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(Dedicated to Professor H. M. Srivastava on his 62nd Birthday)

A COMMON FIXED POINT THEOREM INVOLVING BEST APPROXIMATION

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ABSTRACT

A fixed-point theorem due to Jungck is utilized to derive a common fixed-point theorem for six mappings in compact metric spaces, which is also used to prove another common fixed-point theorem involving best approximation. In process results due to Brosowski, Singh, Hicks-humphris and Sahab et. al. are generalized and improved.

1. Introduction. A self mapping T of a normed space X is said to be non-expansive (*resp.* I-non expansive) if $\|Tx-Ty\|\le\|x-y\|\{resp.\|Tx-Ty\|\le\|Ix-Iy\|\}$ for all x,y in X. If $\bar{x}$ is a point of X and C a subset of X, then set $B_c(\bar{x})$ of best C-approximant to $\bar{x}$ consists of the point y in C such that $\|y-\bar{x}\| = inf\ \{\|z-\bar{x}\| : z \in C\}$. A subset C of X is said to be starshaped (cf.[2]) with respect to a point $q\in C$ if for all x in C and all $0\le\lambda\le1$, $\lambda x+(1-\lambda)q$ is in C. Clearly a convex set is starshaped with respect to each of its points but the converse is not always true.

Brosowski [1] proved that if T is non-expansive with $\bar{x}\in F(T)$, $T(C)\subset C$ and $B_C(\bar{x})$ is nonempty, compact convex, then T has a fixed point in $B_C(\bar{x})$. Subrahmanyam [13] substituted the nonempty requirement of $B_C(\bar{x})$ with the finite dimensionality of C (as a subspace of X) whereas Singh [10,11] noted that Brosowski's result remains true if $B_C(\bar{x})$ is only starshaped, but soon noticed that non-expansive property of T on $B_C(\bar{x})\cup\{\bar{x}\}$ is enough for his earlier result. In this continuation Hicks-Humphries [4] observed that Singh's first result remains true if one replaces $T(C) \subset C$ by $T(\delta C)\subset C$. where δC denotes the boundary of C in X. Smoluk [12] substituted 'finite dimensionality of C' in Subrahmaniyam's result by 'linearity of T and compactness of $\overline{T(D)}$' for every bounded subset D of C, which was later improved by Habiniak [3], by relaxing the linearity of T.

In what follows, $F(I,T)$ denotes the set of common fixed points of I and T whereas $F(A,B,S,T,I,J)$ denotes the set of common fixed point of the mappings A,B,S,T,I and J.

In an attempt to unify and generalize the results due to Hicks-Humphries [4] and Singh [10], Sahab et al. [9] proved the following:

Theorem 1.1 ([9]). Let X be a normed space, I and T self-maps of X with $\bar{x} \in F(T,I)$, $C \subset X$ with $T(\delta C) \subset C$ and $q \in F(I)$. If $D = B_C(\bar{x})$ is compact and q-star shaped, $I(D)=D$, I is continuous and linear on D, I and T are commuting on D and T is I-non expansive on $D \cup \{\bar{x}\}$, then I and T have a common fixed point in D.

We essentially require the following definition.

Definition 1.2 ([6]). A pair of self-mappings (B,I) of a normed space X is said to be compatible if

$$\lim_{n\to\infty} \|BIx_n - IBx_n\| = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Ix_n = t \in X$.

Definition 1.3 ([8]). A pair of self-mappings (B,I) on X is said to be coincidently commuting if (B,I) commute at the coincidence points of B and I.

In this note using a variant of a fixed point theorem due to Jungck [7],we first derive a common fixed point theorem in compact metric spaces involving six mappings, which is then used to prove yet another extension of Theorem 1.1. In process relevant results due to Brosowski [1], Singh [10,11], Hick-Humphries [4] and Sahab et al. [9] are generalized and improved.

2. Main Results. Motivated from the observations explained in Jungck and Rhoades [8], one can state the following variant of a fixed point theorem, which is due to Jungck [7].

Theorem 2.1. ([7]). Let A,S,I and J be continuous self-mappings of a compact metric space (X,d) with $A(X) \subset J(X)$ and $S(X) \subset I(X)$. If either (A,I) are compatible and (S,J) coincidently commuting or (S,J) are compatible and (A,I) coincidently commuting and

$$d(Ax, Sy) < M(x,y),$$

where $M(x,y) = \max\{d(Ix,Jy), d(Ix,Ax), d(Jy,Sy), \tfrac{1}{2}[d(Ix,Sy)+d(Jy,Ax)]\}$, for all $x,y \in X$ with $M(x,y) > 0$, then A,S,I and J have a unique common fixed point.

Proof. The proof is almost the same as that of Jungck's theorem ([7]), hence it is omitted.

Remark 2.2 Theorem 2.1 was originally proved with compatibility of both the pairs in Jungck [7].

As an application of Theorem 2.1 we derive a common fixed point theorem for six mappings, which runs as follows:

Theorem 2.3 Let A,B,S,T,I and J be self-mappings of a compact metric space (X,d) such that $A(X)\subset TJ(X)$, $S(X)\subset BI(X)$ with A,S,TJ and BI being continuous. If either (A, BI) are compatible and (S,TJ) coincidently commuting or (ST,J) are compatible and (A,BI) coincidently commuting and

$$d\,(Ax,\ Sy) < M\,(x,y),$$

where $M(x,y) = max\{d(BIx,TJy),d(BIx,Ax),d(TJy,Sy),\ ½[d(BIx,Sy)+ d(TJy,Ax)\}$, for all $x,y\in X$ with $M(x,y) > 0$ then A,S,BI and TJ have a uniqe common fixed point z in X. Moreover, if the pairs (B,I), (T,J) (A,B), (A,I),(S,T) and (S,J) commute at the fixed point z, then z remains the unique common fixed point of A, B, S, T, I and J separately.

Proof. We begin by noting that the continuity of BI (*resp.* TJ) does not demand the continuity of B or I or both (*resp.* T or J or both). But for maps A,S,BI and TJ all the conditions of Theorem 2.1 are satisfied ensuring the existence of unique common fixed point z of A, S, BI and TJ . Here it is worth noting that z is the common fixed point of both the pairs (A,BI) and (S,TJ) respectively.

Now it remains to show that z is also a common fixed point of A, B, S, T, I and J . For this let z is the unique common fixed point of both the pairs (A,BI) and (S,TJ) , then

$$\begin{aligned}
Bz &= B\,(BIz) = B(IBz)=BI(Bz), & Bz &= B\,(Az) = A\,(Bz),\\
Iz &= I\,(BIz) = IB(Iz)=BI(Iz), & Iz &= I\,(Az) = A\,(Iz),\\
Tz &= T\,(TJz) = T(JTz)=TJ(Tz), & Tz &= T(Sz) = S\,(Tz),\\
Jz &= J\,(TJz) = JT(Jz)=TJ(Jz), & Jz &= J\,(Sz) = S\,(Jz),
\end{aligned}$$

which shows that Bz and Iz (*resp.* Tz and Jz) are other fixed points of the pair (A,BI) (*resp.*(S,TJ)). Now in view of the uniqueness of common fixed point of the pairs (A,BI) and (S,TJ), we get

$$z = Bz = Iz = Tz = Jz = BIz= TJz=Az=Sz.$$

which shows that z also remains the common fixed point of $A,B,S,T,I,$ and J separately. This completes the proof.

Remark 2.4 By restricting $A,B,S,T,I,$ and J suitably and modifying the remaining hypotheses accordingly, one can derive a multitude of known and unknown fixed point theorems. So far we are not familiar of any fixed point theorem involving five or six mappings in compact metric spaces.

As an application of Theorem 2.3, we prove the following fixed point theorem (employing the notion of best approximation) which generalizes earlier results due to Brosowski [1], Hicks-Humphries [4], Singh [10]. Sahab et al. [9] and others.

Theorem 2.5. Let A, B, S,T, I and J be self-mappings of a normed space X and C be a subset of X such that $A,S : \delta C \to C$ with $\bar{x} \in F\,(A,B,S,T,I,J)$.

If $A,B,S,T,I,$ and J satisfy the condition

$$\|Ax - Sy\| < M(x,y),$$

with A and S being continuous where

$$M(x,y) = max\{\|BIx - TJy\|, \|BIx - Ax\|, \|TJy - Sy\|, ½[\|BIx - Sy\| + \|TJy - Ax\|]\}$$

for all $x,y \in D' = D \cup \{\bar{x}\}$. (2.5.1)

Further suppose that the pairs (A,BI) and (S,TJ) are compatible with BI and TJ being linear and continuous on D. If D be a nonempty, compact and starshaped with respect to a point $q \in D$ and $BI(D) = D = TJ(D)$ then

$$D \cap F(A,B,S,T,I,J) \neq \phi,$$

provided the pairs (B,I), (T,J), (A,B), (S,T), (A,I) and (S,J) commute at the common fixed point of BI, TJ, A and S.

Proof . Let $y \in D$; then $BIy \in D$ as $BI(D) = D$. Also if $y \in \delta C$ then $Ay \in C$ as $A(\delta C) \subset C$. Using condition (2.5.1), we obtain

$$\|Ay - \bar{x}\| = \|Ay - S\bar{x}\| < M(y,\bar{x}),$$

giving thereby $Ay \in D$. Thus A is a self-mapping of D. Similarly S is also a self-mapping of D.

Let $\{t_n\}$ be a sequence of real number such that $0 \leq t < 1$ and converging to 't'. Define sequence $\{A_n\}$ and $\{S_n\}$ of mappings by

$$A_n x = t_n Ax + (1-t_n)q, \qquad S_n x = t_n Sx + (1-t_n)q$$

for all $x \in D$ and for each n.

Since D is starshaped with respect to q hence $\{A_n\}$ maps D into itself and also so does $\{S_n\}$. Since I is linear one can have

$$(A_n) Ix_n = t_n (AIx_n) + (1-t_n) Iq, \text{ and } I(A_n) x_n = t_n (IAx_n) + I(1-t_n) q.$$

Since (A, BI) are compatible, therefore

$$0 \leq \lim_{n\to\infty} \|(BI)A_n x_n - A_n(BI)x_n\| \leq \lim_{n\to\infty} \|(BI)Ax_n - A(BI)x_n\| + \lim_{n\to\infty} (1-t_n)\|q - Aq\| = 0$$

whenever $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} BIx_n = t \in D$, for all n.

Hence (BI,A_n) are compatible on D. Similarly it can be shown that (TJ, S_n) are compatible on D.

Further from

$$\|A_n x - S_n y\| = t_n \|Ax - Sy\| < t_n M(x,y) < M(x,y),$$

for all $x, y \in D$. Since I and J are continuous and D is compact, therefore by Theorem 2.3

$$F(A_n) \cap F(BI) \cap F(S_n) \cap F(TJ) = \{x_n\},$$

for each n. Also since D is compact $\{x_n\}$ has convergent subsequence $\{x_{n_i}\}$

converging to z in D.

Now

$$x_{n_i} = A_{n_i} x_{n_i} = t_{n_i} A x_{n_i} + (1 - t_{n_i}) q,$$

which on letting $n \to \infty$ reduces to $Az = z$, giving thereby $z \in D \cap F(A)$. Similarly it can be shown that $z \in D \cap F(S)$. Since BI and TJ are continuous, we have

$$BIz = BI \lim_{i\to\infty} x_{n_i} = \lim_{i\to\infty} BIx_{n_i} = \lim_{i\to\infty} x_{n_i} = z,$$

$$TJz = TJ \lim_{i\to\infty} x_{n_i} = \lim_{i\to\infty} TJx_{n_i} = \lim_{i\to\infty} x_{n_i} = z,$$

yielding thereby $BIz = TJz = Az = Sz = z$.

Let the pairs (A, BI) and (S, TJ) have different fixed point u and v respectively, then

$$\|u - v\| = \|Au - Sv\| < max\{\|BIu - TJv\|, \|BIu - Au\|, \|TJv - Sv\|, ½[\|BIu - Sv\| + \|TJv - Au\|]\}$$

which is a contradiction, implying thereby $u = v$. Thus both the pairs have same common unique fixed point $u = v = z$.

Now on the lines of the proof of Theorem 2.3 it can be easily shown that z remains the unique common fixed point of A, B, S, T, I and J.

Hence, we conclude that

$$D \cap F(A,B,S,T,I,J) \neq \phi.$$

This completes the proof.

Remarks 2.6(i) Theorem 2.5 extends the results of Sahab et al. [9] as we use generalized contractions along with compatibility (cf. [6]) sinstead of commutativity. Also Theorem 2.5 involves six mappings instead of two mappings. In process related results due to Hicks-Humphries [4], Singh [10]. Brosowski [1] and others are modified and improved either partially or completely.

(ii) If we use a fixed point theorem in complete spaces corresponding to Theorem 2.3 then the continuity requirement of any one of the maps A, S, BI or TJ can serve that purpose which is possible due to the fact the compact metric spaces are always complete. But due to a shorter proof we opt to utilize Theorem 2.3.

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