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(Dedicated to Professor H. M. Srivastava on his 62nd Birthday)

A NOTE ON ABSOLUTE SUMMABILITY FACTORS OF INFINITE SERIES

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1. Introduction. Let Σa_n be a given infinite series with (s_n) as the sequence of its nth partial sums. Let t_n denote the nth $(C, 1)$ mean of the sequence (na_n). A series Σa_n is said to be summable $|C, 1|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \frac{|t_n|^k}{n} < \infty. \tag{1.1}$$

Let the nth $(\bar{N}, p_n)$ mean of series Σa_n be difined by

$$u_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v ,$$

where (p_n) is a sequence of positive constants such that $P_n = p_0 + p_1 + ... + p_n \to \infty$ as $n \to \infty$. The series Σa_n is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$,if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |u_n - u_{n-1}|^k < \infty . \tag{1.2}$$

For $p_n=1$, $|\bar{N}, p_n|_k$ reduces to $|C,1|_k$ summability. Very recently, Mazhar [1] has proved the following

Theorem 1. If (X_n) is an almost increasing sequence such that the conditions

$$|\lambda_m| X_m = o(1), \qquad m \to \infty \tag{1.3}$$

$$\sum_{n=1}^{m} nX_n |\Delta^2 \lambda_n| = o(1), \qquad m \to \infty \tag{1.4}$$

$$\sum_{n=1}^{m} \frac{|t_n|^k}{n} = o(X_m), \qquad m \to \infty \tag{1.5}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = o(X_m), \qquad m \to \infty \tag{1.6}$$

and

$$\sum_{n=1}^{m} \frac{P_n}{n} = o(P_m), \qquad m \to \infty \tag{1.7}$$

are satisfied, then the series $\Sigma a_n \lambda_n$ is summable $|C, 1|_k$, $k \geq 1$.

The aim of this paper is to give the following improvement

Theorem. If (X_n) is an almost increasing sequence and (1.3), (1.4) and (1.7) hold and if

$$\sum_{n=1}^{m} \frac{|t_n|^k}{n} = o\,(X_m^k), \quad m \to \infty \tag{1.8}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = o\,(X_m^k), \quad m \to \infty \tag{1.9}$$

are also satisfied, then the series $\Sigma\, a_n \lambda_n$ is summable $|C, 1|_k$, $k \geq 1$.
Clearly that conditions (1.8) and (1.9) are weaker than (1.5) and (1.6) respectively.

2. Lemmas.

Lemma 2.1. [1]. If (X_n) is an almost increasing sequence, then under the condition (1.3) and (1.4)

$$n|\Delta \lambda_n| = o\,(1), \qquad n \to \infty \tag{2.1}$$

$$nX_n|\Delta \lambda_n| = o\,(1), \qquad n \to \infty \tag{2.2}$$

$$\sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty. \tag{2.3}$$

Lemma 2.2 Let (β_n) be a non-negative, non-increasing sequence, then for $k \geq 1$.

$$\Delta\,(\beta_n^k) = o(1)\beta_n^{k-1} \max_{n \leq v \leq n+1} |\Delta\beta_v|. \tag{2.4}$$

Proof. We have, by the mean value theorem

$$-\Delta\, \beta_n^k = \frac{\beta_{n+1}^k - \beta_n^k}{(n+1)-n} = (\beta_v^k)^1 \text{for some } v, \; n \leq v \leq n+1$$

$$= k\, \beta_v^k\, d\,(\beta_v)$$

$$\approx k\, \beta_v^{k-1}\, \Delta\beta_v$$

$$= o(1)\beta_n^{k-1} \max_{n \leq v \leq n+1} |-\Delta\beta_v|.$$

3. Proof of the Theorem. Let T_n denote the nth $(\bar{N}, p_n)$ mean of the series $\Sigma\, a_n \lambda_n$. Then by definition, via Abel's transformation,

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v$$

$$= \frac{p_n t_n \lambda_n}{P_n} - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v t_v \lambda_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v \Delta\lambda_v t_v$$

$$+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{p_v \lambda_{v+1} t_v}{v+1}$$

$$= L_1 + L_2 + L_3 + L_4, \quad \text{say} .$$

In order to prove the theorem, by Minkowski's inequality, it is suffices to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |L_j|^k < \infty, \qquad j = 1,2,3,4 .$$

Applying Hölder's inequality,

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|L_1|^k \le \sum_{n=1}^{m}\frac{p_n\ |t_n|^k|\lambda_n|^k}{P_n}$$

$$= o(1)\sum_{n=1}^{m-1}\left(\sum_{v=1}^{n}\frac{p_v}{P_v}\ |t_v|^k\right)\Delta|\lambda_n|^k + o(1)\left(\sum_{n=1}^{m}\ \frac{p_n}{P_n}\ |t_n|^k\right)|\lambda_m|^k$$

$$= o\ (1)\ \sum_{n=1}^{m}\ X_n^k\ |\lambda_n|^{k-1}\ |\Delta\lambda_n|\ + o(1)\ X_m^k\ |\lambda_m|^k,$$

by (2.1) and (2.4)

$$= o\ (1)\ \sum_{n=1}^{m}\ X_n\ |\Delta\lambda_n|\ + o(1)$$

$$= o\ (1)$$

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|L_2|^k = \sum_{n=1}^{m}\frac{p_n}{P_n\,P_{n-1}^k}\ |\sum_{v=1}^{n-1}\ p_v t_v \lambda_v|^k$$

$$\le \sum_{n=1}^{m}\frac{p_n}{P_n\,P_{n-1}}\ \sum_{v=1}^{n-1}\ p_v|t_v|^k|\lambda_v|^k\left(\sum_{v=1}^{n-1}\frac{p_v}{P_{n-1}}\right)^{k-1}$$

$$= o(1)\ \sum_{v=1}^{m}\ p_v|t_v|^k|\lambda_v|^k\ \sum_{n=v+1}^{m+1}\frac{p_n}{P_n\,P_{n-1}}$$

$$= o(1)\ \sum_{v=1}^{m}\ \frac{p_v}{P_v}|t_v|^k|\lambda_v|^k$$

$$= o(1).$$

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|L_3|^k = \sum_{n=1}^{m+1}\frac{p_n}{P_n\,P_{n-1}^k}\ |\sum_{v=1}^{n-1}\ p_v\Delta\lambda_v t_v|^k$$

$$\le \sum_{n=1}^{m+1}\frac{p_n}{P_n\,P_{n-1}^k}\ \sum_{v=1}^{n-1}\ v^{k-1}p_v|\Delta\lambda_v|^k|t_v|^k\left(\sum_{v=1}^{n-1}\frac{P_v}{v}\right)^{k-1}$$

$$= o(1)\ \sum_{n=1}^{m+1}\ \frac{p_n}{P_n\,P_{n-1}}\sum_{v=1}^{n-1}\ v^{k-1}p_v|\Delta\lambda_v|^k|t_v|^k$$

$$= o(1)\ \sum_{v=1}^{m}\ v^{k-1}p_v|\Delta\lambda_v|^k|t_v|^k\ \sum_{n=v+1}^{m+1}\frac{p_n}{P_n\,P_{n-1}}$$

$$= o\ (1)\ \sum_{v=1}^{m}\ v^k|\Delta\lambda_v|^k\frac{|t_v|^k}{v}$$

$$= o(1)\ \sum_{v=1}^{m-1}\ \sum_{r=1}^{v}\frac{|t_r|^k}{r}\ \Delta\,(v^k|\Delta\lambda_v|^k) + \sum_{v=1}^{m}\ \frac{|t_v|^k}{v}\ m^k|\Delta\lambda_m|^k$$

$$= o(1)\sum_{v=1}^{m} X_v^k|\Delta v^k|\Delta\lambda_v|^k + (v+1)^k\Delta|\Delta\lambda_v|^k| + o(1)X_m^k\ m^k|\Delta\lambda_m|^k$$

$$=o(1)\sum_{v=1}^{m} v^{k-1}X_v^k|\Delta\lambda_v|^k+o(1)\sum_{v=1}^{m} X_v^k v^k|\Delta\lambda_v|^{k-1}|\Delta^2\lambda_v|+o(1)$$

$$=o(1)\sum_{v=1}^{m} X_v|\Delta\lambda_v|+o(1)\sum_{v=1}^{m} vX_v\,|\Delta^2\lambda_v|+o(1)$$

$$=o(1)$$

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|L_4|^k=\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left|\sum_{v=1}^{n-1}\frac{P_v\lambda_{v+1}t_v}{v+1}\right|^k$$

$$\leq \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\sum_{v=1}^{n-1}\frac{P_v|\lambda_v|^k|t_v|^k}{v}\left(\sum_{v=1}^{n-1}\frac{P_v}{v}\right)^{k-1}$$

$$=o(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\frac{P_v|\lambda_v|^k|t_v|^k}{v}$$

$$=o(1)\sum_{v=1}^{m}\frac{P_v|\lambda_v|^k|t_v|^k}{v}\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}$$

$$=o(1)\sum_{v=1}^{m}\frac{|t_v|^k}{v}|\lambda_v|^k$$

$$=o(1)\sum_{v=1}^{m-1}\left(\sum_{r=1}^{v}\frac{|t_r|^k}{r}\right)\Delta|\lambda_v|^k+0(1)\left(\sum_{v=1}^{m}\frac{|t_v|^k}{v}\right)|\lambda_m|^k$$

$$=o(1)\sum_{v=1}^{m} X_v^k|\lambda_v|^{k-1}+|\Delta\lambda_v|+0(1)X_n^k\,|\lambda_m|^k$$

$$=o(1)\sum_{v=1}^{m} X_v|\Delta\lambda_v|+0(1)$$

$$=o(1)$$

REFERENCE

[1] S.M. Mazhar, Absolute Summability factors of infinite series, *Kyungpook Math. J.* **1** (1999), 67-73