Jñānābha, Vol. 31/32, 2002
(Dedicated to Professor H. M. Srivastava on his 62nd Birthday)

FIXED POINT THEOREMS FOR ϕ– CONTRACTIVE MAPPINGS IN *D*-METRIC SPACES

By

Anil Rajput and Smrati Arya (Harda)
Department of Mathematics
Sadhu Vaswani College, Bairagarh, Bhopal, 462008, M.P., India

(Received : May 20, 2002)

ABSTRACT

Our main object in this paper is to obtain some results on fixed point for ϕ–contractive mappings in *D*-metric space with rational inequality.

1. Introduction. In 1992, Dhage [3] introduced the notion of a *D*-metric space which is as follows:

Definition 1.1: A nonempty set X together with a *D*-metric D is called a *D*-metric space where *D*-metric is a function
$D: X \times X \times X \to R$ satisfying the following properties:

1) $D(x,y,z) \geq 0$ for all $x, y, z \in X$ and equality holds if and only if $x=y=z$ (non-negativity)
2) $D(x,y,z) = D(x,z,y) = \ldots$ (symmetry)
3) $D(x,y,z) \leq D(x,y,a)+D(x,a,z)+D(a,y,z)$ for all $x, y, z, a \in X$. (Tetrahedral inequality)

Below in the following we give some definitions due to Dhage [3] which will be useful in the sequel for proving certain fixed point theorems in *D*-metric spaces.

Definition 1.2: A sequence $\{x_n\}$ of points of a *D*-metric space X is called *D*-convergent and converges to a point x in X, if for $\varepsilon>0$, there exists an $n_0 \in N$ such that for all

$$m,n \geq n_0,\ D(x_m, x_n, x)<\varepsilon.$$

Definition 1.3: A sequence $\{x_n\}$ of points of a *D*-metric space X is called *D*-Cauchy if for $\varepsilon>0$, there exists an $n_0 \in N$ such that for all $m>n$, $p \geq n_0$. $D(x_m, x_n, x_p)<\varepsilon$. A complete *D*-metric space X is one in which every *D*-Cauchy sequence $\{x_n\}$ in X converges to a point x in X. A mapping $T:X \to X$ is said to be continuous if and only if $Tx_n \to Tx$ whenever $x_n \to x$.

Definition 1.4: A *D*-metric space (X, D) is said to be bounded if there exists a number $M>0$ such that $D(x,y,z) \leq M$ for all $x, y, z \in X$ and the constant M is *D*-bound of X.

A *D*-metric space (X,D) is said to be unbounded if it is not bounded in that case $D(x, y, z)$ takes values as large as we please.

2002 Mathematics Subject Classification : 47H10.
Key Words and phrases. *D*-metric space, common fixed point theorem, etc.

Definition 1.5 : Let $T:X \to X$, then an orbit of T at a point $x \in X$ is a set $\theta(x)$ in X given by

$$\theta(x) = \{x, Tx, T^2x, ...\}.$$

An orbit $\theta(x)$, $x \in X$ of T in a D-metric space X is said to be bounded if there exists a constant k such that $D(u, v, w) \leq k$ for all $u, v, w \in \theta(x)$.

Definition 1.6 : X is called T-orbitally bounded if the orbit $\theta(x)$ for each $x \in X$ is bounded and X is called T-orbitally complete if every D-Cauchy sequence $\{x_n\}$ in $\theta(x)$, $x \in X$ converges to a point in X.

Every bounded D-metric space is T-orbitally but converse may not be true.

It is shown in Dhage [3] that the D-metric D is a continuous function in the topology of D-metric convergence which is Hausdorff in nature.

2. Priliminaries. In 1992, Dhage [3] proved the following:

Theorem 2.1: Let T be a self map of complete and bounded D-metric space (X,D) satisfying

$$D(Tx, Ty, Tz)\, q \leq D(x, y, z)$$

for all $x, y, z, \in X$ and $0 \leq q < 1$. Then T has a unique fixed point.

From the Thesis of Ajij M. Pthan

Theorem 2.2: Let T be a self map of T-orbitally bounded and T-orbitally complete D-metric space X satisfying

$$D(Tx, Ty, Tz) \leq q \max \{D(x, y, z,), D(x, Tx, Ty), D(y, Ty, Tz), D(z, Tz, Tx)\}$$

for all $x, y, z \in X$ and $0 \leq q<1$. Then T has a unique fixed point $p \in X$ and T is continuous at p.

Corollary 2.3: Let T be a self map of T-orbitally bounded and T-orbitally complete D-metric space X satisfying

$$D(Tx, Ty, Tz) \leq \alpha D(x,y,z) + \beta D(x, Tx, Ty) + \gamma D(y, Ty, Tz), + \delta D(z, Tz, Tx)$$

for all $x, y, z \in X$ where $\alpha, \beta, \gamma, \delta$ are non negative real numbers such that $\alpha+\beta+\gamma+\delta<1$. Then T has a unique fixed point.

In a recent paper Dhage, B.E. Rhoades and Ajij M. Pathan [6] proved

Theorem 2.4: Let T be a self map of a T-orbitally bounded and T-orbitally complete D-metric space satisfying

$$D(Tx, Ty, Tz) \leq \alpha \left\{\frac{[1+D(x, Tx, z)]\, D(y, Ty, z)}{[1+D(x, Tx, z)]} + \beta D(x, y, z)\right\}$$

for all $x, y \in X$ and $z \in \overline{O_T(x) \cup O_T(y)}$ where $\alpha \geq 0$, $\beta \geq 0$, satisfy $\alpha + \beta <1$. Then T has a unique fixed point $u \in X$ and T is continuous at u.

3. Fixed Points Theorems for ϕ-Contraction Mappings.

Theorem 3.1: Let S, T be a self map of a S, T-orbitally bounded and S, T-orbitally complete D-metric space X satisfying the following:

$$D(Sx, Ty, Tz) \leq \phi \max \{(D(x,y,z), D(x,Sx,z), D(y,Ty,Tz),$$

$$\frac{[1+D(x,Sx,z)]D(y,Ty,Tz)}{[1+D(x,y,z)]} \frac{[1+D(y,Ty,Tz)]D(x,Sx,z)}{[1+D(x,y,z)]}\} \tag{3.1}$$

for all $x, y, z \in X$, where $\phi : [0,\infty)^5 \to [0,\infty)$ satisfying
ϕ is upper semi-continuous and decreasing in each co-ordinate variable.
$\phi(t, t, t, t, t,) < t$ for all $t > 0$,
Then S, T have a unique fixed point in X.

Proof: Let $x \in X$ be arbitrary, then we can find x_n's in X such that

$$x_n = Sx_{n-1} \text{ and } x_{n+1} = Tx_n \text{ for } n = 0,1,2,...$$

if $x_r = x_{n+1}$ for some $r \in N$, then clearly x_r is a fixed point of T. So let $x_r \neq x_{r+1}$ for any $r \in N$. Condition (3.1) implies that

$$\begin{aligned} D(x_n, x_{n+1}, x_{p+1}) &= D(Sx_{n-1}, Tx_n, Tx_p) \\ &\leq \phi\, max\, \{D(x_{n-1}, x_n, x_p), D(x_{n-1}, Sx_{n-1}, x_p), D(x_n, Tx_n, Tx_p), \\ &\quad \frac{[1 + D(x_{n-1}, Sx_{n-1}, x_p)]D(x_n, Tx_n, Tx_p)}{[1 + D(x_{n-1}, x_n, x_p)]} \\ &\quad \frac{[1 + D(x_n, Tx_n, Tx_p)]D(x_{n-1}, Sx_{n-1}, x_p)}{[1 + D(x_{n-1}, x_n, x_p)]} \end{aligned}$$

$$\begin{aligned} &\leq \phi\, max \Big\{ D(x_{n-1}, x_n, x_p), D(x_{n-1}, x_n, x_p), D(x_n, x_{n+1}, x_{p+1}), \\ &\quad \frac{[1 + D(x_{n-1}, x_n, x_p)]D(x_n, x_{n+1}, Tx_{p+1})}{[1 + D(x_{n-1}, x_n, x_p)]} \\ &\quad \frac{[1 + D(x_n, x_{n+1}, x_{p+1})]D(x_{n-1}, x_n, x_p)}{[1 + D(x_{n-1}, x_n, x_p)]} \Big\} \end{aligned}$$

Let $D_{n,p} = D(x_n, x_{n+1}, x_p)$. We want to show that $D_{n,\, p+1} \leq D_{n-1,\, p}$
For this, assume that $D_{n,\, p+1} > D_{n-1,\, p}$. Then we get

$$\begin{aligned} D_{n,\, p+1} &\leq \phi\, max\{D_{n-1,\, p}, D_{n-1,\, p}, D_{n,\, p+1}, \\ &\quad \frac{[1 + D_{n-1,p}]}{[1 + D_{n-1,p}]} D_{n,p+1}, \frac{[1 + D_{n-1,p}]}{[1 + D_{n-1,p}]} D_{n,p+1} \Big\} \\ &\leq \phi\, max\{D_{n-1,\, p}, D_{n-1,\, p}, D_{n,\, p+1}, D_{n,p+1}, D_{n,\, p+1}\} \\ &\leq \phi\, max\{D_{n,\, p+1}, D_{n,\, p+1}, D_{n,\, p+1}, D_{n,p+1}, D_{n,\, p+1}\} \\ &< D_{n,p+1} \end{aligned}$$

a contradiction. Hence $D_{n,p+1} \leq D_{n-1,p}$. Thus the sequence $\{D_{n,p}\}$ is a non-increasing in R^+ bounded below by 0. Therefore $D_{n,\, p} \to 0$.

Now we shall show that sequence $\{x_n\}$ is D-Cauchy. If not then there exists an $\varepsilon >$

0 and sequence of positive integers $\{m(k)\},\{n(k)\}$ and $\{p(k)\}$ such that

$k \le n(k) \le p(k) \le m(k)$ and

$$c_k = D(x_{m(k)}, x_{n(k)}, x_{p(k)}) \ge \varepsilon.$$

where $k = 1, 2, 3, \ldots$.

Let $m(k)$ be the least integer exceeding $n(k)$ and $p(k)$ for which above inequality is true, then by well ordering principle, we get

$$D(x_{m(k)-1}, x_{n(k)}, x_{p(k)}) \ge \varepsilon.$$

Now

$$\varepsilon \le c_k$$
$$= D(x_{m(k)}, x_{n(k)}, x_{p(k)})$$
$$\le \left\{D\left(x_{m(k)-1}, x_{n(k)}, x_{p(k)}\right) + D\left(x_{m(k)}, x_{n(k)-1}, x_{p(k)}\right) + D\left(x_{m(k)}, x_{n(k)}, x_{m(k)-1}\right)\right\}$$
$$< \varepsilon + D_{m(k)-1,\, p(k)} + D_{m(k)-1,\, n(k)}.$$

Taking limit as $k \to \infty$ we get $c_k \to \varepsilon$.

Further c_k can have the following eight different cases:

(1) m and n are even and p is odd

(2) m and p are even and n is odd

(3) n and p are even and m is odd

(4) m is even and n and p are odd

(5) n is even and m and p are odd

(6) p is even and m and n are odd

(7) m,n and p are all even

(8) m, n and p are all odd

Case 1. When m and n are even and p is odd. Let

$$C_k = D(x_{2m}, x_{2n}, x_{2p-1})$$
$$\le \{D(x_{2m}, x_{2m+1}, x_{2p-1}) + D(x_{2m+1}, x_{2n}, x_{2p-1}) + D(x_{2m}, x_{2n}, x_{2m+1})\}$$
$$\le \{D_{2m,2p-1} + D_{2m,2n} + D(x_{2m+1}, x_{2n}, x_{2n+1}) + D(x_{2n+1}, x_{2n}, x_{2p-1}) + D(x_{2m+1}, x_{2n+1}, x_{2p-1})\}$$
$$\le \{D_{2m,2p-1} + D_{2m,2n} + D_{2n,2m+1} + D_{2n,\,2p-1} + D(Sx_{2m}, Tx_{2m}, Tx_{2p-2})\}$$

using condition (3.1) we get

$$D(Sx_{2m}, Tx_{2n}, Tx_{2p-2})$$
$$\le \phi\, max\{D(x_{2m}, x_{2n}, x_{2p-2}), D(x_{2m}, Sx_{2m}, x_{2p-2}), D(x_{2n}, Tx_{2n}, Tx_{2p-2}),$$
$$\frac{\left[1 + D\left(x_{2m}, Sx_{2m}, x_{2p-2}\right)\right] D\left(x_n, Tx_{2n}, xT_{2p-2}\right)}{\left[1 + D\left(x_{2m}, x_{2n}, x_{2p-2}\right)\right]}$$

$$\frac{\left[1+D\left(x_{2n},Tx_{2n},Tx_{2p-2}\right)\right]D\left(x_{2m},Sx_{2m},x_{2p-2}\right)}{\left[1+D\left(x_{2m},x_{2n},x_{2p-2}\right)\right]}\Bigg\}$$

$$\leq \phi\, max\{D(x_{2m}, x_{2n}, x_{2p-2}), D(x_{2m}, x_{2m+1}, x_{2p-2}), D\ (x_{2m}, x_{2n+1}, x_{2p-1}),$$

$$\frac{\left[1+D\left(x_{2m},x_{2m+1},x_{2p-2}\right)\right]D\left(x_{2n},x_{2n+1},x_{2p-1}\right)}{\left[1+D\left(x_{2m},x_{2n},x_{2p-2}\right)\right]}$$

$$\frac{\left[1+D\left(x_{2n},x_{2n+1},x_{2p-1}\right)\right]D\left(x_{2m},x_{2m+1},x_{2p-2}\right)}{\left[1+D\left(x_{2m},x_{2n},x_{2p-2}\right)\right]}\Bigg\}$$

$$\leq \phi\, max\{D(x_{2m}, x_{2n}, x_{2p-1})+D(x_{2p-1}, x_{2n}, x_{2p-2})+D\ (x_{2m}, x_{2p-1}, x_{2p-2})),$$
$$D(x_{2m}, x_{2m+1}, x_{2p-2}), D(x_{2n}, x_{2n+1}, x_{2p-1}),$$

$$\frac{\left[1+D\left(x_{2m},x_{2m+1},x_{2p-2}\right)\right]D_{2n,2p-1}}{\left[1+D\left(x_{2m},x_{2n},x_{2p-1}\right)+D\left(x_{2p-1},x_{2n},x_{2p-2}\right)+D\left(x_{2m},x_{2p-1},x_{2p-2}\right)\right]},$$

$$\frac{\left[1+D\left(x_{2n},x_{2n+1},x_{2p-1}\right)\right]D_{2m,2p-2}}{\left[1+D\left(x_{2m},x_{2n},x_{2p-1}\right)+D\left(x_{2p-1},x_{2n},x_{2p-2}\right)+D\left(x_{2m},x_{2p-1},x_{2p-2}\right)\right]}\Bigg\}$$

$$\leq \phi\, max\{(C_k+D_{2p-2,2n}+D_{2p-2,2m}), D_{2m,2p-2}, D_{2n,2p-1},$$

$$\frac{\left[1+D_{2m,2p-2}\right]D_{2n,2p-1}}{\left[1+c_k+D_{2p-2,2n}+D_{2p-2,2m}\right]}\frac{\left[1+D_{2n,2p-1}\right]D_{2m,2p-2}}{\left[1+c_k+D_{2p-2,2n}+D_{2p-2,2m}\right]}\Bigg\}$$

Using the above inequality (3.2) reduces to

$$C_k=\{D_{2m,2p-1}+D_{2m,2n}+D_{2n,2m+1}+D_{2n,2p-1}+\ \phi\ max\ (C_k+D_{2p-2,2n}+D_{2p-2,2m,})$$
$$D_{2m,2p-2}, D_{2n,2p-1},$$

$$\frac{\left[1+D_{2m,2p-2}\right]D_{2n,2p-1}}{\left[1+C_k+D_{2p-2,2n}+D_{2p-2,2m}\right]}\frac{\left[1+D_{2n,2p-1}\right]D_{2m,2p-2}}{\left[1+c_k+D_{2p-2,2n}+D_{2p-2,2m}\right]}\Bigg\}.$$

Since ϕ is upper semi-continuous so taking limit as $m,n,p \to\infty$, we obtain

$C_k \leq 0 + \phi \max(\in, 0,0,0,0) \leq \in$.

Hence $\in = 0$. Similarly in other cases we get $\in = 0$. Hence $\{x_n\}$ is a D-Cauchy sequence in complete D-metric space X, there exists a point u in X such that $\lim_{n\to\infty} x_n = u$.

Now we want to show that u is the fixed point of T. For this, we have

$$D(x_{n+1}, Tu, Tu) = D(Sx_n, Tu, Tu)$$
$$\leq \phi \max\{D(x_n, u, u), D(x_n, Sx_n, u), D(u, Tu, Tu),$$
$$\frac{D(u,Tu,Tu)[1+D(x_n,Sx_n,u)]}{[1+D(x_n,u,u)]} \frac{D(x_n,Sx_n,u)[1+D(u,Tu,Tu)]}{[1+D(x_n,u,u)]}\Bigg\}$$
$$\leq \phi \max\{D(x_n, u, u), D(x_n, x_{n+1}, u), D(u, Tu, Tu),$$
$$\frac{D(u,Tu,Tu)[1+D(x_n,x_{n+1},u)]}{[1+D(x_n,u,u)]} \frac{D(x_n,x_{n+1},u)[1+D(u,Tu,Tu)]}{[1+D(x_n,u,u)]}\Bigg\}$$

Taking limit when $n\to\infty$, we get

$$D(u, Tu, Tu) \leq \phi \max\{0,0\ D(u, Tu, Tu), D(u, Tu, Tu), D(u, Tu, Tu)\}$$

a contradiction. Hence $u = Tu$.

$$D(u, Su, u) = D(Su, Tu, Tu)$$
$$\leq \phi \max\{D(u,u,u), D(u, Su, u), D(u, Tu, Tu),$$
$$\frac{D(u,Su,Tu)[1+D(u,Tu,u)]}{[1+D(u,u,u)]} \frac{D(u,Tu,Tu)[1+D(u,Su,Tu)]}{[1+D(u,u,u)]}\Bigg\}$$
$$\leq D(u,Su,u)$$

a contradiction . Hnce $u = Su$.

To prove uniqueness let $v \neq u$ be another fixed point of T. Then by condition (3.1), we get

$$D(u, u, v) = D(Su, Tu, Tv)$$
$$\leq \phi \max\{D(u,u,v), D(u, u, v), D(u, u, v), D(u, u, v), D(u, u, v),$$
$$D(u, u, v)\}$$
$$< D(u, u, v),$$

gives a contradiction. Hence $v = u$. Thus u is the unique fixed point of T.

Corollary 3.1. Let T be a self map of a T-orbitally bounded and T-orbitally complete D-metric space X satisfying the following:

$$D(Tx, Ty, Tz) \leq \phi \max\{D(x,y,z)D(x, Tx, z), D(y, Ty, Tz),$$

$$\left.\frac{\left[1+D(x,Tx,z)\right]D(y,Ty,Tz)}{\left[1+D(x,y,z)\right]}\frac{\left[1+D(y,Ty,Tz)\right]D(x,Tx,z)}{\left[1+D(x,y,z)\right]}\right\} \quad (3.1.1)$$

for all $x, y, z \in X$, where $\phi:[0,\infty)^5 \to [0,\infty)$ satisfying

ϕ is upper semi-continuous and decreasing in each co-ordinate variable.

$\phi\ (t,t,t,t,t)<t$ for all $t>0$.

Then T has a unique fixed point in X.

Proof: The proof is similar to the theorem 3.1 on taking $S= T$.

Corollary 3.2: Let S, T be a self map of a, S, T-orbitally bounded and S, T-orbitally complete D-metric space X satisfying the following:

$D(Sx,\ Ty,\ Tz) \leq \phi\ max\ \{D(x,\ y,\ z),\ D(x,\ Sx,\ z),\ D(y,\ Ty,\ Tz),$

$$\left.\frac{\left[1+D(x,Sx,z)\right]D(y,Ty,Tz)}{\left[1+D(x,y,z)\right]}\right\} \quad (3.2.1)$$

for all $x, y, z \in X$, where $\phi:[0,\infty)^4 \to [0,\infty)$ satisfying

ϕ is upper semi-continuous and decreasing in each co-ordinate variable.

$\phi\ (t,t,,t,t)<t$ for all $t>0$.

Then S, T have a unique fixed point in X.

Proof. The proof is similar to the theorem 3.1.

Corollary 3.3: Let T be a self map of a T-orbitally bounded and T-orbitally complete D-metric space X satisfying the following:

$D(Tx,\ Ty,\ Tz) \leq \phi\ max\ \{D(x,\ y,\ z),\ D(x,\ Tx,\ z),\ D(y,\ Ty,\ Tz),$

$$\left.\frac{\left[1+D(x,Tx,z)\right]D(y,Ty,Tz)}{\left[1+D(x,y,z)\right]}\right\} \quad (3.3.1)$$

for all $x, y, z \in X$, where $\phi:[0,\infty)^4 \to [0,\infty)$ satisfying

ϕ is upper semi-continuous and decreasing in each co-ordinate variable.

$\phi\ (t,t,,t,t)<t$ for all $t>0$

Then T has a unique fixed point in X.

Proof: The proof to the Corollary 3.2 on taking $S=T$.

Corollary 3.4: Let S, T be a self map of a S, T-orbitally bounded and S,T-orbitally complete D-metric space X satisfying the following:

$D(Sx,\ Ty,\ Tz) \leq \phi\ max\ \{D(x,\ y,\ z),\ D(x,\ Sx,\ z),\ D(y,\ Ty,\ Tz)\}$ (3.4.1)

for all $x, y, z \in X$, where $\phi:[0,\infty)^3 \to [0,\infty)$ satisfying

ϕ is upper semi-continuous and decreasing in each co-ordinate variable.

$\phi\,(t,t,,t,) < t$ for all $t > 0$.

Then S, T have a unique fixed point in X.

Proof. The proof is similar to the theorem 3.1.

Corollary 3.5: Let T be a self map of a T-orbitally bounded and T-orbitally complete D-metric space X satisfying the following:

$$D(Tx, Ty, Tz) \leq \phi\ max\ \{D(x, y, z), D(x, Tx, z), D(y, Ty, Tz)\} \qquad (3.5.1)$$

for all $x, y, z \in X$, where $\phi:[0,\infty)^3 \rightarrow [0,\infty)$ satisfying

ϕ is upper semi-continuous and decreasing in each co-rdinate variable.

$\phi\,(t,t,,t) < t$ for all $t > 0$.

Then T has a unique fixed point in X.

Proof. The proof is similar to the theorem (3.1) if we take $S = T$.

REFERENCES

[1] D. R. Bansal, R. Chug and R. Kumar, Fixed Points for ϕ-Contractive Mappings in D-Metric Spaces, *East Asia Math. Comm.* **1** (1998), 9-15.

[2] E.T. Copson, *Metric Spaces*, Cambridge Univ., Press, 1968.

[3] B.C. Dhage, Generalized metric spaces and mappings with fixed point, *Bull. Calcutta Math. Soc.*, **84** (4) (1992), 329-336.

[4] B.C. Dhage, Generalized metric spaces and topological structure II, *Pure. Appl. Math. Sci.*, **41** (1994), 37-41.

[5] B. C. Dhage, On continuity of mappings in D-metric spaces, *Bull. Calcutta Math. Soc.* **86** (1994), 503-508.

[6] B. C. Dhage, A. M. Pathan and B.E. Rhoades, A general existence principle for fixed point theorem in D-metric spaces, *Intern. J. Math. and Math. Sci.* **21** (1998).

[7] T. Veerapandi and R. Chandrasekher, Fixed point in Dhage Metric spaces, *Pure Appl. Math. Sci.*, **15**, No. 1-2, Math., 1996.