Jñānābha, Vol. 31/32, 2002
(Dedicated to Professor H. M. Srivastava on his 62nd Birthday)

INTEGRALS INVOLVING MULTIPLE HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES THROUGH DIFFERENCE OPERATOR APPROACH

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(Received : December 29, 2002 ; final form: August 20, 2003)

ABSTRACT

In the present paper, making an appeal to difference operators, we evaluate certain integrals involving multiple hypergeometric functions of Chandel-Gupta [3], Exton [6,7] and Karlssoń [10]. We also apply same technique to evaluate integrals involving Appell's functions [1] and their confluent forms due to Humbert [8] and hypergeometric functions of four variables due to Sharma and Parihar [12,13].

1. Introduction. Recently, making an appeal to difference operator E_α defined by

(1.1) $E_\alpha f(\alpha) = f(\alpha+1), \qquad E_\alpha^m(f(\alpha)) = f(\alpha+m),$

and integral due to Erdélyi [5.p.224]

$$(1.2) \quad \int_{-\infty}^{\infty} \frac{\sin[(m+1)\pi x]dx}{\sin \pi x \Gamma(\alpha_1+x)\Gamma(\alpha_2-x)} = \frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}, \qquad Re\,(\alpha_1+\alpha_2)>1,$$

Joshi and Bhati [9] evaluated some integrals involving hypergeometric functions of three and four variables and discussed some special cases.

Very recently, making an appeal to difference operators Chandel [2] obtained various transformations of multiple hypergeometric functions of several variables due to Chandel-Gupta [3], Chandel-Vishwakarma [4] and discussed their interesting special cases.

In the present paper, making an appeal to difference operators, we evaluate certain interesting integrals involving multiple hypergeometric functions of several variables $F_A^{(n)}$, $F_C^{(n)}$ and confluent form $\psi_2^{(n)}$ of Lauricella [11], ${}^{(k)}F_{AC}^{(n)}$, ${}^{(k)}F_{AD}^{(n)}$ and confluent form ${}^{(k)}_{(1)}\phi_{AC}^{(n)}$ of Chandel-Gupta [3], ${}^{(k)}_{(1)}E_D^{(n)}$ of Exton [6,7], ${}^{(k)}F_{CD}^{(n)}$ of Karlsson [10]. As byproduct, we also evaluate integrals involving functions F_1, F_2 of two variables due to Appell [1] and their confluent form Ξ_2 due to Humbert [8].

In the last, we also apply difference operational technique to evaluate integral involving hypergeometric functions $F_{29}^{(4)}$ and $F_{58}^{(4)}$ of four variables introduced by Sharma and Parihar [12,13].

2. Applications to evaluate integrals. Multiplying both sides of (1.2) by

$\frac{\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}$ and operating it by the operator $exp\,[u_1 E_{\alpha_1}+\ldots+u_n E_{\alpha_n}]$

we have

$$\int_{-\infty}^{\infty} exp[u_1E_{\alpha_1}+\ldots+u_nE_{\alpha_n}]\frac{sin(2n'+1)\pi x}{sin\,\pi x\Gamma(\alpha_1+x)\Gamma(\alpha_2-x)}\frac{\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}dx$$

$$=exp[u_1E_{\alpha_1}+\ldots+u_nE_{\alpha_n}]\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\frac{\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}\right\}$$

from which making an appeal to (1.1), we finally derive

$$(2.1)\quad \int_{-\infty}^{\infty}\frac{sin(2n'+1)\pi x}{sin\,\pi x\Gamma(\alpha_1+x)\Gamma(\alpha_2-x)}\psi_2^{(n)}(\alpha_1+\ldots+\alpha_n;\alpha_1+x,\alpha_2-x,\alpha_3,..,\alpha_n;u_1,\ldots,u_n)dx$$

$$=\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\psi_2^{(n-1)}(\alpha_1+\ldots+\alpha_n;\alpha_1+\alpha_2-1,\alpha_3,..,\alpha_n;2(u_1+u_2),u_3,\ldots,u_n)$$

where $Re(\alpha_1+\alpha_2)>1, n'$ is integer and $\psi_2^{(n)}$ is confluent hypergeometric form of Lauricella's multiple hypergeometric function [11].

For brevity, we consider the integral operator

$$(2.2)\quad S\{\;\}=\frac{\Gamma(\alpha_1+\alpha_2-1)}{2^{\alpha_1+\alpha_2-2}}\int_{-\infty}^{\infty}\frac{sin(2n'+1)\pi x}{sin\,\pi x\Gamma(\alpha_1+x)\Gamma(\alpha_2-x)}\{\;\}dx$$

where n' is an integer and $Re(\alpha_1+\alpha_2)>1$.

Therefore,

$$(2.3)\quad S\{1\}=1,$$

and (2.1) can be written as

$$(2.4)\quad S\{\psi_2^{(n)}(\alpha_1+\ldots+\alpha_n;\alpha_1+x,\alpha_2-x,\alpha_3,..,\alpha_n;u_1,\ldots,u_n)\}$$

$$=\psi_2^{(n-1)}(\alpha_1+\ldots+\alpha_n;\alpha_1+\alpha_2-1,\alpha_3,..,\alpha_n;2(u_1+u_2),u_3,\ldots,u_n),\ Re(\alpha_1+\alpha_2)>1.$$

Considering

$$(1-u_1E_{\alpha_1})^{-\beta_1}\ldots(1-u_nE_{\alpha_n})^{-\beta_n}S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\right\}\frac{\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)},$$

we derive

$$(2.5)\quad S\{F_A^{(n)}(\alpha_1+\ldots+\alpha_n,\beta_1+\ldots+\beta_n;\alpha_1+x,\alpha_2-x,\alpha_3,..,\alpha_n;u_1,\ldots,u_n)\}$$

$$={}^{(2)}F_{AD}^{(n)}(\alpha_1+\ldots+\alpha_n,\beta_1+\ldots+\beta_n;\alpha_1+\alpha_2-1,\alpha_3,..,\alpha_n;2u_1,2u_2,u_3,\ldots,u_n),$$

$$Re(\alpha_1+\alpha_2)>1 \quad Re(\alpha_i)>0, i=3,\ldots,n,\ |u_1|+\ldots+|u_n|<1,$$

$F_A^{(n)}$ is Lauricella's multiple hypergeometric function [11] and ${}^{(2)}F_{AD}^{(n)}$ is Intermediate Lauricella's multiple hypergeometric function due to Chandel-Gupta [3] for $k=2$.

Similarly, Considering

$$\left(1-u_1E_{\alpha_1}-u_2E_{\alpha_2}\right)^{-\beta}\left(1-u_3E_{\alpha_3}\right)^{-\beta_3}\ldots\left(1-u_nE_{\alpha_n}\right)^{-\beta_n}S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\frac{\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}\right\},$$

we derive

(2.6) $$S\left\{{}^{(2)}F_{AC}^{(n)}(\alpha_1+\ldots+\alpha_n,\beta,\beta_3\ldots\beta_n;\alpha_1+x,\alpha_2-x,\alpha_3,..,\alpha_n;u_1,\ldots,u_n)\right\}$$

$$=F_A^{(n-1)}(\alpha_1+\alpha_2+\ldots+\alpha_n,\beta,\beta_3+\ldots+\beta_n;\alpha_1+\alpha_2-1,\alpha_3,..,\alpha_n;u_1+u_2,u_3,\ldots,u_n),$$

$Re(\alpha_1+\alpha_2)>1$, $Re(\alpha_i)>0, i=3,\ldots,n,;|u_1+u_2|+|u_3|+\ldots+|u_n|<1$, and ${}^{(2)}F_{AC}^{(n)}$ is intermediate Lauricella's multiple hypergeometric function of Chandel-Gupta [3] for $k=2$.

Further considering

$$\left(1-u_1E_{\alpha_1}\right)^{-\beta_1}\left(1-u_2E_{\alpha_2}\right)^{-\beta_2}{}_0F_1\left[-;\beta;u_3E_{\alpha_3}+\ldots+u_nE_{\alpha_n}\right]$$

$$S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\Gamma(\alpha_1+\ldots+\alpha_n)\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)\right\},$$

we obtain

(2.7) $$S\left\{{}^{(n-2)}F_{AD}^{(n)}(\alpha_1+\ldots+\alpha_n,\alpha_3,\ldots,\alpha_n,\beta_1,\beta_2,\beta;\alpha_1+x,\alpha_2-x;u_3,\ldots,u_{n-2},u_1,u_2)\right\}$$

$$={}^{(2)}_{(1)}F_D^{(n)}(\alpha_1+\ldots+\alpha_n,\beta_1,\beta_2,\alpha_3+\ldots+\alpha_n;\alpha_1+\alpha_2-1,\beta;2u_1,2u_2,u_3,\ldots,u_n),$$

$Re(\alpha_1+\alpha_2)>1$, $Re(\alpha_i)>0, i=3,\ldots,n,$ and ${}^{(2)}_{(1)}E_D^{(n)}$ is multiple hypergeometric function of Exton [6,7] for $k=2$.

If we consider

$$exp\left(u_1E_{\alpha_1}+u_2E_{\alpha_2}\right)\left(1-u_3E_{\alpha_3}\right)^{-\beta_3}\ldots\left(1-u_nE_{\alpha_n}\right)^{-\beta_n}$$

$$S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\frac{\Gamma(\alpha_1+\ldots+\alpha_n)\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}\right\},$$

we again derive (2.5) specially for $\beta_1 = \alpha_1,\ \beta_2 = \alpha_2$.
Further considering

$$\left[1-\left(u_1 E_{\alpha_1}+\ldots+u_n E_{\alpha_n}\right)\right]^{-\alpha}\left[1-\left(u_{k+1} E_{\alpha_{k+1}} E_{\beta_{k+1}}+\ldots+u_n E_{\alpha_n} E_{\beta_n}\right)\right]^{-\beta}$$

$$S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\,\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)\right\},$$

and choosing $k=2$, we finally establish

(2.8) $$S\left\{F_2\left(\alpha,\alpha_1,\alpha_2;\alpha_1+x,\alpha_2-x;u_1,u_2\right)\right\}=F_1\left(\alpha,\alpha_1,\alpha_2;\alpha_1+\alpha_2-1;2u_1,2u_2\right),$$

$Re(\alpha_1+\alpha_2)>1$ and F_1, F_2 are Appell's hypergeometric functions of two variables [1], $max\ (|u_1|,|u_2|)<1/2$,
which has also been obtained by Joshi and Bhati [9, (3.1)] using other operators.

If we consider

$$\left[1-\left(u_1 E_{\alpha_1}+\ldots+u_n E_{\alpha_n}\right)\right]^{-\beta} S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\,\frac{\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}\right\},$$

we obtain

(2.9) $$S\left\{F_C^{(n)}\left(\alpha_1+\ldots+\alpha_n,\beta;\alpha_1+x,\alpha_2-x,\alpha_3,\ldots,\alpha_n;u_1,\ldots,u_n\right)\right\}$$

$$=F_C^{(n-1)}\left(\alpha_1+\ldots+\alpha_n,\beta;\alpha_1+\alpha_2-1,\alpha_3,\ldots,\alpha_n;2(u_1+u_2),u_3,\ldots,u_n\right),$$

$Re(\alpha_1+\alpha_2)>1$, $Re(\alpha_i)>0, i=3,\ldots,n$, $|u_1|^{1/2}+\ldots+|u_n|^{1/2}<1$ and $F_C^{(n)}$ is Lauricella's multiple hypergeometric function of several variables [11].
Further Considering

$$\left(1-u_1 E_{\alpha_1}\right)^{-\alpha_1}\ldots\left(1-u_n E_{\alpha_n}\right)^{-\alpha_n} S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\,\frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)}{\Gamma(b_1+\alpha_3)\ldots\Gamma(b_n+\alpha_n)}\right\},$$

we derive

(2.10) $$S\left\{{}_2F_1\left(a_1,\alpha_1;\alpha_1+x;u_1\right).{}_2F_1\left(a_2,\alpha_2;\alpha_2-x,u_2\right)\right\}$$

$$=F_3\left(a_1,a_2,\alpha_1,\alpha_2;\alpha_1+\alpha_2;2u_1,2u_2\right),$$

$Re(\alpha_1+\alpha_2)>1$, $max\ (|u_1|,\ |u_2|)<1/2$ and F_3 is Appell's function of two variables [1],
which also suggests that

(2.11) $$S\left\{{}_1F_1\left(\alpha_1;\alpha_1+x;u_1\right).{}_1F_1\left(\alpha_2;\alpha_2-x;u_2\right)\right\}$$

$$= \Xi_2(\alpha_1, \alpha_2; \alpha_1 + \alpha_2; 2u_1, 2u_2),$$

$Re(\alpha_1 + \alpha_2) > 1$, $|u_1| < 1/2, |u_2| < \infty$ and Ξ_2 is confluent form due to Humbert [8] of Appell's function [1].

Again Considering

$$\left[1 - \left(u_1 E_{\alpha_1} + \ldots + u_n E_{\alpha_n}\right)\right]^{-\alpha} S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1 + \alpha_2 - 1)} \frac{1}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}\right\},$$

we arrive at

$$(2.12) \quad S\left\{\psi_2^{(n)}(\alpha; \alpha_1 + x, \alpha_2 - x, \alpha_3, \ldots, \alpha_n; u_1, u_2, \ldots, u_n)\right\}$$

$$= \psi_2^{(n-1)}(\alpha; \alpha_1 + \alpha_2 - 1, \alpha_3, \ldots, \alpha_n; 2(u_1 + u_2), u_3, \ldots, u_n),$$

$Re(\alpha_1 + \alpha_2) > 1$ and $\psi_2^{(n)}$ is confluent hypergeometic form of Lauricella's multiple hypergeometric function [11].

Specially for $\alpha = \alpha_1 + \ldots + \alpha_n$, (2.12) reduces to (2.4).

Further considering

$$\left(1 - u_{k+1} E_{\alpha_{k+1}}\right)^{-b_{k+1}} \ldots \left(1 - u_n E_{\alpha_n}\right)^{-b_n} \left[1 - \left(u_1 E_{\alpha_1} + \ldots + u_n E_{\alpha_n}\right)\right]^{-a}$$

$$S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1 + \alpha_2 - 1)} \frac{\Gamma(\alpha_1 + \alpha_2 + \ldots + \alpha_n)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}\right\},$$

we derive

$$(2.13) \quad S\left\{{}^{(k)}_{(1)}\phi^{(n)}_{AC}(\alpha; \alpha_1 + \ldots + \alpha_k; \alpha_1 + x, \alpha_2 - x, \alpha_3, \ldots, \alpha_n; u_1, \ldots, u_n)\right\}.$$

$$= {}^{(k)}_{(1)}\phi^{(n)}_{AC}(\alpha; \alpha_1 + \ldots + \alpha_k; \alpha_1 + \alpha_2 - 1, \alpha_3, \ldots, \alpha_n; 2(u_1 + u_2), u_3, \ldots, u_n),$$

$Re(\alpha_1 + \alpha_2) > 1$, $Re(\alpha_i) > 0, i = 3, \ldots, n$, and ${}^{(k)}_{(1)}\phi^{(n)}_{AC}$ is confluent form of intermediate Lauricella's function due to Chandel-Gupta [3].

If we consider

$$\left(1 - u_1 E_{\alpha_1}\right)^{-b_1} \ldots \left(1 - u_k E_{\alpha_k}\right)^{-b_k} \left[1 - \left(u_{k+1} E_{\alpha_{k+1}} + \ldots + u_n E_{\alpha_n}\right)\right]^{-a}$$

$$S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1 + \alpha_2 - 1)} \frac{\Gamma(\alpha_1 + \alpha_2 + \ldots + \alpha_n)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}\right\},$$

we arrive at for $k=2$

(2.14) $S\left\{ {}^{(n-2)}F_{AC}^{(n)}(\alpha_1+\ldots+\alpha_n, a, b_1, b_2; \alpha_3,\ldots,\alpha_n, \alpha_1+x, \alpha_2-x; u_3,\ldots,u_n,u_1,u_2)\right\}$

$= {}^{(2)}F_{CD}^{(n)}(\alpha_1+\ldots+\alpha_n, a, b_1, b_2; \alpha_1+\alpha_2-1, \alpha_3,\ldots,\alpha_n; 2u_1, 2u_2, u_3,\ldots,u_n)$,

$Re(\alpha_1+\alpha_2)>1$, $Re(\alpha_i)>0, i=3,\ldots,n;$ ${}^{(k)}F_{AC}^{(n)}$ is intermediate Lauricella multiple hypergeometric fucntion due to Chandel Gupta [3] while ${}^{(k)}F_{CD}^{(n)}$ is intermediate Lauricella multiple hypergeometric function due to Karlsson [10].

Further Considering

$$\left[1-\left(u_1E_{\alpha_1}+\ldots+u_kE_{\alpha_k}\right)\right]^{-a}\left(1-u_{k+1}E_{\alpha_{k+1}}\right)^{-\alpha_{k+1}}\ldots\left(1-u_nE_{\alpha_n}\right)^{-\alpha_n}$$

$$S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)}{\Gamma(\alpha_3+\ldots+\alpha_n)}\right\},$$

we finally derive for $k=2$

(2.15) $S\left\{ {}^{(n-2)}F_{CD}^{(n)}(\alpha_1+\ldots+\alpha_n, a, a_{n-1}, a_n; \alpha_3,\ldots,\alpha_n, \alpha_1+x, \alpha_2-x; u_{n-1}, u_n, u_1\ldots,u_{n-2})\right\}$

$= {}^{(n-2)}F_{CD}^{(n-1)}(\alpha_1+\alpha_2+\ldots+\alpha_n, a, a_{n-1}, a_n; \alpha_3,\ldots,\alpha_n, \alpha_1+\alpha_2-1; u_3,\ldots,u_n, 2(u_1+u_2))$,

$Re(\alpha_1+\alpha_2)>1$, $Re(\alpha_3+\ldots+\alpha_n)>0$ and ${}^{(k)}F_{CD}^{(n)}$ is intermediate Lauricella's function due to Karlsson [10].

If we consider

$$exp\left(u_1E_{\alpha_3}E_{\alpha'_3}+u_2E_{\alpha_1}+u_3E_{\alpha_2}+u_4E_{\alpha_4}E_{\alpha'_4}\right)$$

$$S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\frac{\Gamma(\alpha_1+\alpha_3)\Gamma(\alpha_2+\alpha_4)\Gamma(\alpha_2+\alpha_3)\Gamma(\alpha_1+\alpha_4)}{\Gamma(\alpha'_3+\alpha'_4)}\right\},$$

we derive

(2.16) $S\{F_{29}^{(4)}(\alpha_1+\alpha_3, \alpha_1+\alpha_3, \alpha_2+\alpha_4, \alpha_2+\alpha_4, \alpha_2+\alpha_3, \alpha_1+\alpha_4, \alpha_2+\alpha_3, \alpha_1+\alpha_4;$

$\alpha'_3+\alpha'_4, \alpha_1+x, \alpha_2-x, \alpha'_3+\alpha'_4; u_1, u_2, u_3, u_4)\}$

$= F_{58}^{(4)}(\alpha_1+\alpha_3, \alpha_1+\alpha_3, \alpha_2+\alpha_4, \alpha_2+\alpha_4, \alpha_2+\alpha_3, \alpha_1+\alpha_4, \alpha_2+\alpha_3, \alpha_1+\alpha_4;$

$\alpha'_3+\alpha'_4, \alpha_1+\alpha_2-1, \alpha_1+\alpha_2-1, \alpha'_3+\alpha'_4; u_1, 2u_2, 2u_3, u_4)$,

$Re(\alpha_1+\alpha_2)>0$, $Re(\alpha_i)>0, i=3,4$ and $F_{29}^{(4)}, F_{58}^{(4)}$ are hypergeometric functions of four variables defined by Sharma and Parihar [12].

Making similar difference operational approach, several other interesting integrals involving multiple hypergeometric functions of different variables can be evaluated.

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