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*(Dedicated to Professor H. M. Srivastava on his 62nd Birthday)*

# A CERTAIN CLASS OF MULTIPLE GENERATING FUNCTIONS INVOLVING MITTAG-LEFFLER'S FUNCTIONS

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## ABSTRACT

A set of certain class of multiple generating function involving Mittag-Leffler's functions $E_\alpha$, $E_{\alpha,\beta}$ and related function $\phi(\alpha,\beta;z)$ of Wright is given. Some interesting (known and new) multiple generating functions are also obtained as special cases.

**1. Introduction and Definition.** In the usual notation, let ${}_pF_q$ denote a generalized hypergeometric function of one variable with $p$ and $q$ parameters (positive integer or zero), defined by [6; p. 42(1)]

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_q; \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n z^n}{(b_1)_n (b_2)_n \cdots (b_q)_n n!} = {}_pF_q[a_1, \ldots, a_p; b_1, \ldots, b_q; z], \tag{1.1} $$

$(b_j \neq 0, -1, -2, \ldots, j = 1, 2, \ldots, q)$

where $(a)_n$ is the Pochhammer symbol, defined by

$$ (a)_n = \begin{cases} 1, & \text{, if } n=0 \\ a(a+1)(a+2)\ldots(a+n-1), & \text{, if } n=1,2,\ldots \end{cases} \tag{1.2} $$

Series (1.1) is convregent

(i) for $|z|<\infty$ if $p \le q$. (ii) for $|z| < 1$ if $p = q+1$

(iii) absolutely convergent for $|z|<1$ if $p=q+1$ and $Re\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) > 0.$

The function

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad (\alpha > 0) \tag{1.3} $$

was introduced by Mittag-Leffler [5] and was investigated systematically by several

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other authors [3, chapter xviii]. $E_\alpha(z)$ for $\alpha>0$, furnishes important example of entire functions of finite order $1/\alpha$.
The function

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad (\alpha,\beta>0) \tag{1.4}$$

has property very similar to those of Mittag-Leffler's function (see Wiman [7], Agarwal [1]). We have

$$E_{\alpha,1}(z)=E_\alpha(z),\ \ E_1(z)=e^z,\ \ E_2(z^2)=\cosh z \text{ and } E_{1/2}(\sqrt{z})=\frac{2e^{-z}}{\sqrt{\pi}}\,Erfc(-z^{1/2}). \tag{1.5}$$

A function intimately connected with $E_{\alpha,\beta}$ is the entire function

$$\phi(\alpha,\beta;z)=\sum_{k=0}^{\infty}\frac{z^k}{k!\,\Gamma(\alpha k+\beta)},\quad (\alpha,\beta>0) \tag{1.6}$$

which was used by Wright [8] in the asymptotic theory of partition. We can verify easily

$$J_\upsilon(z)=(z/2)^{\upsilon}\phi(1,\upsilon+1;-z^2/4), \tag{1.7}$$

It shows that Wright's function may be regarded as a kind of generalized Bessel function $J_\upsilon(z)$, defined by

$$J_\upsilon(z)=\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\upsilon+2k}}{\Gamma(\upsilon+k+1)k!},\ |z|<\infty, \tag{1.8}$$

which has the particular attention in the diverse field of physics and engineering.

An interesting generating function, due to Humbert [4], is recalled here in the following form:

$$exp\left[\frac{z}{3}\left(x+y\pm\frac{1}{xy}\right)\right]=\sum_{m,n=-\infty}^{\infty}x^m y^n\frac{(z/3)^{m+n}}{\Gamma(m+1)\Gamma(n+1)}\ {}_0F_2\left[-;m+1,n+1;\pm(z/3)^3\right], \tag{1.9}$$

where the Hyper-Bessel function $J_{m,n}(z)$ and modified Hyper-Bessel function $I_{m,n}(z)$ of order 2 are defined by

$$J_{m,n}(z)=\frac{(z/3)^{m+n}}{\Gamma(m+1)\Gamma(n+1)}{}_0F_2\left[-;m+1,n+1;-(z/3)^3\right] \tag{1.10}$$

and

$$I_{m,n}(z)=\frac{(z/3)^{m+n}}{\Gamma(m+1)\Gamma(n+1)}\ {}_0F_2\left[-;m+1,n+1;+(z/3)^3\right],\text{ respectively} \tag{1.11}$$

## 2. Generating function involving Mittag-Leffler's functions

**Result-1**

If $E_{\alpha,\beta}(z)$ is defined by equation (1.4) then

$$E_{\alpha_1,\beta_1}\left(\frac{zx}{3}\right)E_{\alpha_2,\beta_2}\left(\frac{zy}{3}\right)E_{\alpha_3,\beta_3}\left(\frac{-z}{3xy}\right)=\sum_{m,n=-\infty}^{\infty}x^m y^n\ {}_{\beta_1,\beta_2,\beta_3}^{\alpha_1,\alpha_2,\alpha_3}J_{m,n}(z), \tag{2.1}$$

where

$${}_{\beta_1,\beta_2,\beta_3}^{\alpha_1,\alpha_2,\alpha_3}J_{m,n}(z)=\left(\frac{z}{3}\right)^{m+n}\sum_{k=0}^{\infty}\frac{(-1)^k(z/3)^{3k}}{\Gamma(\alpha_1(m+k)+\beta_1)\Gamma(\alpha_2(n+k)+\beta_2)\Gamma(\alpha_3 k+\beta_3)}, \tag{2.2}$$

provided that both sides of equation (2.1) exist.

**Proof of result (2.1):**

If the function

$$V(x,y,z)=E_{\alpha_1,\beta_1}\left(\frac{zx}{3}\right)E_{\alpha_2,\beta_2}\left(\frac{zy}{3}\right)E_{\alpha_3,\beta_3}\left(\frac{-z}{3xy}\right)$$

is expanded by the definition (1.4), we have

$$V=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\alpha_3 k+\beta_3)}\sum_{i=0}^{\infty}\frac{x^{i-k}}{\Gamma(\alpha_1 i+\beta_1)}\sum_{j=0}^{\infty}\frac{y^{j-k}(z/3)^{k+i+j}}{\Gamma(\alpha_2 j+\beta_2)}$$

Now replacing $i$-$k$ and $j$-$k$ by $m$ and $n$ respectively, if we rearrange the resulting triple series (which can be justified by absolute convergence of the series involved), it follows that

$$V=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}x^m y^n\left(\frac{z}{3}\right)^{m+n}\sum_{k=0}^{\infty}\frac{(-1)^k(z/3)^{3k}}{\Gamma(\alpha_1(m+k)+\beta_1)\Gamma(\alpha_2(n+k)+\beta_2)\Gamma(\alpha_3 k+\beta_3)}.$$

Thus the result (2.1) is proved.

A generalization of generating relation (2.1) can be obtained in the following form

$$\prod_{j=1}^{n}\left(E_{\alpha_j,\beta_j}\left(\frac{zx_j}{n+1}\right)\right).E_{\alpha,\beta}\left(\frac{-z/(n+1)}{\prod_{j=1}^{n}(x_j)}\right)=\sum_{m_1\ldots m_n=-\infty}^{\infty}x_1^{m_1}\ldots x_n^{m_n}\ {}_{\beta_1,\ldots,\beta_n,\beta}^{\alpha_1,\ldots,\alpha_n,\alpha}J_{m_1,\ldots,m_n}(z), \tag{2.3}$$

where

$${}_{\beta_1,\ldots,\beta_n,\beta}^{\alpha_1,\ldots,\alpha_n,\alpha}J_{m_1,\ldots,m_n}(z)=\left(\frac{z}{n+1}\right)^{\sum_{j=1}^{n}m_j}\sum_{k=0}^{\infty}\frac{(-1)^k(z^{3})^k}{\Gamma(\alpha_1(m_1+k)+\beta_1)\ldots\Gamma(\alpha_n(m_n+k)+\beta_n)\Gamma(\alpha k+\beta)} \tag{2.4}$$

provided that both member of equation (2.3) exist.

**Result-II**

In a similar way, we obtain

$$E_{\alpha_1,\beta_1}\left(\frac{zx}{3}\right)E_{\alpha_2,\beta_2}\left(\frac{zy}{3}\right)E_{\alpha_3,\beta_3}\left(\frac{4z}{3xy}\right)=\sum_{m,n=-\infty}^{\infty}x^m y^n \,{}^{\alpha_1,\alpha_2,\alpha_3}_{\beta_1,\beta_2,\beta_3}I_{m,n}(z), \tag{2.5}$$

where

$${}^{\alpha_1,\alpha_2,\alpha_3}_{\beta_1,\beta_2,\beta_3}I_{m,n}(z)=\left(\frac{z}{3}\right)^{m+n}\sum_{k=0}^{\infty}\frac{(z/3)^{3k}}{\Gamma(\alpha_1(m+k)+\beta_1)\Gamma(\alpha_2(n+k)+\beta_2)\Gamma(\alpha_3 k+\beta_3)}, \tag{2.6}$$

and thus, a generalization of equation (2.5) can be obtained as follows:

$$\prod_{j=1}^{n}\left(E_{\alpha_j,\beta_j}\left(\frac{zx_j}{n+1}\right)\right).E_{\alpha,\beta}\left(\frac{z/(n+1)}{\prod_{j=1}^{n}(x_j)}\right)=\sum_{m_1\ldots m_n=-\infty}^{\infty}x_1^{m_1}\ldots x_n^{m_n}\,{}^{\alpha_1,\ldots,\alpha_n,\alpha}_{\beta_1,\ldots,\beta_n,\beta}I_{m_1,\ldots,m_n}(z), \tag{2.7}$$

where

$${}^{\alpha_1,\ldots,\alpha_n,\alpha}_{\beta_1,\ldots,\beta_n,\beta}I_{m_1,\ldots,m_n}(z)=\left(\frac{z}{n+1}\right)^{\sum_{j=1}^{n}m_j}\sum_{k=0}^{\infty}\frac{\left[(z/(n+1))^{n+1}\right]^k}{\Gamma(\alpha_1(m_1+k)+\beta_1)\ldots\Gamma(\alpha_n(m_n+k)+\beta_n)\Gamma(\alpha k+\beta)} \tag{2.8}$$

provided that both sides of equations (2.5) and (2.7) exist.

The proof of result II is similar to result-I.

**Result-III**: The above results may be extended for the entire function, defined by

$$(1.6)\ \phi(\alpha_1,\beta_1;zx_1)\ldots\phi(\alpha_n,\beta_n;zx_n)\phi(\alpha,\beta)\left(z\Big/\prod_{j=1}^{n}(x_j)\right)$$

$$=\sum_{m_1,\ldots,m_n=-\infty}^{\infty}\frac{x_1^{m_1}\ldots x_n^{m_n}}{m_1!\ldots m_n!}\,{}^{\alpha_1,\ldots,\alpha_n,\alpha}_{\beta_1,\ldots,\beta_n,\beta}G_{m_1,\ldots,m_n}(z), \tag{2.9}$$

where

$${}^{\alpha_1,\ldots,\alpha_n,\alpha}_{\beta_1,\ldots,\beta_n,\beta}G_{m_1,\ldots,m_n}(z)$$

$$=z^{\sum_{j=1}^{n}m_j}\sum_{k=0}^{\infty}\frac{\left[(z)^{n-1}\right]^k}{(m_1+1)_k\ldots(m_n+1)_k\Gamma(\alpha_1(m_1+k)+\beta_1)\ldots\Gamma(\alpha_n(m_n+k)+\beta_n)\Gamma(\alpha k+\beta)k!} \tag{2.10}$$

provided that both sides of equation (2.9) exist. The proof of result (2.9) is similar to the result (2.1).

**Special Cases.**

For $\alpha_i=\beta_i=\alpha=\beta=1, \{i=1,2,\ldots,n\}$, equations (2.3) and (2.7) reduce to well known generating function [2] of Hyper-Bessel function $J_{m_1,\ldots,m_n}(z)$ of order $n$ and its modified case $I_{m_1,\ldots,m_n}(z)$

$$exp\left[\frac{z}{n+1}\left(x_1+\ldots+x_n-\frac{1}{x_1+\ldots x_n}\right)\right]=\sum_{m_1\ldots m_n=-\infty}^{\infty} x_1^{m_1}\ldots x_n^{m_n} J_{m_1,\ldots,m_n}(z), \tag{3.1}$$

where

$$J_{m_1,\ldots,m_n}(z)={}_{1,\ldots,1.1}^{1,\ldots,1.1}J_{m_1,\ldots,m_n}(z)=\frac{(z/n+1)^{\sum_{j=1}^{n} m_j}}{(m_1)!\ldots(m_n)!}{}_0F_n\left[-;m_1+1,\ldots,m_n+1;-\left(\frac{z}{n+1}\right)^{n+1}\right] \tag{3.2}$$

and

$$exp\left[\frac{z}{n+1}\left(x_1+\ldots+x_n-\frac{1}{x_1\ldots x_n}\right)\right]=\sum_{m_1\ldots m_n=-\infty}^{\infty} x_1^{m_1}\ldots x_n^{m_n} I_{m_1,\ldots,m_n}(z), \tag{3.3}$$

where

$$I_{m_1,\ldots,m_n}(z)={}_{1,\ldots,1.1}^{1,\ldots,1.1}I_{m_1,\ldots,m_n}(z)=\frac{(z/n+1)^{\sum_{j=1}^{n} m_j}}{(m_1)!\ldots(m_n)!}{}_0F_n\left[-;m_1+1,\ldots,m_n+1;-\left(\frac{z}{n+1}\right)^{n+1}\right], \tag{3.4}$$

respectively.
For $n=2$, equations (2.1) and (2.5) give equation (1.9), respectively.

Replacing $\frac{z}{n+1}\to z^2$ and $x_i\to x_i^2$, $i=\{1,2,\ldots,n\}$, respectively, in equations (2.3) and (2.7), taking $\alpha_i=2$ and $\beta_i=1$ $i=\{1,2,\ldots,n\}$, and using the relation (1.5), we get

$$cosh(zx_1)\ldots cosh(zx_n).cos\left(\frac{z}{x_1\ldots x_n}\right)=\sum_{m_1\ldots m_n=-\infty}^{\infty}\frac{x_1^{m_1}\ldots x_n^{m_n}}{(2m_1)!\ldots(2m_n)!}$$

$$\cdot{}_0F_{2n+1}\left[-;m_1+1/2,\ldots,m_n+1/2,m_1+1,\ldots,m_n+1;1/2;(-z/4)^{n+1}\right] \tag{3.5}$$

and

$$cosh(zx_1)\ldots cosh(zx_n).cos\left(\frac{z}{x_1\ldots x_n}\right)=\sum_{m_1\ldots m_n=-\infty}^{\infty}\frac{x_1^{m_1}\ldots x_n^{m_n}}{(2m_1)!\ldots(2m_n)!}$$

$$\cdot{}_0F_{2n+1}\left[-;m_1+1/2,\ldots,m_n+1/2,m_1+1,\ldots,m_n+1;1/2;(-z/4)^{n+1}\right] \tag{3.6}$$

On setting $\alpha_i=\alpha=1/2$ and $\beta_i=\beta=1$ in (2.7) replacing $\frac{z}{n+1}\to\sqrt{z}$, $x_i\to\sqrt{x_i}$,

$i=\{1,2,...,n\}$ respectivley, and using the relation (1.5), we get

$$\left(\frac{2}{\sqrt{\pi}}\right)^{n+1} exp\left[-\left(zx_1+\ldots+zx_n+\frac{z}{x_1\ldots x_n}\right)\right]$$

$$Erfc\left(-\sqrt{zx_1}\right)\ldots Erfc\left(-\sqrt{zx_n}\right)Erfc\left(-\sqrt{\frac{z}{x_1\ldots x_n}}\right)$$

$$=\sum_{m_1\ldots m_n=-\infty}^{\infty}\left(\sqrt{x_1}\right)^{m_1}\ldots\left(\sqrt{x_n}\right)^{m_n}\left(\sqrt{z}\right)^{\sum_{j=1}^{n} m_j}\sum_{k=0}^{\infty}\frac{\left[\left(\sqrt{z}\right)^{n+1}\right]^k}{\left(\frac{m_1+k}{2}\right)!\ldots\left(\frac{m_n+k}{2}\right)!\left(\frac{k}{2}\right)!} \tag{3.7}$$

Now putting $\alpha_i=\alpha=1,\ \beta_i=\lambda_i+1\ \{i=1,2,...,n\}$ and $\beta=\lambda+1$ and replacing $z/n+1$ by $-z^2/4$ and $x_i$ by $x_i^2$ $\{i=1,2,...,n\}$, respectively, in (2.9) and using the relation (1.7), we get a multiple generating relation involving Bessel's functions:

$$J_{\lambda_1}(zx_1)\ldots J_{\lambda_n}(zx_n).J_\lambda\left(\frac{z}{x_1+\ldots x_n}\right)=\left(\frac{zx_1}{2}\right)^{\lambda_1}\ldots\left(\frac{zx_n}{2}\right)^{\lambda_n}\left(\frac{z}{2\prod_{j=1}^{n}x_j}\right)^{\lambda}$$

$$\sum_{m_1\ldots m_n=-\infty}^{\infty}\frac{x_1^{2m_1}\ldots x_n^{2m_n}\left(-z^2\right)^{(m_1+\ldots m_n)}}{(m_1)!\ldots(m_n)!\,4^{m_1+\ldots m_n}\prod_{j=1}^{n}\left(\Gamma\left(\lambda_j+m_j+1\right)\right)\Gamma(\lambda+1)}$$

$$\cdot{}_0F_{2n+1}\left[-;m_1+1,\ldots,m_n+1,\lambda_1+m_1+1,\ldots,\lambda_n+m_n+1,\lambda+1;\left(-z/4\right)^{n+1}\right]. \tag{3.8}$$

For $n=2$, equation (3.8) reduces to

$$J_{\lambda_1}(zx_1)\ldots J_{\lambda_n}(zx_n).J_\lambda\left(\frac{z}{x_1x_2}\right)=\left(\frac{zx_1}{2}\right)^{\lambda_1}\left(\frac{zx_n}{2}\right)^{\lambda_2}\left(\frac{z}{2x_1x_2}\right)^{\lambda}$$

$$\sum_{m_1,m_2=-\infty}^{\infty}\frac{x_1^{2m_1}x_2^{2m_2}\left(-z^2\right)^{m_1+m_2}}{(m_1)!(m_2)!\,\Gamma(\lambda_1+m_1+1)\Gamma(\lambda_2+m_2+1)\Gamma(\lambda+1)}$$

$$\cdot{}_0F_5\left[-;m_1+1,m_2+1,\lambda_1+m_1+1,\lambda_2+m_2+1,\lambda+1;z^6/64\right] \tag{3.9}$$

For $\lambda_1=\ldots=\lambda_n=\lambda=\pm 1/2$, equation (3.9) reduce to the following multiple generating relations:

$$sin(zx_1)\ldots sin(zx_n)\,sin\left(\frac{z}{x_1\ldots x_n}\right) = z^{n+1}\sum_{m_1\ldots m_n=-\infty}^{\infty}\frac{x_1^{2m_1}\ldots x_n^{2m_n}\left(-z^2/4\right)^{m_1+\ldots+m_n}}{(m_1)!\ldots(m_n)!(3/2)_{m_1}\ldots(3/2)_{m_n}}$$

$$\cdot{}_0F_{2n+1}\left[-;m_1+1,\ldots,m_n+1,m_1+3/2,\ldots,m_n+3/2;3/2;\left(-z^2/4\right)^{n+1}\right] \qquad (3.10)$$

and

$$cos(zx_1)\ldots cos(zx_n)\,cos\left(\frac{z}{x_1\ldots x_n}\right) = \sum_{m_1\ldots m_n=-\infty}^{\infty}\frac{x_1^{2m_1}\ldots x_n^{2m_n}\left(-z^2/4\right)^{m_1+\ldots+m_n}}{(m_1)!\ldots(m_n)!(1/2)_{m_1}\ldots(1/2)_{m_n}}$$

$$\cdot{}_0F_{2n+1}\left[-;m_1+1,\ldots,m_n+1,m_1+1/2,\ldots,m_n+1/2;1/2;\left(-z^2/4\right)^{n+1}\right] \qquad (3.11)$$

respectively.

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