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(*Dedicated to Professor H. M. Srivastava on his 62nd Birthday*)

DETERMINATION OF SOME MORE GENERATING FUNCTIONS FOR BIORTHOGONAL POLYNOMIALS THROUGH LIE GROUP TECHNIQUES

By

Hemant Kumar, Ram Kumar Maurya and Shiv Sahay Pandey
Department of Mathematics, D.A.V. College, Kanpur, U.P., India

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ABSTRACT

In the present paper, generating functions are evaluated through Lie group techniques for the biorthogonal function $Y_n^{\alpha}(x,k)y^{kn}$, which are believed to be new.

1. Introduction. The biorthogonal polynomials are very useful in solving many physical problems, in the theory of Approximation, Queueing, Coding and several branches of applied Mathematics (see Thakare [12]).

Srivastava and Manocha [11] have studied and evaluated various generating functions for different hypergeometric functions as using Lie group techniques. Pathan, Goyal and Shahwan [8] have obtained the generating functions for Bessel polynomials by Lie theory. Chatterjea [2], Mukherjee and Chongdar [7] and Weisner [14-16] have determined the generating functions for Hermite. Laguerre and Bessel functions etc. by Lie Algebra.

Konhauser [1967] was perhaps the first to build up a systematic study of general properties of biorthogonal polynomial sets in the Hilbert space $L^2\ (a,b)$, where (a,b) is a real interval. He suggested that if $s(x)$ be real polynomial in x of degree $k>0$ and *let* $S_n(x)$ denotes the polynomial of degree n in $s(x)$. Thus $S_n(x)$ be the polynomial of degree nk in x and the biorthogonal polynomials $Z_n^{\alpha}(x,k)$ of degree n are defined as

$$Z_n^{\alpha}(x,k)=\frac{\Gamma(kn+\alpha+1)}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{x^{kj}}{\Gamma(kj+\alpha+1)}. \tag{1.1}$$

Carlitz [1] (see Srivastava and Manocha [11] eq. (73) p. 432) defined $Y_n^{\alpha}(x,k)$ as

$$Y_n^{\alpha}(x,k)=\frac{1}{n!}\sum_{r=0}^{n}\frac{x^r}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(\frac{1+\alpha+s}{k}\right)_n. \tag{1.2}$$

Particularly, from (1.2) for $k=1$ and $k=2$, we have

$$Y_n^{\alpha}(x,1)=L_n^{\alpha}(x), \tag{1.3}$$

and the polynomial obtained by Spencer and Fano [10] in their calculation of penetration of gamma rays through matter respectively. (1.4)

Here, in this paper, we introduce a biorthogonal type polynomial which satifies the linear differntial equation, is given by

$$x\frac{d^2}{dx^2}Y_n^\alpha(x,k)+(1+\alpha-x)\frac{d}{dx}Y_n^\alpha(x,k)+knY_n^\alpha(x,k)=0 \tag{1.5}$$

and for formal series $\sum_{n=0}^{\infty}Y_n^\alpha(x,k)y^{kn}$ the differential recurrence relations are satisfied for $k>0$, as

$$x\frac{d}{dx}Y_n^\alpha(x,k)=knY_n^\alpha(x,k)-(kn+\alpha)Y_{n-1/k}^\alpha(x,k) \tag{1.6}$$

and

$$x\frac{d}{dx}Y_n^\alpha(x,k)=(x-kn-\alpha-1)Y_n^\alpha(x,k)+(kn+1)Y_{n+1/k}^\alpha(x,k). \tag{1.7}$$

2. Group of Operators. To find the group of operators, we define the first order differential operators A, B and C for the eigen function $Y_n^\alpha(x,k)y^{kn}$ those follow the relations

$$A\left[Y_n^\alpha(x,k)y^{kn}\right]=a_{n,k}Y_n^\alpha(x,k)y^{kn}\ , \tag{2.1}$$

$$B\left[Y_n^\alpha(x,k)y^{kn}\right]=b_{n,k}Y_{n-1/k}^\alpha(x,k)y^{kn-1}, \tag{2.2}$$

$$C\left[Y_n^\alpha(x,k)y^{kn}\right]=c_{n,k}Y_{n+1/k}^\alpha(x,k)y^{kn+1}, \tag{2.3}$$

where $a_{n,k}$, $b_{n,k}$ and $c_{n,k}$ are the expressions in n and k and are indepenent of x and y but not necessarily α.

Now, let $A=A_1\frac{\partial}{\partial y}+(\alpha+1)/2$, (2.4)

where $A_1=A_1(x,y)$, and expression of x and y which is independent of n and k but not necessarily α.

Then we have by (2.4) as

$$A\left[Y_n^\alpha(x,k)y^{kn}\right]=A_1Y_n^\alpha(x,k)kny^{kn-1}+\frac{1}{2}(\alpha+1)Y_n^\alpha(x,k)y^{kn}. \tag{2.5}$$

On choosing $A_1=y$ in (2.5), we find an equivalent relation to (2.1) for $a_{n,k}=\{kn+(\alpha+1)/2\}$ as

$$A\left[Y_n^\alpha(x,k)y^{kn}\right]=\{kn+(\alpha+1)/2\}Y_n^\alpha(x,k)y^{kn} \tag{2.6}$$

and then, the operator (2.4) becomes

$$A = y\frac{\partial}{\partial y} + (\alpha+1)/2. \tag{2.7}$$

Further, let $B = B_1\frac{\partial}{\partial x} + B_2\frac{\partial}{\partial y} + B_3,$ (2.8)

where $B_i = B_i\ (x,y)$, $i = 1,2,3$; the expressions of x and y are independent of n and k but not necessarily of α.

Thus, by (2.8) we obtain

$$B\left[Y_n^\alpha(x,k)y^{kn}\right] = B_1\frac{\partial}{\partial x}\left[Y_n^\alpha(x,k)y^{kn}\right] + B_2knY_n^\alpha(x,k)y^{kn-1} + B_3knY_n^\alpha(x,k)y^{kn}. \tag{2.9}$$

On making an appeal to (1.6) and (2.9), we find

$$B\left[Y_n^\alpha(x,k)y^{kn}\right] = B_1y^{kn}\frac{1}{x}\left[knY_n^\alpha(x,k) - (kn+\alpha)Y_{n-1/2}^\alpha(x,k)\right] + B_3Y_n^\alpha(x,k)y^{kn}. \tag{2.10}$$

On setting $B_1 = xy^{-1}$, $B_2 = -1$, and $B_3 = 0$ in (2.10) we find

$$B\left[Y_n^\alpha(x,k)y^{kn}\right] = (-kn-\alpha)Y_{n-1/k}^\alpha(x,y)y^{kn-1}, \tag{2.11}$$

which is equivalent to the relation (2.2) for $b_{n,k} = (-kn-\alpha)$ and then the operator (2.8) becomes

$$B = xy^{-1}\frac{\partial}{\partial x} - \frac{\partial}{\partial y}; \tag{2.12}$$

Again, let $C = C_1\frac{\partial}{\partial x} + C_2\frac{\partial}{\partial y} + C_3,$ (2.13)

where $C_i = C_i\ (x,y)$, $i = 1,2,3$; the expressions of x and y are independent of n and k but not necessarily of α.

Then by (2.13), we obtain

$$C\left[Y_n^\alpha(x,y)y^{kn}\right] = C_1\frac{\partial}{\partial x}\left[Y_n^\alpha(x,y)y^{kn}\right] + C_2knY_n^\alpha(x,k)y^{kn-1} + C_3Y_n^\alpha(x,y)y^{kn}. \tag{2.14}$$

On making an application of (1.7) in (2.14) we get

$$C\left[Y_n^\alpha(x,k)y^{kn}\right] =$$

$$C_1y^{kn}\frac{1}{x}\left[(x-kn-\alpha-1)Y_n^\alpha(x,k) + (kn+1)Y_{n+1/k}^\alpha(x,k)\right] + C_2knY_n^\alpha(x,k)y^{kn-1} + C_3Y_n^\alpha(x,k)y^{kn} \tag{2.15}$$

On choosing $C_1 = xy$, C_2y^2 and $C_3 = (1+\alpha-x)y$ in (2.15) we get

$$C\left[Y_n^\alpha(x,y)y^{kn}\right] = (kn+1)Y_{n+1/k}^\alpha(x,k)y^{kn+1}, \tag{2.16}$$

which is equivalent to the relation (2.3) for $C_{n,k} = (kn+1)$.

Then, the operator (2.13) becomes

$$C = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + (1+\alpha - x)y\,. \tag{2.17}$$

The commutation relations due to the operators (2.7), (2.12) and (2.17) are

$$[A,B] = AB - BA = -B \tag{2.18}$$

$$[A,C] = AC - CA = C \tag{2.19}$$

and

$$[C,B] = CB - BC = 2A \tag{2.20}$$

(Also, see Srivastava and Manocha [11, 6.4(8) p. 321]). Again we introduce an operator

$$D = CB + AA - A = CB + A^2 - A$$

$$= x\left[x\frac{\partial^2}{\partial x^2} + (1+\alpha - x)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right] + \frac{1}{4}(\alpha^2 - 1) \tag{2.21}$$

which commutes with A, B and C.

Further on replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$ and nk by $y\frac{\partial}{\partial y}$ in the equation (1.5) we find that

$$\left[x\frac{\partial^2}{\partial x^2} + (1+\alpha - x)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right]f(x,y) = 0\,. \tag{2.22}$$

Then with the aid of (2.21) and (2.22), we get

$$Df(x,y) = \frac{1}{4}(\alpha^2 - 1)f(x,y), \tag{2.23}$$

and from (2.7), we have

$$Af(x,y) = [(\alpha+1)/2 + kn]f(x,y). \tag{2.24}$$

The relation (2.23) and (2.24) may be written as

$$\left[x\frac{\partial^2}{\partial x^2} + (1+\alpha - x)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right]f(x,y) = 0, \tag{2.25}$$

and

$$\left[y\frac{\partial}{\partial y} - kn\right]f(x,y) = 0\,. \tag{2.26}$$

Hence, with the aid of (2.25) and (2.26) we find that

$$f(x,y) = y^{kn}Y_n^{\alpha}(x,k)\ . \tag{2.27}$$

Again, we know that $L_\beta \in G'$ is an Isomorphic image of $\beta \in G$ then T is a multiplier representation of G on the representation space of A_{x^β}, where all $x^\beta \in C$

and then

$$\left[T\left(e^{\beta t}\right)f\right](x) = exp\left[tL_{\beta}\right]f(x) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\left[L_{\beta}^{n}f\right](x) \tag{2.28}$$

where t lies in a sufficiently small neighbourhood of $0 \in C$.

Hence making an use of (2.28) for the operators A, B and C, we get following relations (see Srivastava and Manocha [11, p 322]).

$$\left[T(\exp aA)f\right](x,y) = \exp\left[a(\alpha+1)/2\right]f\left(x, ye^{a}\right), \tag{2.29}$$

$$\left[T(\exp bB)f\right](x,y) = f\left[\frac{x}{1-b/y}, y-b\right], |b/y|<1, \tag{2.30}$$

$$\left[T(\exp cC)f\right](x,y) = (1-cy)^{-\alpha-1}\exp\left[\frac{-cxy}{1-cy}\right]f\left\{\frac{x}{1-cy}, \frac{y}{1-cy}\right\}, \tag{2.31}$$

and

$$\left[T\{exp(aA)\,exp(bB).\,exp(cC)\}f\right](x,y) = \left[T(exp(aA))T(exp(bB)).\,T(exp(cC))f\right](x,y)$$

$$= exp\left[\frac{a}{2}(\alpha+1)\right](1-cy)^{-\alpha-1}\,exp\left(\frac{-cxy}{1-cy}\right)f\left(\frac{xy}{(1-cy)(y-b+bcy)}, \frac{(y-b+cb)e^{a}}{1-cy}\right) \tag{2.32}$$

3. Generating Functions. To evaluate the generating functions for $Y_n^{\alpha}(x,k)$, we have to find

$$A^2\left[Y_n^{\alpha}(x,k)y^{kn}\right] = AA\left[Y_n^{\alpha}(x,k)y^{kn}\right] \tag{3.1}$$

The relation (3.1) as using (2.6) becomes

$$A^2\left[Y_n^{\alpha}(x,k)y^{kn}\right] = \{kn+(\alpha+1)/2\}^2\,Y_n^{\alpha}(x,k)y^{kn}. \tag{3.2}$$

Then, on generalization of (3.1), we get

$$A^p\left[Y_n^{\alpha}(x,k)y^{kn}\right] = \{kn+(\alpha+1)/2\}^p\,Y_n^{\alpha}(x,k)y^{kn}. \tag{3.3}$$

Therefore, from (3.3) we have

$$A^p = \{kn+(\alpha+1)/2\}^p. \tag{3.4}$$

Now, making an appeal to (2.28), (2.29) and (3.4) we derive a generating function for $f(x,y) = Y_n^{\alpha}(x,k)y^{kn}$ as

$$\exp\left[\frac{a}{2}(\alpha+1)\right]Y_n^{\alpha}(x,k)\left(ye^{a}\right)^{kn} = \sum_{p=0}^{\infty}\frac{a^p\{kn+(\alpha+1)/2\}^p}{p!}Y_n^{\alpha}(x,k)y^{kn}. \tag{3.5}$$

Again, with the aid of (2.11), we get

$$B^2\left[Y_n^\alpha(x,k)y^{kn}\right] = B.B\left[Y_n^\alpha(x,k)y^{kn}\right] = (-kn-\alpha)_2 Y_{n-2/k}^\alpha(x,k)y^{kn-2}. \tag{3.6}$$

On generalization of (3.6). we find that

$$B^p\left[Y_n^\alpha(x,k)y^{kn}\right] = (-kn-\alpha)_p Y_{n-p/k}^\alpha(x,k)y^{kn-p}, \tag{3.7}$$

which is correct version of the result due to Kumbhat and Singhal [eq. (2.7) p. 80]. Now making an appeal to (2.28), (2.30) and (3.7), we obtain another generating function for $f(x,y) = Y_n^\alpha(x,k)y^{kn}$ as

$$Y_n^\alpha\left(\frac{x}{1-b/y},k\right)(y-b)^{kn} = \sum_{p=0}^{\infty}\frac{b^p(-kn-\alpha)_p}{p!}Y_{n-p/k}^\alpha(x,k)y^{kn-p}, |b/y|<1. \tag{3.8}$$

Now, with the use of (2.16), we find that

$$C^2\left[Y_n^\alpha(x,k)y^{kn}\right] = C.C\left[Y_n^\alpha(x,k)y^{kn}\right] = (kn+1)_2\left[Y_{n+2/k}^\alpha(x,k)y^{kn+2}\right]. \tag{3.9}$$

On generalization of (3.9), we get

$$C^p\left[Y_n^\alpha(x,k)y^{kn}\right] = (kn+1)_p Y_{n+p/k}^\alpha(x,k)y^{kn+p}. \tag{3.10}$$

Now, making an appeal to (2.28), (2.32), (3.3), (3.7) and (3.10), we obtain another generating function as

$$exp\left[\frac{1}{2}(\alpha+1)a\right](1-cy)^{-\alpha-1} exp\left[\frac{-cxy}{1-cy}\right]Y_n^\alpha\left(\frac{xy}{(1-cy)(y-b+bcy)},k\right).\left\{\frac{(y-b+cb)e^a}{1-cy}\right\}^{kn}$$

$$=\sum_{r=0}^{\infty}\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\frac{c^p\ b^q}{p!\ q!}\frac{\left\{(kn+p-q)a+\frac{a}{2}(\alpha+1)\right\}^r}{r!}$$

$$.(kn+1)_p(-kn-p-\alpha).Y_{n+(p-q)/k}^\alpha(x,k)y^{kn+p-q}. \tag{3.11}$$

Again setting $e^{a/2} = \frac{1}{\delta}, c = -\beta/\delta$ and $b = -\theta\delta$ in (3.11) and using the fact that $\lambda\delta - \beta\theta = 1$, we get

$$(\beta y+\delta)^{-\alpha-kn-1} exp\left[\frac{\beta xy}{\delta+\beta y}\right]Y_n^\alpha\left(\frac{xy}{(\lambda y+\theta)(\beta y+\delta)},k\right).(\theta+\lambda y)^{kn}$$

$$=\sum_{r=0}^{\infty}\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-\beta/\delta)^p}{p!}\frac{(-\theta\delta)^q}{q!}\frac{(-1)^r}{r!}\left\{(kn+p-q)\log\delta^2+\frac{(\alpha+1)}{2}\log\delta^2\right\}^r.$$

$$.(kn+1)_p(-kn-p-\alpha)_q Y^{\alpha}_{n+(p-q)/k}(x,k)y^{kn+p-q}. \tag{3.12}$$

4. Particular Cases. Setting $\theta = 0$, $\delta=\lambda=1$ and $y=-1$ in (3.12), we find that

$$(1-\beta)^{-\alpha-kn-1}\exp\left[\frac{-\beta x}{1-\beta}\right]Y_n^{\alpha}\left(\frac{x}{1-\beta},k\right)=\sum_{p=0}^{\infty}\frac{(kn+1)_p}{p!}Y^{\alpha}_{n+p/k}(x,k)\beta^p. \tag{4.1}$$

Now putting $k=1$ in (4.1), we get a relation due to Mc Bride [6]

$$(1-\beta)^{-\alpha-n-1}\,exp\left[\frac{-\beta x}{1-\beta}\right]L_n^{\alpha}\left(\frac{x}{1-\beta}\right)=\sum_{p=0}^{\infty}\frac{(n+1)_p}{p!}.\,L^{\alpha}_{n+p}(x)\beta^p$$

provided that $Y_n^{\alpha}(x,1)=L_n^{\alpha}(x)$. (4.2)

Further, rearranging (3.12) in the form

$$\left(1+\frac{\beta y}{\delta}\right)^{-\alpha-kn-1}\left(1+\frac{\theta}{\lambda y}\right)^{kn}\exp\left[\frac{\beta xy}{\delta+\beta y}\right]Y_n^{\alpha}\left(\frac{xy}{(\lambda y+\theta)(\beta y+\delta)},k\right)$$

$$=\sum_{r=0}^{\infty}\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-\beta/\delta)^p}{p!}\frac{(-\theta\delta)^q}{q!}\frac{(-1)^r}{r!}\delta^{\alpha+1}\left(\frac{\delta}{\lambda}\right)^{kn}\left\{(kn+p-q)\log\delta^2+\frac{(\alpha+1)}{2}\log\delta^2\right\}^r.$$

$$.(kn+1)_p(-kn-p-\alpha)_q Y^{\alpha}_{n+(p-q)/k}(x,k)y^{p-q} \tag{4.3}$$

and then setting $\beta = 0$, $\delta=\lambda=1$ and $y=-1$ in (4.3) we find that

$$(1-\theta)^{kn}Y_n^{\alpha}\left(\frac{x}{1-\theta},k\right)=\sum_{q=0}^{\infty}\frac{(-kn-\alpha)_q}{q!}Y^{\alpha}_{n-p/k}(x,k)\theta^q,|\theta|<1. \tag{4.4}$$

Again, setting $k=1$ in (4.4), we find a result due to McBride [p. 35, eq. (1)] (see also Truesdell [p. 85, eq. (9)])

$$(1-\theta)^n L_n^{\alpha}\left(\frac{x}{1-\theta}\right)=\sum_{q=0}^{\infty}\frac{(-n-\alpha)_q}{q!}L^{\alpha}_{n-q}(x)\theta^q,|\theta|<1. \tag{4.5}$$

Other particular cases may also be obtained from the result (3.11). Due to lack of space we omit them.

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