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(Dedicated to Professor H. M. Srivastava on his 62nd Birthday)

ON A *KP* CLASS OF EQUATIONS AND MIURA TRANSFORMATION

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ABSTRACT

A Miura transformation is presented for a kadomtsev petrviashvili (*KP*) class of equations, which can be considered as generalization of the *KP* equation. The Bäcklund transformation and the recurrence formula for solutions are derived for this *KP* calss of equations. Also, the fundamental equations of the inverse scattering scheme for these equations is given.

1. Introduction. The search for possible connections among different approaches to partial differential equations may help to clarify some aspects concerning the complete integrability of equations under consideration.

The Miura transformation(*MT*) [11] is the well-known example of a transformation of one partial differential equation (*PDE*) into another. The *MT* has made it possible to discover the existence of the infinite sequence of conservation laws for the *KDV* equation [12] and the method to integrate the *KDV* equation [7]. Moreover, the *MT* is closely linked with the existence of the hierarchies of completely integrable nonlinear *PDEs*, the Bäcklund transformation [13], the bi-Hamiltonian structures [6] and makes it possible to connect certain *PDE's* formerly considered to be not connected. Also, the *MT* has played an important role both in the discovery of the inverse scattering method and in the further understanding of the properties of completely integrable nonlinear *PDEs* [1, 9, 14].

The *KP* equation and the modified *KP* equation, which has been introduced within the different approaches are connected by the two-dimensional Miura transforamtion [8]. In this paper we study the *KP* class of equations

$$(u_t + uu_x + uu_{xxx})_x + Au_x + Bu_y + Cu_{yy} + Du_{xy} + Eu_{xx} = 0 \qquad \text{... (1.1)}$$

or

$$u_t + uu_x + u_{xxx} + Au + B\partial_x^{-1} u_y + C\partial_x^{-1} u_{yy} + Du_y + Eu_x = 0 \qquad \text{... (1.2)}$$

where $u = u\ (x, y, t)$, A and C are arbitrary functions of y, t and symmetric with respect to y,

$$B = \frac{1}{2} C_y,\ D = C^{1/2}(2F + G),\ E = F^2 + \int (F_y G + C^{1/2} F_t)\, dy,$$

$$F = \int A C^{1/2}\, dy \text{ and } G = \frac{1}{2} \int C^{3/2} C_t\, dy$$

and subscripts denote partial derivatives.

Equation (1.2), for particular values of the coefficients, appears in many physical systems [3,10]. In [5], it was found that the KP class of equations under consideration possesses the painleve property and allows a set of Bäcklund transformations obtained by truncating the series expansions of the solutions about the singularity manifold.

In section 2 a generalized Miura transformation for (1.2) is presented. In section 3 a Bäcklund transformation of the KP class of equations is derived. In this context, some theorems are also proved. In section 4 we derive a recurrence formula for solutions of these equations. Finally in section 5 the fundamental equations of the inverse scattering method for (1.2) is given.

2. Generalized Miurra Transformation and Modified *KP* Class of Equations. In this section we shall prove that there is a Miura transformation for KP class of equations (1.2).

Theorem 1. Suppose v is a solution of the equation :

$$v_t + v_{xxx} + 6v^2 v_x - 2(3)^{1/2} C^{1/2} v_x \partial_x^{-1} v_y - 2(3)^{1/2} F v v_x + Av + B\partial_x^{-1} v_y + C\partial_x^{-1} v_{yy} + Dv_y + E_x v = 0, \quad \text{...(2.1)}$$

then u defined by

$$u = -2(3)^{1/2} C^{1/2} \partial_x^{-1} v_y - 2(3)^{1/2} Fv - 6v_x - 6v^2 \quad \text{... (2.2)}$$

is a solution of (1.2).

Proof. Substitution of (2.2) into (1.2) yields

$$(-2(3)^{1/2} C^{1/2} \partial_x^{-1} - 2(3)^{1/2} F - 6\partial_x - 12v)(v_t + v_{xxx} - 6v^2 v_x - 2(3)^{1/2} C^{1/2} v_x \partial_x^{-1} v_y - 2(3)^{1/2} F v v_x + Av + B\partial_x^{-1} v_y + C\partial_x^{-1} v_{yy} + Dv_y + E_x v) = 0. \quad \text{... (2.3)}$$

Thus, if v is a solution of the equation

$$v_t + v_{xxx} + 6v^2 v_x - 2(3)^{1/2} C^{1/2} v_x \partial_x^{-1} v_y - 2(3)^{1/2} F v v_x + Av + B\partial_x^{-1} v_y + C\partial_x^{-1} v_{yy} + Dv_y + E_x v = 0, \quad \text{...(2.4)}$$

then equation (2.2) defines a solution of (1.2) and we complete the proof.

Equation (2.2) links the KP class of equation (1.2) and (2.4) when $F=0$ and u is independent of y, (2.2) is the Miura transformation between the KDV equation and the modified KDV equation. We call (2.4) the modified KP class of equations and (2.2) the generalized Miura transformation.

3. Bäcklund Transformation. In this section we present a Bäcklund transformation for (1.2) defined by the relation

$$u = -2(3)^{1/2}C^{1/2}\,\partial_x^{-1}\,v_y - 2\,(3)^{1/2}\,Fv - 6v_x - 6v^2\eta\,, \qquad \text{... (3.1)}$$

where η is any real parameter. Then

$$u_t + uu_x + u_{xxx} + Au + B\partial_x^{-1}\,u_y + C\partial_x^{-1}\,u_{yy} + Du_{xy} + Eu_{xx} =$$

$$(-2(3)^{1/2}C^{1/2}\,\partial_x^{-1}\partial_y - 2\,(3)^{1/2}\,F - 6\partial_x - 12v)\,(v_t + v_{xxx} - \eta^2 v_x - 6v^2 v_x - 2\,(3)^{1/2}$$

$$C^{1/2}v_x\partial_x^{-1}\,v_y - 2(3)^{1/2}Fvv_x + Av + B\partial_x^{-1}\,v_y + C\partial_x^{-1}\,v_{yy} + Dv_y + E_x v)\,. \qquad \text{... (3.2)}$$

Thus, (1.2) is satisfied by u if v satisfies the equation

$$v_t + v_{xxx} + \eta v_x - 6v^2 v_x - 2\,(3)^{1/2}\,C^{1/2}v_x\partial_x^{-1}\,v_y - 2(3)^{1/2}Fvv_x + Av + B\partial_x^{-1}\,v_y$$

$$+ C\partial_x^{-1}\,v_{yy} + Dv_y + E_x v = 0. \qquad \text{... (3.3)}$$

Further, if we change (v, y) to $(-v, -y)$, equation (3.3) is invariant and (1.2) is invariant as well when y is changed to $-y$. Hence, for this new pair $(-v, -y)$ there is a corresponding solution u' of (1.2) such that

$$u' = -2(3)^{1/2}C^{1/2}\,\partial_x^{-1}\,v_y - 2\,(3)^{1/2}\,Fv - 6v_x + 6v^2 + \eta \qquad \text{... (3.4)}$$

for given v and η. Equations (3.1) and (3.4) imply that

$$u - u' = 12v_x \qquad \text{... (3.5a)}$$

and

$$u + u' = -4\,(3)^{1/2}\,C^{1/2}\partial_x^{-1}\,v_y - 4(3)^{1/2}Fv + 12v^2 + 2\eta. \qquad \text{... (3.5b)}$$

We introduce the additional transformation

$$u = -w_x\,, \qquad \text{... (3.6a)}$$

$$u' = -w'_x\,. \qquad \text{... (3.6b)}$$

It follows from (1.2) that w and w' satisfied the equations

$$w_t - \tfrac{1}{2}w_x^2 + w_{xxx} + Aw + B\partial_x^{-1}\,w_y + C\partial_x^{-1}\,w_{yy} + Dw_{xy} + Ew_{xx} = 0, \qquad \text{... (3.7)}$$

and

$$w'_t - \tfrac{1}{2}w'^2_x + w'_{xxx} + Aw' + B\partial_x^{-1}\,w'_y + C\partial_x^{-1}\,w'_{yy} + Dw'_{xy} + Ew'_{xx} = 0, \qquad \text{... (3.8)}$$

respectively. So that (3.5a) and (3.5b) become

$$w - w' = 12\,v \qquad \text{... (3.9)}$$

and

$$w_x + w'_x = -2\eta + 3^{1/2}C^{1/2}\partial_x^{-1}\,(w-w')y + 3^{1/2}F(w-w') + \tfrac{1}{2}\,(w-w')^2 \qquad \text{...(3.10)}$$

respectively. Equation (3.10) constitutes part of the Bäcklund transformation for w and w' which, in turn, generate solutions of (1.2) via (3.6). Using equations (3.5a) - (3.9), equation (3.3) can be written as

$$(w - w')_t + (w-w')_{xxx} - \tfrac{1}{2}\,(w_x^2 - w'^2_x) + A\,(w - w') + B\,\partial_x^{-1}(w - w')_y +$$

$$C\,\partial_x^{-1}\,(w - w')_{yy} + D\,(w - w')_y + E(w - w')_x = 0, \qquad \text{... (3.11)}$$

which together with equation (3.8) forms the Bäcklund transformation (1.2). To conclude, we have the following theorem:

Theorem 2. The *KP* class of equation (1.2) possesses the Bäcklund transformation

(3.10) and (3.11) (with (3.6)).

Now using (3.8) and (3.10), (3.11) can be rewritten as

$$(w+w')_t = \frac{1}{3}(w_x^2 + w_x w'_x - w'^2_x) - \frac{1}{6}(w-w')(w-w')_{xx} -$$
$$3^{1/2}C^{1/2}(w-w')_{xy} - 3^{1/2}F(w-w')_{xx} - A(w+w') -$$
$$B\,\partial_x^{-1}(w+w')_y - C\,\partial_x^{-1}(w+w')_{yy} - D(w+w')_y - E(w+w')_x = 0. \qquad ...(3.12)$$

Hence, we have proved the following proposition

Proposition 1. The second part of Bäcklund transformation for (1.1) can be written in the form (3.12).

However, instead of (3.12), a different expression for the second part of Bäcklund transformation can be given as:

Inserting (3.7) and (3.10) into (3.12) we find

$$(w-w')_t = [2w_{xx} - \frac{1}{3}(w-w')w_x + \frac{2}{3}\eta(w-w') + E(w-w') - 3^{1/3}C^{1/2}(w-w')\,\partial_x^{-1}(w-w')_y$$
$$- \frac{1}{4}3^{3/2}F(w-w')^2 - 3^{1/2}F(w-w')_x - 3^{1/2}C^{1/2}(w-w')_y\,]_x - \frac{1}{2}3^{1/2}C^{1/2}(w-w')(w-w')_y$$
$$+ A(w-w') + B\,\partial_x^{-1}(w-w')_y + C\,\partial_x^{-1}(w-w')_{yy} + D(w-w')_y\ . \qquad ...(3.13)$$

Thus, (3.10) and (3.12) or (3.13) (with(3.6)) constitutes the Bäcklund transformation for (1.2).

The method we deduce the Bäcklund transformation can be considered as a development of Chen's in [2].

4. Superposition Formula. The advantage of Bäcklund transformation is the possibility of deriving a superposition formula for solutions. this superposition enables us to construct more complicated solutions by algebraic means only. No more integration quadrature is needed then. it also implies that solutions obtained are stable. They do not lose their identities after colliding with each other.

To derive the superposition formula, we suppose that we generate two solutions, w_1 and w_2, from the Bäcklund transformation, by using the same given solution (w_0, say) but two different values of (η_1, η_2, say) . Thus, in particular, we can write (3.10) in the two forms:

forms:

$$(w_1 + w_0)_x = 2\eta_1 - 3^{1/2}C^{1/2}\partial_x^{-1}(w_1 - w_0)_y - 3^{1/2}F(w_1 - w_0) + \frac{1}{12}(w_1 - w_0)^2 \qquad (4.1a)$$

and

$$(w_2 + w_0)_x = 2\eta_2 - 3^{1/2}C^{1/2}\partial_x^{-1}(w_2 - w_0)_y - 3^{1/2}F(w_2 - w_0) + \frac{1}{12}(w_2 - w_0)^2. \qquad (4.1b)$$

Now we construct another solution, w_{12} from w_1 and η_2, and similary a solution w_{21} from w_2 and η_1, so that

$$(w_{12} + w_1)_x = 2\eta_2 - 3^{1/2}C^{1/2}\partial_x^{-1}(w_{12} - w_1)_y - 3^{1/2}F(w_{12} - w_1) + \frac{1}{12}(w_{12} - w_1)^2 \quad (4.2a)$$

and

$$(w_{21} + w_1)_x = 2\eta_1 - 3^{1/2}C^{1/2}\,\partial_x^{-1}(w_{21} - w_2)_y - 3^{1/2}F(w_{21} - w_2) + \frac{1}{12}(w_{21} - w_2)^2. \quad (4.2b)$$

Since Binanchi's theorem of permutability [4] from the theory of Bäcklund transformation, states that, if w_{12} and w_{21} are defined as above, then

$$w_{12} = w_{21}. \quad \dots (4.3)$$

Subtracting the difference of equations (4.1) from the difference of equations (4.2), and use equation (4.3), so as to produce zero on the left hand side of the resulting equation

$$0 = 4(\eta_2 - \eta_1) - 3^{1/2}C^{1/2}\partial_x^{-1}(w_{21} - w_2)_y - 3^{1/2}F(w_{21} - w_2) + \frac{1}{12}(w_{21} - w_2)^2 +$$
$$3^{1/2}C^{1/2}\partial_x^{-1}(w_{12} - w_1)_y + 3^{1/2}F(w_{12} - w_1) - \frac{1}{12}(w_{12} - w_1)^2 +$$
$$3^{1/2}C^{1/2}\partial_x^{-1}(w_{12} - w_1)_y + 3^{1/2}F(w_2 - w_0) - \frac{1}{12}(w_2 - w_0)^2 -$$
$$3^{1/2}C^{1/2}\partial_x^{-1}(w_1 - w_0)_y - 3^{1/2}F(w_1 - w_0) + \frac{1}{12}(w_1 - w_0)^2. \quad \dots (4.4)$$

Again using (4.3) we derive

$$w_{12} = w_0 - 24(\eta_2 - \eta_1)/(w_1 - w_2) + 4(3)^{1/2}C^{1/2}\,\partial_x^{-1}(w_1 - w_2)_y/(w_1 - w_2)$$
$$- 4(3)^{1/2}F, \quad \dots (4.5)$$

which describes w_{12} in algebraic terms.

Equation (4.5) constitutes a nonlinear superposition principle for the generation of solutions.

To conclude, we have the following theorem:

Theorem 3. For the *KP* class of equations (1.2) there exists superposition formula for solutions, (4.5)

$$w_{12} = w_0 - 24(\eta_2 - \eta_1)/(w_1 - w_2) + 4(3)^{1/2}C^{1/2}\,\partial_x^{-1}(w_1 - w_2)_y/(w_1 - w_2) - 4(3)^{1/2}F, \quad \dots (4.5)$$

where

$$u_0 = -w_{0x}, \quad u_1 = -w_{1x}, \quad u_2 = -w_{2x} \text{ and } u_{12} = -w_{12x}.$$

5. The Fundamental Equations of Inverse Scattering Method. In this section we find the corresponding inverse scattering problem for the *KP* class of equations (1.2), by using standard procedure of linearisation.

Define v by the relation

$$u = -2(3)^{1/2}C^{1/2}\partial_x^{-1}v_y - 2(3)^{1/2}Fv - 6v^2 + \eta, \quad \dots (5.1)$$

where η is any real parameter. Then

$$u_t + uu_x + u_{xxx} + Au + B\partial_x^{-1}u_y + C\partial_x^{-1}u_{yy} + Du_y + Eu_x =$$
$$(-2(3)^{1/2}C^{1/2}\,\partial_x^{-1} - 2(3)^{1/2}F - 6\partial_x - 12v)(v_t + v_{xxx} - \eta v_x - 6v^2v_x - 2(3)^{1/2}$$
$$C^{1/2}v_x\partial_x^{-1}v_y - 2(3)^{1/2}Fvv_x + Av + B\partial_x^{-1}v_y + C\partial_x^{-1}v_{yy} + Dv_y + E_xv) \quad \dots (5.2)$$

Thus, if v is a solution of

$$v_t + v_{xxx} - \eta v_x - 6v^2v_x - 2(3)^{1/2}C^{1/2}v_x\partial_x^{-1}v_y - 2(3)^{1/2}Fvv_x + Av + B\partial_x^{-1}v_y$$

$$+C\partial_x^{-1} v_{yy} + Dv_y + E_x v = 0, \qquad \text{... (5.3)}$$

then (5.1) defines a solution of (1.2).

We recognize (5.10 as generalized Riccati equation for v which therefore may linearised by the substitution

$$v = \psi_x/\psi . \qquad \text{...(5.4)}$$

For some differentiable function $\psi(x,y,t) = 0$. Equation (5.1), upon the introduction of (5.4), becomes

$$\psi_{xx} + 3^{1/2}C^{1/2}\psi_y + 3^{1/2}F\psi_x + \frac{1}{6}(u-\eta)\psi = 0 . \qquad \text{... (5.5)}$$

Equation (5.3), upon the introduction of (5.4) and (5.5), becomes

$$\frac{\partial}{\partial x}(\psi_t/\psi + 4\psi_{xxx}/\psi + u\psi_x/\psi - 2FC^{1/2}\psi_y/\psi - F^2\psi_x/\psi - \frac{1}{2}(3^{1/2}C^{1/2}u_y + 3^{1/2}Fu)$$

$$+\frac{1}{2}u_x + D\psi_y/\psi + E\psi_x/\psi) = 0 \qquad \text{...(5.6)}$$

i.e

$$\psi_t + 4\psi_{xxx} + u\psi_x + \frac{1}{2}u_x\psi + (D-2FC^{1/2})\psi_y + (E-F^2)\psi_x - \frac{1}{2}3^{1/2}C^{1/2}u_y$$

$$-\frac{1}{2}3^{1/2}Fu\psi + \alpha\psi = 0,$$

where α is constant.

The above discussion are summarized in as the following:

Theorem 4. The *KP* class of equations (1.2) has the fundamental equation of the inverse scattering method, (5.5) and (5.6).

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