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(Dedicated to Professor H. M. Srivastava on his 62nd Birthday)

ANALYSIS OF BULK QUEUEING MODEL $M/G^K/2$ FOR NON-IDENTICAL SERVERS WITH A GRAND VACATION POLICY

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ABSTRACT

In this paper, we consider a bulk queueing model $M/G^k/2$ for non-identical servers with batch service and a grand vacation process. Previous works did not consider the concept of non-identical servers with fixed batch service alongwith a grand vacation queueing medel. In this chapter, we have discussed the behaviour of the queuing model $M/G^k/2$ as some important performance measures which include staionary queue size distribution, departure point queue size distribution, queue size distribution at the stationary point of vacation period and joint distribution of the queue size and a vacation period. The study of this type of the models are very useful in various practical situations such as Computers, FMS and Inventory, manufacturing systems etc.

1. Introduction. Now-a-days, in many real-life queueing situations, service is rendered in batches. Apart from interest, from the point of view of theoretical structures. bulk queues have been used for modelling in various situations. In this sequence of works, Bailey [1] considered that units are served in groups of size less than the fixed capacity of the server. Borthakur [2] has studied bulk queueing models and obtained a closed form expression for the busy period of the system. Gohain and Borthakur [9] have discussed difference equation technique in a comprehensive manner which could easily be applied on bulk service queues for analysing variuos performance measures such as queue length and waiting time distribution etc. Waiting time distribution for Poisson bulk queues has been analysed by Medhi [20].

Kella [14] has studied the optimal control of the vacation scheme in the *M/G/1* queue whereas Levy and Yechiali [17] discussed about the utilization of the idle time in the *M/G/1* queueing system. In connection with the vacation policy, Minh [21] has obtained the transient solutions for some exhaustive *M/G/1* queues with generalised independent vacations. Also, Fuhrman and Cooper [8] have studied the stochastic decompositions in the *M/G/1* queue with generalised vacations.

The *M/G/1* queueing system with vacations, multiple as well as single, under a threshold policy was considered by Loris-Teghem [18], where the first customer in each busy period receives an exceptional service. Kella [13] set a milestone in the theory of vacation queues by studying the threshold policy in the $M/G/1$ queue with server vacation. Recently. Lee et. al. [16] have extended this $M/G/1$ queue for the more general class of threshold policy for a batch arrival queue with multiple vacations. In this approach, they introduced the notion of a grand vacation process. Also, they have demonstrated, by an ingenious combination of supplimentary variable technique and semi-Markov technique, as how to solve problems in a broad class of stochastic processes including queueing process. Recently, Singh et. al. [23] have obtained the transient and stationary solutions for $M/G/s$ queueing system with vacationing.

Very recently, Chaudhury and Baruah [7] have studied the $M/M/1$ queueing model with a threshold policy and a grand vacation process, where the server takes a sequence of vacations till he returns to find at least some prespecified number of customers observed after each grand vacation. Using an analytic approach, they have obtained the queue size distributions at stationary as well as departure point of time, queue size distribution at the stationary point of vacation period including some other important performance measures. Thus, the series of works did not concentrate on multiple non-identical servers with fixed bulk services. The queueing models of this type have wide ranging potential for applications in various fields such as computers manufacturing. *FMS* and inventory systems etc. where service capacities of different servers are not identical to each other. In such practical situations sometimes, demands also arise for fixed batch services in view of vacations to be taken by servers.

In this paper, a fresh attempt has been made to consider the bulk queueing moderl $M/G^k/2$ for non-identical servers with fixed bulk service and a grand vacation process. By considering two non-identical servers, various operating characteristics including stationary queue size distribution, departure point queue size distribution, queue size distribution at the stationary point of vacation period and joint distribution of the queue size and a vacation period have been discussed in the present paper.

2. Stationary Queue Size Distribution. Let $N_Q(t)$ be the number of customers present in the system at time t. We further assume that the arrival of customer in the system follows Poisson process with constant rate λ. The arriving customers are served by two non-identical servers in batches of fixed size, say, K. The two non-identical servers have their different service capacities, say, μ_1 and μ_2 respectively which are independently and non-identically distributed and μ be the average service rate of μ_1 and μ_2.

Under the bulk service discipline, the batches are of fixed size k taken up for the service by two non-identical servers. If after the completion of a service, the servers find customer less than k, then they will go on vacation for random duration and after vacation, if they finds k or more than k customers waiting, they takes a batch of k customers in order of their arrival for service while others wait. This generates a sequence of vacation periods which are independently and non-identically distributed random variables, say, V_1, V_2, V_3, ..., respectively. Thus,

$V_i = \{V_1, V_2, ..., V_{m1}\}$ is a generic r.v. of the sequence of vacation periods for the first server and $i = 1, 2, 3, ..m_1$, and

$V_j = \{V_1, V_2, ..., V_{m1}\}$ is a generic r.v. of the sequence of vacation periods for the second server and $j = 1,2,3,...,m_2$.

Hence, V_i and V_j are considered as grand vacations comprises of m_1 and m_2 vacations respectively and the probabilities of k or more arrival during the m_1-th and m_2-th vacations; $m_1, m_2 = 1,2,3,...$, are given by

$$p_{m_1} = \int_0^{\infty} \frac{e^{-\lambda x}(\lambda x)^{m1}}{m_1!} dV_i(x); \quad m_1 = 0,1,2,... \qquad ...(1)$$

$$p_{m_2} = \int_0^{\infty} \frac{e^{-\lambda x}(\lambda x)^{m2}}{m_2!} dV_j(x); \quad m_2 = 0,1,2,... \qquad ...(2)$$

where $V_i(x)$ and $V_j(x)$ are the distribution functions of V_i and V_j respectively. Now, we introduce an indicator function

$$I(t) = \begin{cases} 1, \text{ If either one or both servers are busy} \\ 0, \text{ otherwise.} \end{cases} \qquad ...(3)$$

Clearly, $\{N_Q(t), I(t)\}$ is a continuous time Markov process, where $V_i^0(t)$ and $V_j^0(t)$ are the elapsed times at time t.

For n customers in the system with vacation periods, the system may have anyone of the following states as-

$S_1 = \{n, 0, 1\}$, $S_2 = \{n, 1, 0\}$, $S_3 = \{n, 1, 1\}$, and $S_4 = \{n, 0, 0\}$,

These four states will also be formed when there are no customers in the system ($n=0$). Here, 1 stands for busy state whereas 0 for vacation state in S_1, S_2, S_3 and S_4 defined above.

Now, for two non-identical servers, we will define the following probabilities under the steady state conditions as-

$P_{n,1,1}$ =Probability that there are n customers present in the system when both servers are busy.

$P_{n,0,1}$ =Probability that there are n customers present in the system when first server is on vacation while another busy.

$P_{n,1,0}$ =Probability that there are n customers present in the system when first server is busy while another on vacation.

and

$P_{n,0,0}$ =Probability that there are n customers present in the system when both servers are on vacation.

Let

$$P_{n,0,1}(x)dx = \lim_{t\to\infty} Pr\left[N_Q(t) = n, x < V_i^0(t) \le x + dx; I(t) = 1\right], \quad x > 0, n \ge 0 \quad \ldots(4)$$

$$P_{n,1,0}(x)dx = \lim_{t\to\infty} Pr\left[N_Q(t) = n, x < V_j^0(t) \le x + dx; I(t) = 1\right], \quad x > 0, n \ge 0 \quad \ldots(5)$$

$$P_{n,1,1}(x)dx = \lim_{t\to\infty} Pr\left[N_Q(t) = n; I(t) = 1\right], \quad x > 0, n \ge 0 \quad \ldots(6)$$

and

$$P_{n,0,0}(x)dx = \lim_{t\to\infty} Pr\left[N_Q(t) = n, x < V_i^0(t) \le x + dx, x < V_j^0(t) \le x + dx; I(t) = 0\right]$$

$$x > 0, n \ge 0 \quad \ldots(7)$$

$$Q_{n,0,1} = \lim_{t\to\infty} Pr\left[N_Q(t) = n; I(t) = 1\right], n \ge 0 \quad \ldots(8)$$

$$Q_{n,1,0} = \lim_{t\to\infty} Pr\left[N_Q(t) = n; I(t) = 1\right], n \ge 0 \quad \ldots(9)$$

$$Q_{n,1,1} = \lim_{t\to\infty} Pr\left[N_Q(t) = n; I(t) = 1\right], n \ge 0 \quad \ldots(10)$$

and

$$Q_{n,0,0} = \lim_{t\to\infty} Pr\left[N_Q(t) = n; I(t) = 1\right], n \ge 0 \quad \ldots(11)$$

Under the steady-state, we also have,

$$P_{n,0,1} = \int_0^\infty P_{n,0,1}(x)dx \quad \ldots(12)$$

$$P_{n,1,0} = \int_0^\infty P_{n,1,0}(x)dx \quad \ldots(13)$$

$$P_{n,1,1} = \int_0^\infty P_{n,1,1}(x)dx \quad \ldots(14)$$

and

$$P_{n,0,0} = \int_0^\infty P_{n,0,0}(x)dx \quad \ldots(15)$$

Further, let us introduce the hazard rate function for first server,

$$\eta_1(x)dx = \frac{dV_i(x)}{1 - V_i(x)} \quad \ldots(16)$$

and for second server

$$\eta_2(x)dx = \frac{dV_j(x)}{1 - V_j(x)} \quad \ldots(17)$$

as the probability density functions for vacation times V_i and V_j respectively under the condition that V_i, $V_j > x$, *i.e.*

$$V_i(x) = \frac{d}{dx} V_j(x) = \eta_1(x) exp\left(-\int_0^x \eta_1(y)dy\right) \quad ...(18)$$

$$V_j(x) = \frac{d}{dx} V_j(x) = \eta_2(x) exp\left(-\int_0^x \eta_2(y)dy\right) \quad ...(19)$$

It is also assumed that $V_i(0)=V_j(0)=0$ and $V_i(\infty)=V_j(\infty)=1$ and $V_i(x), V_j(x)$ both are continuous at $x=0$.

Now, from defined probabilities and in the light of equations (16), (17), (18) and (19), we have the following state equations obtained for the load factor $\rho = \lambda/k\mu < 1$

$$\frac{d}{dx} P_{n,0,1}(x) + [\lambda + \eta_1(x)] P_{n,0,1}(x) = \lambda P_{n-1,0,1}(x)dx, \quad x > 0,\ n \geq 0 \quad ...(20)$$

$$\frac{d}{dx} P_{n,1,0}(x) + [\lambda + \eta_2(x)] P_{n,1,0}(x) = \lambda P_{n-1,1,0}(x)dx, \quad x > 0,\ n \geq 0 \quad ...(21)$$

$$\frac{d}{dx} P_{n,1,1}(x) + \lambda P_{n,1,1}(x) = \lambda P_{n-1,1,1}(x)dx, \quad x > 0,\ n \geq 0 \quad ...(22)$$

$$\frac{d}{dx} P_{n,0,0}(x) + \lambda P_{n,0,0}(x) = \lambda P_{n-1,0,0}(x)dx, \quad x > 0,\ n \geq 0 \quad ...(23)$$

and

$$(\lambda + k\mu_1) Q_{n,0,1} + \lambda P_{n+1} P_{0,0,1} = \lambda Q_{n-1,0,1} + k\mu_1 Q_{n-1,0,1} + \int_0^\infty \eta_1(x) P_{n-1,0,1}(x)dx$$
$$n = 0,1,2,...,m_1 \quad ...(24)$$

$$(\lambda + k\mu_2) Q_{n,1,0} + \lambda P_{n+1} P_{0,1,0} = \lambda Q_{n-1,1,0} + k\mu_2 Q_{n+1,1,0} + \int_0^\infty \eta_2(x) P_{n+1,1,0}(x)dx,$$
$$n = 0,1,2,...,m_2 \quad ...(25)$$

$$(\lambda + k\mu) Q_{n,1,1} + \lambda P_{n+1} P_{0,1,1} = \lambda Q_{n-1,1,1} + k\mu Q_{n+1,1,1} + \int_0^\infty P_{n+1,1,1}(x)dx,$$
$$n = 0,1,2,...,m_1 + m_2 \quad ...(26)$$

$$\lambda Q_{n,0,0} + \lambda P_{n+1} P_{0,0,0} = \lambda Q_{n-1,0,0} + \int_0^\infty P_{n+1,0,0}(x)dx \quad n = 0,1,2,...,m_1 + m_2 \quad ...(27)$$

and

$$(\lambda + k\mu_1) Q_{n,0,1} = \lambda Q_{n-1,0,1} + k\mu_1 Q_{n+1,0,1} + \int_0^\infty \eta_1(x) P_{n+1,0,1}(x)dx, \quad n \geq m_1 \quad ...(28)$$

$$(\lambda + k\mu_2) Q_{n,1,0} = \lambda Q_{n-1,1,0} + k\mu_2 Q_{n+1,1,0} + \int_0^\infty \eta_2(x) P_{n+1,1,0}(x)dx, \quad n \geq m_2 \quad ...(29)$$

$$(\lambda + k\mu) Q_{n,1,1} = \lambda Q_{n-1,1,1} + k\mu Q_{n+1,1,1} + \int_0^\infty P_{n,1,1}(x)dx, \quad n \geq (m_1 + m_2) \quad ...(30)$$

$$\lambda Q_{n,0,0} = \lambda Q_{n-1,0,0} + \int_0^\infty P_{n+1,0,0}(x)dx, \quad n \geq (m_1 + m_2) \qquad ...(31)$$

and

$$\lambda p_0 P_{0,0,1} = k\mu_1 Q_{0,0,1} + \int_0^\infty \eta_1(x) P_{0,0,1}(x)dx \qquad ...(32)$$

$$\lambda p_0 P_{0,1,0} = k\mu_2 Q_{0,1,0} + \int_0^\infty \eta_2(x) P_{0,1,0}(x)dx \qquad ...(33)$$

$$\lambda p_0 P_{0,1,1} = k\mu Q_{0,1,0} + \int_0^\infty P_{0,1,1}(x)dx \qquad ...(34)$$

$$\lambda p_0 P_{0,0,0} = \int_0^\infty P_{0,0,0}(x)dx \qquad ...(35)$$

where $P_{0,0,0}(x) = Q_{0,0,0}(x) = 0$.

These equations are to be solved by using the following boundary conditions at $x=0$;

$$P_{n,l,m}(0) = \begin{cases} = \lambda\, p_n P_{0,l,m}; n = 0,1,2,\ldots \\ = 0 \end{cases} \qquad ...(36)$$

and the normalizing condition

$$\sum_{n=0}^{\infty} \int_0^\infty P_{n,l,m}(x)dx + \sum_{n=0}^{\infty} Q_{n,l,m} = 1 \qquad ...(37)$$

where l and m having the value either 0 or 1 depending on the state of the system.

Let us now consider the following probability generating function ($PGFS$)-

$$P(x; z_k) = \sum_{n=0}^{\infty} z_k^n P_{n,l,m} \qquad (: |z_k| \leq 1:) \qquad ...(38)$$

$$P(0; z_k) = \sum_{n=0}^{\infty} z_k^n P_{n,l,m} \qquad (: |z_k| \leq 1:); \quad k = 1,2,3,4\,. \qquad ...(39)$$

After some algebraic manipulations with equations (20)-(37), we have

$$P(x; z_1) = \lambda P_{0,0,1} G(z_1)[1 - V_i(x)]e^{-\lambda(1-z_1)x}; x > 0 \qquad ...(40)$$

$$P(x; z_2) = \lambda P_{0,1,0} G(z_2)[1 - V_j(x)]e^{-\lambda(1-z_2)x}; x > 0 \qquad ...(41)$$

$$P(x; z_3) = \lambda P_{0,1,1} G(z_3) e^{-\lambda(1-z_3)x}; x > 0 \qquad ...(42)$$

$$P(x; z_4) = \lambda P_{0,0,0} G(z_4)[1 - (V_i(x) + V_j(x))]e^{-\lambda(1-z_4)x}; x > 0 \qquad ...(43)$$

and

$$Q(z_1) = \frac{\rho P_{0,0,1} G(z_1)\left[1 - V_i^*(\lambda - \lambda z_1)\right]}{(1 - z_1)(1 - \rho z_1)} \qquad \text{...(44)}$$

$$Q(z_2) = \frac{\rho P_{0,1,0} G(z_2)\left[1 - V_j^*(\lambda - \lambda z_2)\right]}{(1 - z_2)(1 - \rho z_2)} \qquad \text{...(45)}$$

$$Q(z_3) = \frac{\rho P_{0,1,1} G(z_3)}{(1 - z_3)(1 - \rho z_3)} \qquad \text{...(46)}$$

$$Q(z_4) = \frac{\rho P_{0,0,0} G(z_4)\left[1 - \left(V_i^*(\lambda - \lambda z_4) + V_j^*(\lambda - \lambda z_4)\right)\right]}{(1 - z_4)(1 - \rho z_4)} \qquad \text{...(47)}$$

where $G(z_k) = \sum_{r=0}^{k} z_k^r g_r \qquad (:|z_k| \le 1:); \; k=1,2,3,4$ ...(48)

and $V_i^*(\lambda - \lambda z_k)$ and $V_j^*(\lambda - \lambda z_k)$ are the transforms of V_i and V_j respectively.

Thus, we have

$$P(z_1) = \int_0^\infty P(x; z_1)dx = P_{0,0,1} G(z_1)\left[1 - V_i^*(\lambda - \lambda z_1)\right]/(1 - z_1) \qquad \text{...(49)}$$

$$P(z_2) = P_{0,1,0} G(z_2)\left[1 - V_j^*(\lambda - \lambda z_2)\right]/(1 - z_2) \qquad \text{...(50)}$$

$$P(z_3) = P_{0,1,1} G(z_3)/(1 - z_3) \qquad \text{...(51)}$$

$$P(z_4) = P_{0,0,0} G(z_4)\left[1 - \left(V_i^*(\lambda - \lambda z_4) + V_j^*(\lambda - \lambda z_4)\right)\right]/(1 - z_4) \qquad \text{...(52)}$$

Let $\psi(z_k) = P(z_k) + Q(z_k)$ be the *PGF* of the stationary queue size distribution for $k=1,2,3,4$. Then

$$\psi(z_k) = \frac{P_{0,1,m} G(z_k)\left[1 + (1 - z_k)\rho\right]\left[1 - V_s^*(\lambda - \lambda z_k)\right]}{(1 - z_k)(1 - \rho z_k)} \qquad \text{...(53)}$$

where $k=1,2,3$ and 4; V_s^* is either V_i^*, V_j^* or both and $\rho = \lambda / k\mu$

Using the conditions of (37) and $\lim_{z_k \to 1} \psi(z_k) = 1$ we get,

$$P_{0,1,m} = \frac{(1 - \rho)}{\lambda E(V_s) E(G)} \qquad \text{...(54)}$$

where $E(G)=\left(\sum_{r=0}^{k} g_r\right)$ is the expected number of grand vacations during a vacation period.

Thus, the PGF of the stationary queue size distribution becomes as-

$$\psi(z_k)=\left[\frac{(1-\rho)[1+(1-z_k)\rho]}{(1-\rho z_k)}\right]\left[\frac{1-V_s^*(\lambda-\lambda z_k)}{E(V_s)(\lambda-\lambda z_k)}\right]\left[\frac{G(z_k)}{E(G)}\right] \qquad \text{...(55)}$$

3. Departure Point Queue Size Distribution. We shall derive now PGF of the limiting queue size distribution at a departure point of time. In two recent papers Chaudhury and Borthakur [3] and Chaudhury [4,5] have taken recourse to stochastic decomposition property to derive the PGF for the departure point queue size distribution of $M/M/1$ queues with multiple vacation as well as setup time. However, it is also possible to derive the PGF of the departure point queue size distribution for the queueing system $M/G^k/2$ without utilizing the stochastic decomposition property. Hence following the arguments of $PASTA$ [e.g. see Wolf [25]] we state that a batch of k departing customers will see r customers in the queue just after a batch departure if and only if there were r customers in the queue at the departure point of time. Thus, we may have,

$$\pi_r = C_0 Q_r, \quad r=0,1,2,3,\ldots$$

where $\pi_r = Prob$ [r customers in the queue just after a batch departure of fixed size k], and C_0 is a normalizing constant.

Let $\pi(z_k)$ be the PGF of $\{\pi_r;\ r=0,1,2,3\ldots\}$, then

$$\pi(z_k)=\frac{C_0 P_{0,1,m}\rho\left[1-V_s^*(\lambda-\lambda z_k)\right]G(z_k)}{(1-z_k)(1-\rho z_k)} \qquad \text{...(56)}$$

where V_s^* is either V_i^*, V_j^* or both.

Using the normalizing condition of (38) $\lim_{z_k\to 1}\pi(z_k)=1$; $k=1,2,3,4,$ we get

$$C_0=\frac{(1-\rho)}{\rho\lambda E(V_s)E(G)P_{0,1,m}},$$

and hence

$$\pi(z_k)=\frac{(1-\rho)\left[1-V_s^*(\lambda-\lambda z_k)\right]G(z_k)}{E(V_s)(1-\rho z_k)(\lambda-\lambda z_k)E(G)}. \qquad \text{...(57)}$$

4. Queue Size Distribution at the Stationary Point of Vacation Period. For the queueing system $M/G^k/2$, we shall obtain the PGF of the limiting queue size distribution at the stationary point of vacation period by using an appropriate limiting procedure. For this, we define,

R_r = *Prob* [r customers arrive in the queue at the stationary point of vacation periods]: $r \geq 0$

and $R(z_k)$ be the *PGF* of $\{R_r;\ r=0,1,2,\ldots\}$ then for fixed λ, in equation (58) we have,

$$\lim_{\mu_1 \to \infty} \pi(z_1) = \lim_{\mu_1 \to \infty} \psi(z_1) = \frac{\left[1 - V_i^*(\lambda - \lambda z_1)\right]G(z_1)}{E(V_i)(\lambda - \lambda z_1)E(G)} = R(z_1) \qquad \ldots(58)$$

$$\lim_{\mu_2 \to \infty} \pi(z_2) = \frac{\left[1 - V_j^*(\lambda - \lambda z_2)\right]G(z_2)}{E(V_j)(\lambda - \lambda z_2)E(G)} = R(z_2) \qquad \ldots(59)$$

$$\lim_{\mu_3 \to \infty} \pi(z_3) = \frac{G(z_3)}{(\lambda - \lambda z_3)E(G)} = R(z_3) \qquad \ldots(60)$$

$$R(z_4) = \frac{\left[1 - \left(V_i^*(\lambda - \lambda z_4) + V_j^*(\lambda - \lambda z_4)\right)\right]G(z_4)}{E(V_i + V_j)(\lambda - \lambda z_4)E(G)} \qquad \ldots(61)$$

5. Joint Distribution of the Queue Size And A Vacation Period. To obtain the *PGF* of the Laplace Stieltjes Transfrom (*LST*) of the limiting joint distribution of the queue size and a vacation period, we follow the arguments of Choudhury [6] [also see Takagi [24]] and define

$$P_r^*(u) = \int_0^\infty e^{-ux} P_r(x)dx, \quad r = 0,1,2,\ \ldots. \qquad \ldots(62)$$

Let $P^*(u; z_k)$ be the *PGF* of $\{P_r^*(u);\ r = 0,1,2,\ldots\}$, then

$$P^*(u; z_k) = \sum_{r=0}^{\infty} z_k^r P_r^*(u) \quad (\because |z_k| \leq 1) \text{ for } k=1,2,3,4. \qquad \ldots(63)$$

Thus, we have

$$P^*(u; z_1) = \frac{(1 - \lambda/k\mu_1)\left[1 - V_i^*(\lambda - \lambda z_1) + u\right]G(z_1)}{E(V_i)\left[\lambda(1 - z_1) + u\right]E(G)} \qquad \ldots(64)$$

$$P^*(u; z_2) = \frac{(1 - \lambda/k\mu_2)\left[1 - V_j^*(\lambda - \lambda z_2) + u\right]G(z_2)}{E(V_j)\left[\lambda(1 - z_2) + u\right]E(G)} \qquad \ldots(65)$$

$$P^*(u; z_3) = \frac{(1 - \lambda/k\mu)\left[1 + u\right]G(z_3)}{\left[\lambda(1 - z_3) + u\right]E(G)} \qquad \ldots(66)$$

and

$$P^*(u;z_4)=\frac{(1-\lambda/k\mu)\left[1-\left(V_i^*(\lambda-\lambda z_4)+V_j^*(\lambda-\lambda z_4)\right)+u\right]G(z_4)}{E\left(V_i+V_j\right)\left[\lambda(1-z_4)+u\right]E(G)} \qquad ...(67)$$

In particular, if we are taking *lim* $z_k \to 1$; $k=1,2,3,4$ in the above equations, then we have,

$$\lim_{z_1\to 1} P^*(u;z_1)=\frac{\left(1-\frac{\lambda}{k\mu_1}\right)\left[1-V_i^*(u)\right]}{uE(V_i)} \qquad ...(68)$$

$$\lim_{z_2\to 1} P^*(u;z_2)=(1-\lambda/k\mu_2)\left[1-V_j^*(u)\right]/\left[uE(V_j)\right] \qquad ...(69)$$

$$\lim_{z_3\to 1} P^*(u;z_3)=(1-\lambda/(k\mu))/u \qquad ...(70)$$

$$\lim_{z_4\to 1} P^*(u;z_4)=\frac{(1-\lambda/(k\mu))\left[1-\left(V_i^*(u)+V_j^*(u)\right)\right]}{uE\left(V_i+V_j\right)} \qquad ...(71)$$

which are the *LSTs* of the distribution functions of the vacation periods at stationary point of time.

In the similar way, the *PGFs* of the number of customers that arrive during a vacation period are, by taking limit $u\to 0$ in the above equations as-

$$\lim_{u\to 0} P^*(u;z_1)=\frac{(1-\lambda/(k\mu_1))\left[1-V_i^*(\lambda-\lambda z_1)\right]G(z_1)}{E(V_i)(\lambda-\lambda z_1)E(G)} \qquad ...(72)$$

$$\lim_{u\to 0} P^*(u;z_2)=\frac{(1-\lambda/(k\mu_2))\left[1-V_j^*(\lambda-\lambda z_2)\right]G(z_2)}{E\left(V_j\right)(\lambda-\lambda z_2)E(G)} \qquad ...(73)$$

$$\lim_{u\to 0} P^*(u;z_3)=\frac{(1-\lambda/(k\mu_2))G(z_3)}{(\lambda-\lambda z_3)E(G)} \qquad ...(74)$$

and

$$\lim_{u\to 0} P^*(u;z_4)=\frac{(1-\lambda/(k\mu))\left[1-\left(V_i^*(\lambda-\lambda z_4)+V_j^*(\lambda-\lambda z_4)\right)\right]G(z_4)}{E\left(V_i+V_j\right)(\lambda-\lambda z_4)E(G)} \qquad ...(75)$$

Hence the relationships between $\lim_{u\to 0} P^*(u;z_k)$ and $R(z_k)$; $k=1,2,3,4$ are given by

$$\lim_{u \to 0} P^{*}(u; z_k) = (1-\rho)R(z_k);$$

so that

$$Prob \text{ [the server(s) is on vacation]} = \lim_{\substack{z_k \to 1 \\ u \to 0}} P^{*}(u; z_k) = 1-\rho \quad ...(76)$$

6. Conclusion. In this paper, we have succeeded in considering a bulk queueing model with a grand vacation process for two non-identical servers with batch service. Here, we have discussed the behaviour of the queueing model $M/G^k/2$ including stationary queue size distribution, departure point queue size distribution at the stationary point of vacation period and joint distribution of the queue size and a vacation period. This type of queueing models are useful in the study of various problems arising out of *FMS*, inventory systems and computers etc.

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