated to Professor H. M. Srivastava on his 62nd Birthday)

A GENERALIZATION OF A GENERATING FUNCTION BY CHAUNDY

By

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ABSTRACT

The elementary manipulation of series and the binomial theorem are employed to deduce a new generating function for a class of hypergeometric polynomials which gneralizes a result presented by T.W. Chaundy in 1943.

If the Pochhammer symbol (a,n) is given by

$$(a,n)=a(a+1)(a+2)...(a+n-1);\ (a,0)=1, \tag{1}$$

consider the double series, assumed to be absolutely convergent,

$$S=\Sigma[(b,(k-1)m+r)C_m x^m y^r]/[m!r!], \tag{2}$$

where $k = 1, 2, 3, ...$ and C_m is a generalized coefficient.

Replace r by n-km, when

$$S=\Sigma[C_m(b,n-m)x^m y^{n-km}]/[m!(n-km)!]. \tag{3}$$

Compare Srivastava and Manocha (1984) page 100.

The indices of summation run over all of the non-negative indices and any quantities leading to results which do not make sense are tacitly exeluded.

With a slight modification of the notation, we then have

$$\Sigma[C_n x^n(b,(k-1)n)\,{}_1F_0\,[b+(k-1)n;-;y]/n!$$
$$=\Sigma\,(b,n)y^n/[n!]\ \Sigma[C_m(-n,km)(-1)^{k-1}x^m y^{-km}/[m!(1-b-n,\ m)]. \tag{4}$$

The generalized hypergeometric function occurs in the following analysis and is given by

$${}_AF_B\,[a_1,a_2,...,a_A\,;\,b_1,b_2\,,...,b_B\,;x)$$
$$=\Sigma[(a_1,n)\,(a_2,n)...(a_A,n)x^n]/\,[(b_1,n)(b_2,n)...(b_B\,,n)n!]. \tag{5}$$

See Slater (1966), for example.

If the binomial theorem is applied to the ${}_1F_0$ series on the left of (4), we obtain the expression

$$(1\text{-}y)^{-b}\Sigma C_n(b,(k\text{-}1)n)[x(1-y)^{1-k}]^n/n!$$
$$=\Sigma\,(b,n)y^n/[n!]\ \Sigma[C_m(-n,km)(-1)^{km-m}x^m y^{-km}]/[m!(1-b-n,\ m)]. \tag{6}$$

A generating function for a class of hypergeometric polynomials which seems to

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be new can be obtained from (6) if the coefficient C_n is replaced by a product of Pochhammer symbols, namely

$$C_n = [(a_1,n)\,(a_2,n)\ldots(a_A,n)]/\,[(c_1,n)(c_2,n)\ldots(c_C,n)], \tag{7}$$

bearing in mind that

$$(f,km) = (f/k,m)(f/k+1/k,m)(f/k+2/k,m)\ldots(f/k+1-1/k,m)k^k. \tag{8}$$

If x is then replaced by $(-1)^{k-1}xy^k$, it follows that

$$\Sigma\,(b,n)[n!]^{-1}\,y^n\;{}_{A+k}F_{C+1}\left[\begin{matrix}a_1,a_2,\ldots,a_A,\ -n/k,\ 1/k-n/k,\ 2/k-n/k,\ \ldots,\ 1-1/k-n/k; \\ c_1,\ c_2,\ c_C,\quad 1-b-n;\end{matrix}\ x\right]$$

$$= (1-y)^{-b}\;{}_{A+k-1}F_C\left[\begin{matrix}a_1,a_2,\ldots,a_A,b/(k-1),(b+1)/(k-1),\ldots,(b+k-2)/(k-1); \\ c_1,\ c_2,\ c_C;\end{matrix}\ (-1)^{k-1}xy^kk^{-k}\,(k-1)^{k-1}\right] \tag{9}$$

If $k=1$, a result by Chaundy (1943) is recoverd, that is

$$\Sigma\,(b,n)[n!]^{-1}\,y^n\;{}_{A+1}F_{C+1}\left[\begin{matrix}a_1,a_2,\ldots,a_A,\ -n; \\ c_1,\ c_2,\ c_C,\ 1-b-n;\end{matrix}\ x\right]$$

$$= (1-y)^{-b}\;{}_AF_C\left[\begin{matrix}a_1,a_2,\ldots,a_A; \\ c_1,\ c_2,\ c_C;\end{matrix}\ xy\right]. \tag{10}$$

See also Srivastava and Manocha (1984) page 139.

If $k=2$ and the parameters {a} and {c} are suppressed, a known generating relation for the Gegenbauer polynomial is obtained. See Appell et Kampé de Fériet (1926) page 389. The main result (9) can be obtained as a special case of eq. (28), page 144 of Srivastava and Manocha (1984) by suppressing the c_j and d_j parameters and putting

$$\gamma_n = (b,n) \text{ and } \Delta_{n,\,k} = [(1-b-n,k)]^{-1}. \tag{10}$$

See also Srivastava (1984), page 331, eq. (2.2) for the above-mentioned result reproduced in Srivastava and Manocha (1984), page 144, eq. (28), and aslo for q-extensions.

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