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A COMMON FIXED POINT THEOREM FOR A SEQUENCE OF MAPPINGS BY ALTERING DISTANCES

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ABSTRACT

The main purpose of this paper is to obtain a unique common fixed point theorem for a sequence of mappings of a complete metric space by altering distance between the points.

1. Introduction. In recent years, the study of common fixed points for self mappings on a complete metric space by altering distances between the points with the use of control functions, has generated a wide interest and amny interesting theorems have been obtained by various authors. In this direction, Khan, Swaleh and Sessa [1] established fixed point theorem for a single self map. Sastry and Babu [6] proved fixed point theorem for a pair of self maps. Sastry *et al.* [7] proved a unique common fixed point theorem for four mappings by using a control function in order to alter distances between the points. Pant *et al.*[4,5] obtained an answer to the open problem of Sastry et al. [7] by establishing a connection between continuity and reciprocal continuity in the setting of control function.

In the present paper, we prove a common fixed point theorem for a sequence of self mappings in a complete metric space under control function. It is also notable that our result extends the corresponding result of Pant and Padaliya [3], by altering distance between points.

First we give the following definitions and notations : [see [2,3,7]].

(1) A control function ψ is defined as $\psi : \Re_+ \rightarrow \Re_+$ which is continuous at zero, monotonically increasing, $\psi(2t) \leq 2\, \psi(t)$ and $\psi(t)=0$ if and if $t=0$.

(2) Two self mappings A and S of a metric space (X, d) are called compatible if $lim_n d(ASx_n, SAx_n)=0$ whenever $\{x_n\}$ is a sequence such that $lim_n Ax_n = lim_n Sx_n = t$ for some t in X.

(3) Two self mappings A and S of a metric space (X,d) are called ψ - compatible

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if $lim_n \psi(d(ASx_n,SAx_n))=0$ whenever $\{x_n\}$ is a sequence such that $lim_n Ax_n=lim_n Sx_n=t$ for some t in X.

(4) Two self mappings A and S of a metric space (X, d) are said to be reciprocally continuous in X $lim_n ASx_n=At$ and $lim_n SAx_n=St$ whenever $\{x_n\}$ is a sequence such that $lim_n Ax_n=lim_n Sx_n=t$ for some t in X.

(5) If $\{A_i\}$, $i=1,2,3,\ldots,S$ and T are self mappings of a metric space (X,d) in the sequel for each $i>1$, and ψ is a control function on $\Re_+$, we shall denote

$$M_{\psi_i}(x,y)=max\{\psi(d(Sx,Ty)),\psi(d(A_ix,Sx)),\psi(d(A_iy,Ty))\}.$$

2. Results

Theorem. Let $\{A_i\}, i=1,2,3,\ldots,S$ and T are self mappings of a complete metric space (X, d) for some $i>1$, and ψ is a control function on $\Re_+$ as in (1) satisfying:

(i) $A_1X \subset TX$, $A_2X \subset SX$;

(ii) $\psi(d(A_1x,A_2y)) \le h\, M_{\psi_i}(x,y)$, $0 \le h<1$, and

(iii) $\psi(d(A_1x,A_iy)) \le M_{\psi_i}(x,y)$, whenever $M_{\psi_i}(x,y)>0$.

Suppose that $\{A_1,S\}$ and $\{A_2,T\}$ be $-\Psi$ compatible pair of reciprocally continuous mappings. Then all the A_i, S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X such that

(2.1) $y_{2n}=A_1x_{2n}=Tx_{2n+1}$; $y_{2n+1}=A_2x_{2n+1}=Sx_{2n+2}$.

This can be done by virtue of (i). We claim that $\{y_n\}$ is a Cauchy sequence.

We write $\alpha_n = \psi(d(y_n,y_{n+1}))$. Then, using condition (ii), it follows that

$$\begin{aligned}\alpha_{2n} &= \psi(d(y_{2n},y_{2n+1})) = \psi(d(A_1x_{2n},A_2x_{2n+1})) \le h(M_{\psi_2}(x_{2n},x_{2n+1}))\\ &= h\,max\{\psi(d(Sx_{2n},Tx_{2n+1})),\psi(d(A_1x_{2n},Sx_{2n})),\psi(d(A_2x_{2n+1},Tx_{2n+1}))\}\\ &= h\,max\{\psi(d(y_{2n-1},y_{2n})),\psi(d(y_{2n-1},y_{2n})),\psi(d(y_{2n+1},y_{2n}))\},\\ &\le h\,\psi(d(y_{2n-1},y_{2n}))= h\,\alpha_{2n-1}<\alpha_{2n-1}.\end{aligned}$$

that is,

(2.2) $\alpha_{2n}\le h\,\alpha_{2n-1}<\alpha_{2n-1}$.

Similarly, $\alpha_{2n-1}<h\,\alpha_{2n-2}$; $\alpha_{2n-2}<h\,\alpha_{2n-3}$ and so on. That is $\alpha_n\le h\,\alpha_{n-1}\le\ldots\le h^n\alpha_0$.

Moreover, for each integer $p>0$, we get

$$\begin{aligned}\psi(d(y_n,y_{n+p})) &\le \psi[d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+\ldots+d(y_{n+p-1},y_{n+p})]\\ &\le \psi[(1+h+\ldots+h^{p-1})h^n d(y_0,y_1)]\\ &\le \psi[(1/(1-h)h^n d(y_0,y_1)].\end{aligned}$$

So that for a given $\varepsilon>0$, there exists $N\in Z^+$ such that

$\psi[(1/(1-h)h^n d(y_0,y_1)]<\psi(\varepsilon)$foll all $n\ge N$. This implies $d(y_n,y_{n+p})<\varepsilon$ for all $n\ge N$. Hence $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there is a point z in X such that $y_n\to z$ as $n\to\infty$. Also, from (2.1), we have

(2.3) $y_{2n}=A_1x_{2n}=Tx_{2n+1}\to z$ and $y_{2n+1}=A_2x_{2n+1}=Sx_{2n+2}\to z$.

Now suppose that $\{A_1, S\}$ is a ψ-compatible pair of reciprocally continuous

mappings. Then, since A_1 and S are reciprocally continuous, by (2.3), we get $A_1Sx_{2n} \to A_1z$ and $SA_1x_{2n} \to Sz$. Also, ψ–compatibility of A_1 and S implies that $lim_n \psi(d(A_1Sx_{2n}, SA_1x_{2n}))=0$ That is,
$Lim_n d(A_1z, Sz)=0$. Hence
(2.4) $A_1z = Sz$.
Since $A_1X \subset TX$, there is a point w in X such that $Tw=A_1z$. Thus, by (2.4),
(2.5) $Tw = A_1z = Sz$.
Now, we show that $A_1z = A_2w$. Suppose $A_1z \neq A_2w$. Then, using assumption (ii), we have

$$\psi(d(A_1z, A_2w)) \leq h\, M_{\psi_2}(z,w)$$
$$= h\, max\, \{\psi(d(Sz,Tw)),\ \psi(d(A_1z, Sz)), \psi(d(A_2w, Tw))\}.$$
$$= h\psi(d(A_2w, Tw)) < \psi(d(A_2w, A_1z)),$$

a contradiction. Hence $A_1z=A_2w$. Therefore, from (2.4), (2.6) $A_2w=A_1z=Sz=Tw$.
Now, if $A_1z \neq A_iw$ for some $i>2$, then from (iii), we get

$$\psi(d(A_1z, A_iw)) < max\, \{\psi(d(Sz,Tw)),\ \psi(d(A_1z, Sz)),\ \psi(d(A_iw, Tw))\}.$$
$$= \psi(d(A_iw, A_1z)),\ \text{a contradiction.}$$

Hence $A_1z= A_iw = Tw$. We now show that $A_1A_1z=A_1z$.
Suppose that $A_1A_1z \neq A_1z$ Then, by (ii), we get

$$\psi(d(A_1z, A_1A_1z)) = \psi(d(A_2w, A_1A_1z))$$
$$\leq h\ M_{\psi_2}(A_1z, w))$$
$$= h\ \psi(d(A_1z, A_1A_1z)),\ \text{a contradiction.}$$

Hence $A_1A_1z=A_1z$ Also, using (iii), for $i>2$

$$\psi(d(A_1z, A_1A_1z)) = \psi(d(A_1A_1z, A_iw))$$
$$< max\, \{\psi(d(SA_1,Tw)),\ \psi(d(A_1A_1z, SA_1z)),\ \psi(d(A_iw, Tw))\}.$$
$$= \psi(d(A_1A_1z, A_iw)),\ \text{a contradiction.}$$

Hence $A_1A_1z=A_1Sz=SA_1z=SSz=A_1z$; that is , A_1z is a common fixed point of A_1 and S.
Similarly, we can show A_2w to be a common fixed point of A_2 and T. Hence $A_1z=A_2w$ is a common fixed point of all A_1, A_2,S and T.

Moreover, if $A_2w \neq A_iA_2w$ for some $i>2$, then using assumption (iii), we get $\psi(d(A_1z,A_iA_2w) < \psi(d(A_1z,A_iA_2w)$ a contradiction. Hence $A_1z=A_2w$ is a common fixed point of all $\{A_i\}$, S and T.
The uniqueness of a common fixed point follows easily from (ii). The proof is similar when $\{A_2, T\}$ is assumed to be ψ–compatible and reciprocally continuous. This complete the proof of the theorem.

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