Jñānābha, Vol. 31/32, 2002

(Dedicated to Professor H. M. Srivastava on his 62nd Birthday)

THE INTEGRATION OF CERTAIN PRODUCTS PERTAINING TO THE *H*-FUNCTION WITH GENERAL POLYNOMIALS

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(Received : December 13, 2002)

ABSTRACT

The object of the present paper is to establish a finite integral pertaining to the *H*-function and general polynomials. This integral is unified in nature and acts as a key formula from which we can derive as its special cases, integrals pertaining to a large number of simpler special functions and polynomials. For example, we derive a few special cases of our main integral which are also new and of interest by themselves. In the last, we give applications of our main findings by interconnecting them with the Riemann-Liouville type of fractional integral operator. The results established here are basic in nature and are likely to be of useful applications in several fields notably electromagnetic theory, statistical mechanics and probability theory.

1. Introduction. H.M. Srivastava ([12], p. 185, eqn. (7)) has defined and introduced the general polynomials

$$S_{N_1,\ldots,N_S}^{M_1,\ldots,M_S}(x_1,\ldots,x_S) = \sum_{k_1=0}^{[N_1/M_1]} \cdots \sum_{k_S=0}^{[N_S/M_S]} \frac{(-N_1)_{M_1k_1}}{k_1!} \cdots \frac{(-N_S)_{M_Sk_S}}{k_S!}$$

$$A[N_1,k_1;\ldots;N_S,k_S]x_1^{k_1},\ldots,x_S^{k_S}, \qquad \ldots(1)$$

where $M_1, \ldots M_S$ are arbitrary positive integers and the coefficients $A[N_1,k_1, \ldots, N_S,k_S]$ are arbitrary constants, real or complex.

The *H*-function, introduced by Inayat-Hussain ([9], see also [2]) in terms of Mellin-Barnes type contour integral, is defined by

$$H_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j,\alpha_j:A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\phi(\xi)z^{\xi}d\xi, \qquad ...(2)$$

where

$$\phi(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \qquad ...(3)$$

which contains fractional powers of some of the Γ-functions.

Here and throughout the paper a_j $(j=1,...,P)$ and b_j $(j=1,...,Q)$ are complex parameters, $\alpha_j \geq 0$ $(j=1,...,P)$, $\beta_j \geq 0$ $(j=1,...,Q)$, (not all zero simultaneously) and the exponents A_j $(j=1,...,N)$ and B_j $(j=M+1,...,Q)$ can take on non-integer values.

The contour in (2) is imaginary axis $Re(\xi)=0$. It is suitably indented in order to avoid the singularities of the Γ-functions and to keep these singularities on appropriate sides. Again, for A_j $(j=1,...,N)$ not an integer, the poles of the Γ-functions of the numerator in (3) are converted to branch points. However, as long as there is no coincidence of poles from any $\Gamma(b_j-\beta_j\xi)$ $(j=1,...,M)$ and $\Gamma(1-a_j+\alpha_j\xi)$ $(j=1,...,N)$ pair, the branch cuts can be chosen so that the path of integration can be distorted in the usual manner.

For the sake of brevity

$$T=\sum_{j=1}^{M}|\beta_j|+\sum_{j=1}^{N}A_j\alpha_j-\sum_{j=M+1}^{Q}|B_j\beta_j|-\sum_{j=N+1}^{P}\alpha_j>0. \qquad ...(4)$$

2. Main Integral.

$$\int_0^w x^{u-1}(w-x)^{v-1}S_{N_1,...,N_S}^{M_1,...,M_S}\left[e_1x^{u_1}(w-x)^{v_1},...,e_sx^{u_s}(w-x)^{v_s}\right]$$

$$H_{P,Q}^{M,N}\left[zx^b(w-x)^c\left|\begin{matrix}(a_j,\alpha_j:A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dx$$

$$=w^{u+v-1}\sum_{k_1=0}^{[N_1/M_1]}\cdots\sum_{k_S=0}^{[N_S/M_S]}\frac{(-N_1)_{M_1k_1}}{k_1!}\cdots\frac{(-N_S)_{M_Sk_S}}{k_S!}$$

$$.\,A[N_1,k_1;...;N_s,k_s]\,e_1^{k1}\cdots e_s^{k_s}\,w^{k_1(u_1+v_1)+...+k_s(u_s+v_s)}\,H_{P+2,Q+1}^{M,N+2}$$

$$\left[zw^{b+c} \left| \begin{matrix} (1-u-k_1u_1-\ldots-k_su_s,b:1),(1-v-k_1v_1-\ldots-k_s,v_s,c:1):(a_j,\alpha_j:A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}:(1-u-v-k_1(u_1+v_1)-\ldots-k_s(u_s+v_s),b+c:1), \end{matrix} \right. \right]$$

where

(i) $b \geq 0, c \geq 0$

(ii) $Re\,(u + b\,\frac{b_j}{\beta_j}) > 0,\ Re\,(v' + c\,\frac{b_j}{\beta_j}) > 0;$

$j = 1, \ldots, M,\ N_{i''} = 0, 1, 2, \ldots,\ \forall\ i_{''} = 1, \ldots, s,$

(iii) $|arg\,(z)| < \frac{1}{2} T\pi,\ T > 0$

$M_1,\ldots,M_s$ are arbitrary positive integers and the coefficients $A[N_1,k_1; \ldots ; N_s,k_s]$ $[N_1,k_1;\ldots; N_s,k_s \geq 0]$ are arbitrary constants, real or complex.

Proof. To derive the integral in (5), we express the general class of polynomials occurring therein in the series form given by (1) and the H-Function occurring therein in its left hand side in terms of Mellin-Barnes countour integral given by (2) and then interchanging the order of summations and integration and the order of x-and ξ-integrals (which is permissible under the conditions stated with (5)) so that the left-hand side of the integral (5) (say ∇) takes the following form after a slight simplification:

$$\nabla = \sum_{k_1=0}^{[N_1/M_1]} \cdots \sum_{k_s=0}^{[N_s/M_s]} \frac{(-N_1)'_{M_1k_1}}{k_1!} \cdots \frac{(-N_s)_{M_sk_s}}{k_s!}\ A[N_1,k_1;\ldots; N_s,k_s]\ e_1^{k_1}\cdots e_s^{k_s}$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \phi(\xi) z^{\xi}\ \{ \int_0^w x^{u+k_1u_1+\ldots+k_su_s+b\xi-1} .(w-x)^{v+k_1v_1+\ldots+k_sv_s+c\xi-1}\,dx\ \}\ d\xi. \quad \ldots (6)$$

On evaluating the Eulerian integral occurring in (6), we obtain after a little simplification

$$\nabla = w^{u+v-1} \sum_{k_1=0}^{[N_1/M_1]} \cdots \sum_{k_s=0}^{[N_s/M_s]} \frac{(-N_1)_{M_1k_1}}{k_1!} \cdots \frac{(-N_s)_{M_sk_s}}{k_s!}$$

$$.\ A[N_1,k_1;\ldots; N_s,k_s]\ e_1^{k_1}\cdots e_s^{k_s}\ w^{k_1(u_1+v_1)+\ldots+k_s(u_s+v_s)}$$

$$\cdot \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(u+k_1u_1+\ldots+k_su_s+b\xi)\Gamma(v+k_1v_1+\ldots+k_sv_s+c\xi-1)}{\Gamma(u+v+k_1(u_1+v_1)+\ldots+k_s(u_s+v_s)+(b+c)\xi)}\left[zw^{(b+c)}\right]^{\xi}\phi(\xi)d\xi. \quad (7)$$

On expressing the Mellin-Barnes contour integral appearing in (7) in terms of the H-function given by (2), we easily arrive at the desired result (5).

3. Special cases

(i) Taking $M=1$, $N=3=P=Q$ and replacing z by $-z$ in (5), and using

$$g(\gamma,\eta,\tau,p:z)=\frac{k_{d-1}\Gamma(p+1)\Gamma\left(\frac{1}{2}+\frac{\tau}{2}\right)}{(-1)^p 2^{2+p}\pi^{1/2}\Gamma\gamma\Gamma\left(\gamma-\frac{\tau}{2}\right)}$$

$$H_{3,3}^{1,3}\left[-z\left|\begin{matrix}(1-\gamma,1:1),\left(1-\gamma+\frac{\tau}{2},1;1\right),(1-\eta,1;1+p)\\(0,1),\left(-\frac{\tau}{2},1;1\right),(-\eta,1:1+p)\end{matrix}\right.\right], \quad \ldots(8)$$

where $k_d=\frac{2^{1-d}\pi^{-d/2}}{\Gamma(d/2)}$ ([8], p.4121, eqn. (5)).

The above function is connected with a certain class of Feynman integrals. We get

$$\int_0^w x^{u-1}(w-x)^{v-1}g\left(\gamma,\eta,\tau,p;zx^b(w-x)^c\right)S_{N_1,\ldots,N_S}^{M_1,\ldots,M_S}\left[e_1x^{u_1}(w-x)^{v_1},\ldots,e_sx^{u_s}(w-x)^{v_s}\right]dx$$

$$=\frac{k_{d-1}\Gamma(p+1)\Gamma\left(\frac{1}{2}+\frac{\tau}{2}\right)}{(-1)^p 2^{2+p}\pi^{1/2}\Gamma\gamma\Gamma\left(\gamma-\frac{\tau}{2}\right)}w^{u+v-1}\sum_{k_1=0}^{[N_1/M_1]}\cdots\sum_{k_S=0}^{[N_S/M_S]}\frac{(-N_1)_{M_1k_1}}{k_1!}\cdots\frac{(-N_S)_{M_Sk_S}}{k_S!}$$

$$.\,A[N_1,k_1;\ldots;N_s,k_s]\,e_1^{k_1}\cdots e_s^{k_s}\quad w^{k_1(u_1+v_1)+\ldots+k_s(u_s+v_s)}$$

$$H_{5,4}^{1,5}\left[-zw^{b+c}\left|\begin{matrix}(1-u-k_1u_1-\ldots-k_su_s,b;1),(1-v,c;1),\\(0,1),\left(-\frac{\tau}{2},1;1\right),(-\eta,1;1+p),\end{matrix}\right.\right.$$

$$\left.\left.\begin{matrix}(1-v-k_1v_1-\ldots-k_sv_s,c;1),(1-\gamma,1;1),\left(1-\gamma+\frac{\tau}{2},1;1\right),(1-\eta,1;1+p)\\(1-u-v-k_1(u_1+v_1)-\ldots-k_s(u_s+v_s),b+c;1)\end{matrix}\right.\right], \quad \ldots(9)$$

valid under the conditions as obtainable from (5).

(ii) For $v_1=v_2=\ldots=v_s=0$, the integral in (9) reduces to the following form

$$\int_0^w x^{u-1}(w-x)^{v-1}g\left(\gamma,\eta,\tau,p;zx^b(w-x)^c\right)S_{N_1,\ldots,N_S}^{M_1,\ldots,M_S}\left[e_1x^{u_1},\ldots,e_sx^{u_s}\right]dx$$

$$=\frac{k_{d-1}\Gamma(p+1)\Gamma\left(\frac{1}{2}+\frac{\tau}{2}\right)}{(-1)^p 2^{2+p}\pi^{1/2}\Gamma\gamma\Gamma\left(\gamma-\frac{\tau}{2}\right)}w^{u+v-1}$$

$$\sum_{k_1=0}^{[N_1/M_1]}\cdots\sum_{k_S=0}^{[N_S/M_S]}\frac{(-N_1)_{M_1k_1}}{k_1!}\cdots\frac{(-N_S)_{M_Sk_S}}{k_S!}\,.\,A[N_1,k_1;\ldots;N_s,k_s]\,e_1^{k_1}\cdots e_s^{k_s}\;w^{k_1u_1+\ldots+k_su_s}$$

$$H_{5,4}^{1,5}\left[-zw^{b+c}\left|\begin{matrix}(1-u-k_1u_1\ldots-k_su_s,b;1),(1-\upsilon,c;1),(1-\gamma,1,1),\\ (0,1),\left(-\frac{\tau}{2},1;1\right),(-\eta,1;1+p),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(1-\gamma+\frac{\tau}{2},1;1\right),(1-\eta,1;1+p)\\ (1-u-\upsilon-k_1u_1-\ldots-k_su_s,b+c;1)\end{matrix}\right], \quad \ldots(10)$$

valid under the conditions as needed for (9).

(iii) On taking $u_1 = u_2 = \ldots = u_s = 0$, the integral in (9) takes the following form

$$\int_0^w x^{u-1}(w-x)^{\upsilon-1}g\left(\gamma,\eta,\tau,p;zx^b(w-x)^c\right)S_{N_1,\ldots,N_S}^{M_1,\ldots,M_S}\left[e_1(w-x)^{\upsilon_1},\ldots,e_s(w-x)^{\upsilon_s}\right]dx$$

$$=\frac{k_{d-1}\Gamma(p+1)\Gamma\left(\frac{1}{2}+\frac{\tau}{2}\right)}{(-1)^p2^{2+p}\pi^{1/2}\Gamma\gamma\Gamma\left(\gamma-\frac{\tau}{2}\right)}\,w^{u+\upsilon-1}\sum_{k_1=0}^{[N_1/M_1]}\cdots\sum_{k_S=0}^{[N_S/M_S]}\frac{(-N_1)_{M_1k_1}}{k_1!}\cdots\frac{(-N_S)_{M_Sk_S}}{k_S!}$$

$$A[N_1,k_1;\ldots;N_s,k_s]\,e_1^{k_1}\cdots e_s^{k_s}\;w^{k_1\upsilon_1+\ldots+k_s\upsilon_s}\;H_{5,4}^{1,5}\left[-zw^{b+c}\right|$$

$$\left.\begin{matrix}(1-u,b;1),(1-\upsilon-k_1\upsilon_1-\ldots-k_s\upsilon_s,c;1),(1-\gamma,1;1),\\ (0,1),\left(-\frac{\tau}{2},1;1\right),(-\eta,1;1+p),\end{matrix}\quad\begin{matrix}\left(1-\gamma+\frac{\tau}{2},1;1\right),(1-\eta,1;1+p)\\ (1-u-\upsilon-k_1\upsilon_1-\ldots-k_s\upsilon_s,b+c;1)\end{matrix}\right], \quad \ldots(11)$$

valid under the condition as required for (9).

4. Applications.

We shall define the Riemann-Liouville fractional derivative of a function $f(x)$ of order υ (or, alternatively, $-\upsilon th$ order fractional integral) ([3], p. 181, 11, p. 49) by

$$aD_x^{\upsilon}\{f(x)\}=\begin{cases}\dfrac{1}{\Gamma(\upsilon)}\displaystyle\int_a^x(x-t)^{-\upsilon-1}f(t)dt,\mathrm{Re}(\upsilon)\geq 0\\ \dfrac{d^q}{dx^q}\;{}_aD_x^{\upsilon-q}\{f(x)\},(q-1)\leq\mathrm{Re}(\upsilon)<q,\end{cases} \quad \ldots.(12)$$

where q is a positive integer and the integral exists.

When $a = 0$, we have $D_x^{\upsilon}\equiv{}_0D_x^{\upsilon}$.

Now, replacing w by x in the main result (5), it can be rewritten as the following fractional integral formula

$$D_x^{-v}\left\{t^{u-1}H_{P,Q}^{M,N}\left[zt^b(x-t)^c\left|\begin{matrix}(a_j,\alpha_j:A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j:B_j)_{M+1,Q}\end{matrix}\right.\right]S_{N_1,\ldots,N_S}^{M_1,\ldots,M_S}\right.$$

$$\left.\left[e_1t^{u_1}(x-t)^{v_1},\ldots,e_St^{u_S}(x-t)^{v_S}\right]\right\}$$

$$=\frac{x^{u+v-1}}{\Gamma(v)}\sum_{k_1=0}^{[N_1/M_1]}\cdots\sum_{k_S=0}^{[N_S/M_S]}\frac{(-N_1)_{M_1k_1}}{k_1!}\cdots\frac{(-N_S)_{M_Sk_S}}{k_S!}$$

$$.\,A[N_1,k_1;\ldots;N_s,k_s]\,e_1^{k_1}\cdots e_s^{k_s}\;x^{k_1(u_1+v_1)+\ldots+k_s(u_s+v_s)}$$

$$H_{P+2,Q+1}^{M,N+2}\left[zx^{b+c}\left|\begin{matrix}(1-u-k_1u_1-\ldots-k_su_s,b;1),\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1-v-k_1v_1-\ldots-k_sv_s,c;1)(a_j,\alpha_j:A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(1-u-v-k_1(u_1+v_1)-\ldots-k_s(u_s+v_s),b+c:1)\end{matrix}\right],$$

where $Re\ (v)>0$ and all the other conditions of validity mentioned with (5) are satisfied.

The results recently obtained by Gupta and Soni in ([5]. eqn.(2.1), p. 100, (3,1), (3.2), p. 101 and (4.1), p. 102) follow as particular cases from our results in (5), (9) and (13), on assigning suitable values to parameters. the fractional integral formula established in (12) is also quite general in nature and can easily yield Riemann-Liouville fractional integrals of a large number of simpler functions and polynomials merely by specializing the parameters of H and $S_{N_1,\ldots,N_S}^{M_1,\ldots,M_S}$ appearing in it which may find applications in electromagnetic theory and probability.

ACHNOWLEDGEMENTS

The authors are thankful to Professor H.M. Srivastava (University of Victoria, Canada) for his help and suggestions in the preparation of this paper.

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