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(Dedicated to Professor H. M. Srivastava on his 62nd Birthday)

A STUDY ON A GENRALIZATION CONCERNING INCLUSION THEOREMS FOR ABSOLUTE HARMONIC-CESARO METHOD OF SUMMATION

By

W.T. Sulaiman

College of Education,

Ajman University, P. O. Box: 346, Ajman, United Arab Emirates

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ABSTRACT

In this paper two new theorems on absolute harmonic-Cesaro method of summation are proved, one of them gives some generalization to our previous result [3].

1. Introduction. The coefficient $A_n^{\alpha,\beta}$ is defined for $\alpha \geq 2$, by the series

$$\frac{1}{(1-z)^{\alpha-1}}\left(\log\frac{\alpha}{1-z}\right)^{\beta} = \sum_{n=0}^{\infty} A_n^{\alpha,\beta} z^n , \tag{1.1}$$

α,β are assumed to be real numbers. The (Z,α,β)– transformation of $\Sigma\, a_n$ with partial sums s_n is given by [1],

$$\sigma_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha,\beta}} \sum_{\upsilon=0}^{n} A_{n-\upsilon}^{\alpha-1,\beta} s_\upsilon = \sum_{\upsilon=0}^{n} \frac{1}{A_n^{\alpha,\beta}} A_{n-\upsilon}^{\alpha-1,\beta} a_\upsilon . \tag{1.2}$$

The transformation $(Z, \alpha, 0)$ is the same as (C,α) and $(Z, 0,1)$ reduces to a method equivalent to harmonic transformation $\left(N, \frac{1}{n+1}\right)$ (see[1]). We use the following estimates (see [4])

$$A_n^{\alpha,\beta} \approx \frac{n^{\alpha}}{\Gamma(\alpha+1)}(\log n)^{\beta} \qquad (a \neq -1, -2, \ldots) \tag{1.3}$$

$$A_n^{\alpha,\beta} \approx \beta(-1)^{\alpha-1}(|\alpha|-1)!\, n^{\alpha}(\log n)^{\beta-1} \quad (a = -1, -2, \ldots) \tag{1.4}$$

and the following identities which are deducible from (1.1), (see [1]).

$$\sum_{n=0}^{m} A_n^{\alpha,\beta} = A_m^{\alpha+1,\beta} \tag{1.5}$$

$$\sum_{v=0}^{m} A_{n-v}^{\alpha,\beta} A_{v}^{\alpha',\beta'} = A_{m}^{\alpha+\alpha'+1,\beta+\beta'}. \tag{1.6}$$

Let σ_n^δ and η_n^δ denote the *nth* Cesaro means of order $\delta(\delta\rangle-1)$ of the sequences $\{s_n\}$ and $\{na_n\}$ respectively. The series $\Sigma\, a_n$ is said to be summable $|C,\ \delta|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}\left|\sigma_n^\delta - \sigma_{n-1}^\delta\right|^k < \infty$$

or equivalently

$$\sum_{n=1}^{\infty} \frac{1}{n}\left|\eta_n^\delta\right|^k < \infty.$$

Let $T_n^{\alpha,\beta}$ be the $(Z,\ \alpha,\ \beta)$- transform of the sequence $\{na_n\}$. Then the series $\Sigma\, a_n$ is summable $|Z,\ \alpha,\ \beta|$ if (see[2])

$$\sum_{n=1}^{\infty} \frac{1}{n}\left|T_n^{\alpha,\beta}\right| < \infty.$$

The series $\Sigma\, a_n$ is summable $|Z,\ \alpha,\ \beta|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \frac{1}{n}\left|T_n^{\alpha,\beta}\right|^k < \infty.$$

We [3], have proved the following

Theorem. Suppose that $k \geq 1$

(1) Let $\gamma > \alpha > 0$ and $\beta, \delta \in R$. Then $|Z,\alpha,\beta|_k \subset |Z,\gamma,\delta|_k$

(2) Let $\alpha > 0$, $\delta > \beta$. Then $|Z,\alpha,\beta|_k \subset |Z,\gamma,\delta|_k$

(3) Let $\delta > \beta > 0$. Then $|Z,0,\beta|_k \subset |Z,0,\delta|_k$

(4) Let $\delta > 0$, $\beta > 0$. Then $|Z,0,\beta|_k \subset |Z,\gamma,\delta|_k$.

The object of this paper is to prove the following

Theorem 1. Let $1-1/k < \alpha < \gamma < 1$, $\gamma-\alpha > 1/k$, $\left\{\left|\Delta_r A_{n-r}^{\alpha-1,\delta} \in_r\right|\right\}$ is nonincreasing *w.r.t.* r, then sufficient conditions that $\Sigma\, a_n \in_n$ is summable $|Z,\ \gamma,\delta|_k$ whenever $\Sigma\, a_n$ is summable $|Z,\alpha,\beta|_k$ $k \geq 1$, are

$$\in_n = o\left\{(\log n)^\delta n^{\gamma-1}\right\}, \Delta \in_n = o(1/n).$$

Theorem 2. Let $1-1/k < \alpha < 2-1/k$, $0 < \gamma < 1$. Then necessary conditions that $\Sigma\, a_n \in_n$ is summable $|Z,\ \gamma,\delta|_k$ whenever $\Sigma\, a_n$ is summable $|Z,\alpha,\beta|_k$, $k \geq 1$, are

$$\epsilon_n = o\left\{n^{\gamma-\alpha}(\log n)^{\delta-\beta}\right\}, \Delta \epsilon_n = o(1/n).$$

2. Lemmas.

Lemma.1 [3]

$$\sum_{n=\upsilon}^{m} \frac{(n-\upsilon)^{\delta-1}}{n^{\sigma}} \log^{\alpha} n \log^{\beta}(n-\upsilon) = o\left(\upsilon^{\delta-\sigma} \log_{\upsilon}^{\alpha+\beta}\right), \qquad \sigma > \delta > 0 .$$

Lemma. 2. Let $k \geq 1$, $Q_n = q_0 + q_1 + \ldots + q_n \to \infty$, as $n \to \infty$ $\{q_n\}$ is nonincreasing, then

(a) If $Q_n = o(nq_n)$, $\dfrac{1}{\upsilon^{\rho} Q_{\upsilon}^{k}} = o(1)\sum_{n=\upsilon}^{\infty} \dfrac{q_{n-\upsilon}}{n^{\rho} Q_n^{k+1}}$,

(b) If $\Delta q_n = o\left(\dfrac{q_n}{n}\right)$, $\dfrac{q_{\upsilon}^{m}}{\upsilon^{\rho} Q_{\upsilon}^{k}} = O(1)\sum_{n=\upsilon}^{\infty} \dfrac{q_{n-\upsilon}^{m}}{n^{\rho} Q_n^{k+1}}$.

Proof. We consider the case $m > 1$, m and k integers. If m or k are not integer, the result follows by making use of the mean value theorem.

$$\begin{aligned} \Delta Q_n^k &= Q_n^k - Q_{n+1}^k \\ &= \left(Q_n - Q_{n+1}\right)\left(Q_n^{k-1} + Q_n^{k-2} Q_{n+1} + \ldots + Q_{n+1}^{k-1}\right) \\ &\leq k\left|\Delta Q_n\right| Q_n^{k-1}, \end{aligned}$$

then

$$\Delta\left(\frac{1}{Q_n}\right) = \frac{-\Delta Q_n^k}{Q_n^k Q_{n+1}^k} \leq \frac{k q_n}{Q_n^{k+1}},$$

and hence

$$\begin{aligned} \Delta\left(1/n^{\rho} Q_n^k\right) &= \frac{1}{Q_n^k}\Delta\left(1/n^{\rho}\right) + \frac{1}{(n+1)^{\rho}}\Delta\left(1/Q_n^k\right) \\ &= o\left(\frac{1}{n^{\rho+1} Q_n^k}\right) + o\left(\frac{q_n}{n^{\rho} Q_n^{k+1}}\right) = o\left(\frac{1}{n^{\rho} Q_n^{k+1}}\right) \end{aligned}$$

$$\begin{aligned} \Delta\left(\frac{q_n^m}{n^{\rho} Q_n^k}\right) &= \frac{\Delta q_n^m}{n^{\rho} Q_n^k} + q_{n+1}^m \Delta\left(\frac{1}{n^{\rho} Q_n^k}\right) \\ &= \frac{q_n^{m-1}\left|\Delta q_n\right|}{n^{\rho} Q_n^k} + o\left(\frac{q_n^m}{n^{\rho} Q_n^{k+1}}\right) \\ &= o(1)\left(\frac{q_n^m}{n^{\rho+1} Q_n^k}\right) + o\left(\frac{q_n^m}{n^{\rho} Q_n^{k+1}}\right) = o\left(\frac{q_n^m}{n^{\rho} Q_n^{k+1}}\right), \text{ as } Q_n = O(n). \end{aligned}$$

Finally,

$$\frac{q_v^m}{v^\rho Q_v^k} = \sum_{n=v}^{\infty} \Delta\left(\frac{q_n^m}{n^\rho Q_n^k}\right) = o(1)\sum_{n=v}^{\infty} \frac{q_n^m}{n^\rho Q_n^{k+1}}.$$

Lemma3. Let $\beta > 1$, $Q_n = o\ (nq_n)$, $\{q_n\}$ nonicreasing, then

$$1/v^\beta = o(1)\sum_{n=v}^{\infty} q_{n-v} / n^\beta Q_n .$$

Proof.

$$1/v^\beta = o(1)\sum_{n=v}^{\infty} \frac{1}{n^{\beta+1}} = o(1)\sum_{n=v}^{\infty} \frac{q_n}{n^\beta Q_n} = o(1)\sum_{n=v}^{\infty} \frac{q_{n-v}}{n^\beta Q_n}.$$

3. Próof of Theorems 1 and 2

1- **Proof of Theorem 1**. Write

$$T_n^{\gamma,\delta} = \frac{1}{A_n^{\gamma,\delta}} \sum_{v=0}^{n} A_{n-v}^{\gamma-1,\delta} v a_v \in_v,$$

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha,\beta}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1,\beta} v a_v.$$

By inversion formula, we have

$$T_n^{\gamma,\delta} = \frac{1}{A_n^{\gamma,\delta}} \sum_{v=0}^{n} A_{n-v}^{\gamma-1,\delta} \in_v \sum_{r=0}^{n} A_{v-r}^{-\alpha-1,-\beta} A_r^{\alpha,\beta} t_r^{\alpha,\beta}$$

$$= \frac{1}{A_n^{\gamma,\delta}} \sum_{r=0}^{n} A_r^{\alpha,\beta} t_r^{\alpha,\beta} \sum_{v=r}^{n} A_{n-v}^{\gamma-1,\delta} A_{n-v}^{\gamma-1,\delta} A_{v-r}^{-\alpha-1,-\beta} \in_v$$

$$= \frac{1}{A_n^{\gamma,\delta}} \sum_{r=0}^{n} A_r^{\alpha,\beta} t_r^{\alpha,\beta} \sum_{v=0}^{n-r} A_{n-r-v}^{\gamma-1,\delta} A_v^{-\alpha-1,-\beta} \in_{v+r}$$

$$= \frac{1}{A_n^{\gamma,\delta}} \sum_{r=0}^{n} A_r^{\alpha,\beta} t_r^{\alpha,\beta} \left[\sum_{v=0}^{n-r-1} \left(\sum_{k=0}^{v} A_k^{-\alpha-1,-\beta}\right) \Delta_v\left(A_{n-r-v}^{\gamma-1,\delta} \in_{v+r}\right) + \left(\sum_{k=0}^{n-r} A_k^{-\alpha-1,-\beta}\right) A_0^{\gamma-1,\delta} \in_n\right]$$

$$= \frac{1}{A_n^{\gamma,\delta}} \sum_{r=0}^{n} A_r^{\alpha,\beta} t_r^{\alpha,\beta} \left[\sum_{v=0}^{n-r-1} A_{v-r}^{-\alpha,-\beta} \left\{\Delta_v\left(A_{n-r-v}^{\gamma-1,\delta}\right) \in_{v+r} + A_{n-r-v-1}^{\gamma-1,\delta} \Delta_v \in_{v+r}\right\} + A_{n-r}^{-\alpha,-\beta} \in_n\right]$$

$$= \frac{1}{A_n^{\gamma,\delta}} \sum_{r=0}^{n} A_r^{\alpha,\beta} t_r^{\alpha,\beta} \left[\sum_{v=r}^{n-1} \left\{A_{v-r}^{-\alpha,-\beta} \Delta_v\left(A_{n-v}^{\gamma-1,\delta}\right) \in_v + A_{v-r}^{-\alpha,-\beta} A_{n-v-1}^{\gamma-1,\delta} \Delta \in_v\right\} + A_{n-r}^{-\alpha,-\beta} \in_n\right]$$

$= T_1 + T_2 + T_3$, say.

To prove the theorem, by Minkowski's inequlaity, it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left|T_j\right|^k < \infty, \quad j = 1,2,3.$$

Applying *Hölder's* inequality,

$$\sum_{n=1}^{m}\frac{1}{n}|T_1|^k=\sum_{n=1}^{m}\frac{1}{n\left(A_n^{\gamma,\delta}\right)^k}\left|\sum_{r=1}^{n}A_r^{\alpha,\beta}t_r^{\alpha,\beta}\sum_{v=r}^{n-1}A_{v-r}^{-\alpha,-\beta}\Delta_v\left(A_{n-v}^{\gamma-1,\delta}\right)\in_v\right|^k$$

$$\leq\sum_{n=1}^{m}\frac{1}{n\left(A_n^{\gamma,\delta}\right)^k}\left\{\sum_{r=1}^{n}A_r^{\alpha,\beta}\left|t_r^{\alpha,\beta}\right|\left|\Delta_r A_{n-r}^{\gamma-1,\delta}\in_r\right|\left|\sum_{v=r}^{n}A_{v-r}^{-\alpha,-\beta}\right|\right\}^k$$

$$=\sum_{n=1}^{m}\frac{1}{n\left(A_n^{\gamma,\delta}\right)^k}\left\{\sum_{r=1}^{n}A_r^{\alpha,\beta}\left|t_r^{\alpha,\beta}\right|^k\left|\Delta_r A_{n-r}^{\gamma-1,\delta}\right|\left|\in_r\right|\left|A_{n-r}^{1-\alpha,-\beta}\right|\right\}^k$$

$$\leq\sum_{n=1}^{m}\frac{1}{n\left(A_n^{\gamma,\delta}\right)^k}\sum_{r=1}^{n}A_r^{\alpha,\beta}\left|t_r^{\alpha,\beta}\right|^k\left|\Delta_r A_{n-r}^{\gamma-1,\delta}\right|^k\left|\in_r\right|^k\left(A_{n-r}^{1-\alpha,-\beta}\right)^k\left(\sum_{r=0}^{n}A_r^{\alpha,\beta}\right)^{k-1}$$

$$=\sum_{r=1}^{m}A_r^{\alpha,\beta}\left|t_r^{\alpha,\beta}\right|^k\left|\in_r\right|^k\sum_{n=r}^{m}\frac{\left|\Delta_r A_{n-r}^{\gamma-1,\delta}\right|^k\left(A_{n-r}^{1-\alpha,-\beta}\right)^k\left(A_n^{\alpha+1,-\beta}\right)^{k-1}}{n\left(A_n^{\gamma,\delta}\right)^k}$$

$$=o(1)\sum_{r=1}^{m}A_r^{\alpha,\beta}\left|t_r^{\alpha,\beta}\right|^k\left|\in_r\right|^k\sum_{n=r}^{m}\frac{(n-r)^{k(\gamma-\alpha-1)}\left[\log(n-r)\right]^{k(\delta-\beta)}(\log n)^{-\beta-\delta k}}{n^{2+\gamma k-\alpha k+\alpha-k}}$$

$$=o(1)\sum_{r=1}^{m}\frac{1}{r}\left|t_r^{\alpha,\beta}\right|^k\left|\in_r\right|^k.$$

$$\sum_{n=1}^{m}\frac{1}{n}|T_2|^k=\sum_{n=1}^{m}\frac{1}{n\left(A_n^{\gamma,\delta}\right)^k}\left|\sum_{r=1}^{n-1}A_r^{\alpha,\beta}t_r^{\alpha,\beta}\sum_{v=r}^{n-1}A_{v-r}^{-\alpha,-\beta}A_{n-v-1}^{\gamma-1,\delta}\Delta\in_v\right|^k$$

$$\leq\sum_{n=1}^{m}\frac{1}{n\left(A_n^{\gamma,\delta}\right)^k}\left\{\sum_{r=1}^{n}A_r^{\alpha,\beta}\left|t_r^{\alpha,\beta}\right|\left|\Delta\in_r\right|\left|\sum_{v=r}^{n}A_{n-r}^{\gamma-1,\delta}A_{v-r}^{-\alpha,-\beta}\right|\right\}^k$$

$$=\sum_{n=1}^{m}\frac{1}{n\left(A_n^{\gamma,\delta}\right)^k}\left\{\sum_{r=1}^{n}A_r^{\alpha,\beta}\left|t_r^{\alpha,\beta}\right|\left|\Delta\in_r\right|\left|A_{n-r}^{\gamma-\alpha,\delta-\beta}\right|\right\}^k$$

$$\leq\sum_{n=1}^{m}\frac{1}{n\left(A_n^{\gamma,\delta}\right)^k}\sum_{r=1}^{n}\left(A_r^{\alpha,\beta}\right)^k\left|t_r^{\alpha,\beta}\right|^k\left|\Delta\in_r\right|^k A_{n-r}^{\gamma-\alpha,\delta-\beta}\left(\sum_{r=0}^{n}A_{n-r}^{\gamma-\alpha,\delta-\beta}\right)^{k-1}$$

$$=\sum_{n=1}^{m}\frac{1}{n\left(A_n^{\gamma,\delta}\right)^k}\sum_{r=1}^{n}\left(A_r^{\alpha,\beta}\right)^k\left|t_r^{\alpha,\beta}\right|^k\left|\Delta\in_r\right|^k A_{n-r}^{\gamma-\alpha,\delta-\beta}\left(A_n^{1+\gamma-\alpha,\delta-\beta}\right)^{k-1}$$

$$=\sum_{r=1}^{m}\left(A_r^{\alpha,\beta}\right)^k\left|t_r^{\alpha,\beta}\right|^k\left|\Delta\in_r\right|^k\sum_{n=r}^{m}\frac{A_{n-r}^{\gamma-\alpha,\delta-\beta}\left(A_n^{1+\gamma-\alpha,\delta-\beta}\right)^{k-1}}{n\left(A_n^{\gamma,\delta}\right)^k}$$

$$= o(1)\sum_{r=1}^{m} r^{\alpha k}(\log r)^{\beta k}\left|t_r^{\alpha,\beta}\right|^k \sum_{n=r}^{m} \frac{(n-r)^{\gamma-\alpha}\left[\log(n-v)\right]^{\delta-\beta}(\log n)^{-\beta-\beta k-\delta}}{n^{2+\gamma-\alpha+\alpha k-k}}$$

$$= o(1)\sum_{r=1}^{m} \frac{1}{r}\left|t_r^{\alpha,\beta}\right|^k \left|r\Lambda \in_r\right|^k.$$

$$\sum_{n=1}^{m}\frac{1}{n}\left|T_3\right|^k = \sum_{n=1}^{m}\frac{1}{n\left(A_n^{\gamma,\delta}\right)^k}\left|\sum_{r=1}^{n} A_r^{\alpha,\beta} t_r^{\alpha,\beta} A_{n-r}^{-\alpha,-\beta} \in_n\right|^k$$

$$\leq \sum_{n=1}^{m}\frac{\left|\in_n\right|^k}{n\left(A_n^{\gamma,\delta}\right)^k}\sum_{r=1}^{n}\left(A_r^{\alpha,\beta}\right)^k\left|t_r^{\alpha,\beta}\right|^k A_{n-r}^{-\alpha,-\beta}\left(\sum_{r=0}^{n} A_{n-r}^{-\alpha,-\beta}\right)^{k-1}$$

$$= \sum_{r=1}^{m}\left(A_r^{\alpha,\beta}\right)^k\left|t_r^{\alpha,\beta}\right|^k \sum_{n=r}^{m}\frac{A_{n-r}^{-\alpha,-\beta}\left(A_n^{1-\alpha,-\beta}\right)^{k-1}\left|\in_n\right|^k}{n\left(A_n^{\gamma,\delta}\right)^k}$$

$$= o(1)\sum_{r=1}^{m} r^{\alpha k}(\log r)^{\beta k}\left|t_r^{\alpha,\beta}\right|^k\left|\in_r\right|^k \sum_{n=r}^{m}\frac{(n-r)^{-\alpha}\left[\log(n-r)\right]^{-\beta}(\log n)^{-\beta-\beta k-\delta k}}{n^{2+\gamma k+\alpha k-k-\alpha}}$$

$$= o(1)\sum_{r=1}^{m}\frac{1}{r}\left|t_r^{\alpha,\beta}\right|^k r^{k-k\gamma}(\log r)^{-\delta k}\left|\in_r\right|^k.$$

2. Proof of Theorem 2.

For $k \geq 1$, we define

$A = \{\{a_i\}: \Sigma a_i$ is summable $|Z,\alpha,\beta|_k\}$

$B = \{\{a_i\}: \Sigma a_i \in_i$ is summable $|Z,\gamma,\delta|_k\}$.

These sets are BK-spaces if normed by

$$\|a\|_1 = \left\{\sum_{n=1}^{\infty}\frac{1}{n}\left|t_n^{\alpha,\beta}\right|^k\right\}^{\frac{1}{k}} \tag{2.1}$$

$$\|a\|_2 = \left\{\sum_{n=1}^{\infty}\frac{1}{n}\left|T_n^{\gamma,\delta}\right|^k\right\}^{\frac{1}{k}} \tag{2.2}$$

respectively. Since $|Z,\alpha,\beta|_k$ summability of Σa_n implies $|Z,\gamma,\delta|_k\}$ summability of $\Sigma a_n \in_n$ by the hypothesis of the theorem, then

$$\|a\|_1 < \infty \Rightarrow \|a\|_2 < \infty. \tag{2.3}$$

Consider the inclusion map $I: A \rightarrow B$ defined by $I(x) = x$. This map is continuous, follows as A and B are BK-spaces. Therefore there exists a constant C such that

$$\|a\|_2 \leq C\|a\|_1. \tag{2.4}$$

By applying each of t_n and T_n to $a = e_v - e_{v+1}$, where e_v is the vth coordinate vector, we have

$$t_n = \begin{cases} 0 & ,n < \upsilon \\ \dfrac{\upsilon}{A_\upsilon^{\alpha,\beta}} & ,n = \upsilon \\ \dfrac{\Delta_\upsilon\left(\upsilon A_{n-\upsilon}^{\alpha-1,\beta}\right)}{A_n^{\alpha,\beta}} & ,n > \upsilon \end{cases} \quad , \qquad T_n = \begin{cases} 0 & ,n < \upsilon \\ \dfrac{\upsilon \in_\upsilon}{A_\upsilon^{\gamma,\delta}} & ,n = \upsilon \\ \dfrac{\Delta_\upsilon\left(\upsilon A_{n-\upsilon}^{\gamma-1,\delta} \in_\upsilon\right)}{A_n^{\gamma,\delta}} & ,n > \upsilon \end{cases}$$

(2.1) and (2.2) give

$$\|a\|_1 = \left\{ \frac{1}{\upsilon}\left(\frac{\upsilon}{A_\upsilon^{\alpha,\beta}}\right)^k + \sum_{n=\upsilon+1}^{m} \frac{1}{n}\left|\frac{\Delta_\upsilon\left(\upsilon A_{n-\upsilon}^{\alpha-1,\beta}\upsilon\right)}{A_n^{\alpha,\beta}}\right|^k \right\}^{\frac{1}{k}}$$

$$\|a\|_2 = \left\{ \frac{1}{\upsilon}\left|\frac{\upsilon \in_\upsilon}{A_n^{\gamma,\delta}}\right|^k + \sum_{n=\upsilon+1}^{m} \frac{1}{n}\left|\frac{\Delta_\upsilon\left(A_{n-\upsilon}^{\gamma-1,\delta} \in_\upsilon\right)}{A_n^{\gamma,\delta}}\right|^k \right\}^{\frac{1}{k}}$$

(2.4) implies

$$\frac{1}{\upsilon}\left|\frac{\upsilon \in_\upsilon}{A_n^{\gamma,\delta}}\right|^k + \sum_{n=\upsilon+1}^{m} \frac{1}{n}\left|\frac{\Delta_\upsilon\left(\upsilon A_{n-\upsilon}^{\gamma-1,\delta} \in_\upsilon\right)}{A_n^{\gamma,\delta}}\right|^k \le C^k \left\{ \frac{1}{\upsilon}\left(\frac{\upsilon}{A_n^{\alpha,\beta}}\right)^k + \sum_{n=\upsilon+1}^{m} \frac{1}{n}\left|\frac{\Delta_\upsilon\left(A_{n-\upsilon}^{\alpha-1,\beta}\upsilon\right)}{A_\upsilon^{\alpha,\beta}}\right|^k \right\} \qquad (2.5)$$

As $\quad \Delta\left(\upsilon A_{n-\upsilon}^{\alpha-1,\beta}\right) = -A_{n-\upsilon}^{\alpha-1,\beta} + (\upsilon+1)\Delta_\upsilon A_{n-\upsilon}^{\alpha-1,\beta}$,

then, by Minkowski's inequality,

$$\text{R.H.S. of (2.5)} = o\left\{\upsilon^{k-\alpha k-1}(\log \upsilon)^{-\beta k}\right\} + o(1)\sum_{n=\upsilon+1}^{m} \frac{(n-\upsilon)^{k\alpha-k}\left[\log(n-\upsilon)\right]^{\beta k}(\log n)^{-\beta k}}{n^{1+\alpha k}}$$

$$+ o\left(\upsilon^k\right)\sum_{n=\upsilon+1}^{m} \frac{(n-\upsilon)^{k\alpha-2k}\left[\log(n-\upsilon)\right]^{\beta k}(\log n)^{-\beta k}}{n^{1+\alpha k}}$$

$$= o\left(\upsilon^{k-\alpha k-1}(\log \upsilon)^{-\beta k}\right) + o\left(\upsilon^{-k}\right) + o\left(\upsilon^k\right)o\left(\upsilon^{-1-\alpha k}(\log \upsilon)^{-\beta k}\right)\sum_{n=\upsilon+1}^{m}(n-\upsilon)^{k\alpha-2k}\left[\log(n-\upsilon)\right]^{\beta k}.$$

Therefore

$$\frac{1}{\upsilon}\left|\frac{\upsilon \in_\upsilon}{A_n^{\alpha,\beta}}\right|^k = o\left(\upsilon^{k-\alpha k-1}(\log \upsilon)^{-\beta k}\right).$$

that is $\quad \in_\upsilon = o\left(\upsilon^{\gamma-\alpha}(\log \upsilon)^{\delta-\beta}\right)$.

From (2.5),

$$\sum_{n=\upsilon+1}^{m} \frac{1}{n}\left|\frac{\Delta_\upsilon\left(\upsilon A_{n-\upsilon}^{\gamma-1,\delta} \in_\upsilon\right)}{A_n^{\gamma,\delta}}\right|^k = o\left(\upsilon^{k-\alpha k-1}(\log \upsilon)^{-\beta k}\right). \qquad (2.6)$$

Now,

$$\left(\sum_{n=v+1}^{m} \frac{1}{n(n+1)^{1-1/k}} M\right)^k = \left\{\sum_{n=v+1}^{m} \frac{1}{n(n+1)}\left((n+1)^{1/k} M\right)\right\}^k$$

$$\leq \sum_{n=v+1}^{m} \frac{1}{n(n+1)}(n+1)M^k.\left(\sum_{n=v+1}^{m} \frac{1}{n(n+1)}\right)^{k-1} = \sum_{n=v+1}^{m} \frac{1}{n} M^k v^{1-k}.$$

Therefore, with $M = \left|\frac{\Delta_v\left(vA_{n-v}^{\gamma-1,\delta} \in_v\right)}{A_n^{\gamma,\delta}}\right|$, and making use of (2.6), we have

$$\sum_{n=v+1}^{m} \frac{\Delta_v\left(vA_{n-v}^{\gamma-1,\delta} \in_v\right)}{n^{2-\frac{1}{k}} A_n^{\gamma,\delta}} = o\left(v^{-\alpha}(\log v)^{-\beta}\right). \tag{2.7}$$

Since

$$\Delta_v\left(vA_{n-v}^{\gamma-1,\delta} \in_v\right) = -\in_v A_{n-v}^{\gamma-1,\delta} + (v+1)A_{n-v}^{\gamma-1,\delta}(\Delta \in_v) + (v+1) \in_{v+1} \Delta_v A_{n-v}^{\gamma-1,\delta}$$

then

$$v(\Delta \in_v)\sum_{n=v+1}^{m} \frac{A_{n-v}^{\gamma-1,\delta}}{n^{2-1/k} A_n^{\gamma,\delta}} = o\left(v^{-\alpha}(\log v)^{-\beta}\right) + o\left(v^{\gamma-\alpha}(\log v)^{\delta-\beta}\right)\sum_{n=v}^{m} \frac{A_{n-v}^{\gamma-1,\delta}}{n^{2-1/k} A_n^{\gamma,\delta}}$$

$$+ o\left(v^{1+\gamma-\delta}(\log v)^{\delta-\beta}\right)\frac{1}{v^{2-1/k} A_v^{\gamma,\delta}} \sum_{n=v}^{m} \Delta_v\left(A_{n-v}^{\gamma-1,\delta}\right).$$

Making use of Lemma 1 and Lemma 3 or Lemma 2(a) with $q_n = A_n^{\gamma-1,\delta}$, we have

$$\frac{\Delta \in_v}{v^{1-1/k}} = o(1)v^{-\alpha}(\log v)^{-\beta} + o(1)v^{\gamma-\alpha-2+1/k}(\log v)^{\delta-\beta} + o(1)v^{-\alpha-1+1/k}(\log v)^{-\beta} o(1)v^{-\alpha-1+\frac{1}{k}}(\log v)^{-\beta}$$

$$\Delta \in_v = o(1)v^{1-\alpha-1/k}(\log v)^{-\beta} + o(1)v^{\gamma-\alpha-1}(\log v)^{\delta-\beta} + o(1)v^{-\alpha}(\log v)^{-\beta}$$

$$= o\left(v^{1-\alpha-1/k}(\log v)^{-\beta}\right).$$

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