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(Dedicated to Professor H. M. Srivastava on his 62nd Birthday)

INTEGRATION OF CERTAIN PRODUCTS INVOLVING THE *H*-FUNCTION OF TWO VARIABLES AND A DOUBLE HYPERGEOMETRIC FUNCTION

By

Ritu Shrivastava
Department of Mathematics
Govt. Science College, Rewa-486001, M.P, India

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ABSTRACT

In this paper an attempt has been made to evaluate some infinite integrals involving the product of *H*-function of two variables and double hypergeometric function with the help of expansion formulae recently obtained by Srivastava [5]. The integrals obtained here are of general nature and on specialising the parameters are capable of providing many interesting and new particular cases.

1. Introduction. The *H*-function of two variables is given in [6, p. 82] and we shall define here in the following manner:

$$H[x,y] = H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} (a_j;\alpha_j,A_j)_{1,p_1} : (c_j;\gamma_j)_{1,p_2} : (e_j;E_j)_{1,p_3} \\ (b_j;\beta_j,B_j)_{1,q_1} : (d_j;\delta_j)_{1,q_2} : (f_j;F_j)_{1,q_3} \end{matrix}\right]$$

$$= \frac{(-1)}{4\pi^2}\int_{L_1}\int_{L_2} \phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)x^{\xi}y^{\eta}d\xi d\eta, \tag{1.1}$$

where

$$\phi_1(\xi,\eta) = \frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j\xi-A_j\eta)\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j\xi+B_j\eta)},$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_1}\Gamma(d_j-\delta_j\xi)\prod_{j=1}^{n_2}\Gamma(1-c_j+\gamma_j\xi)}{\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j\xi)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\gamma_j\xi)},$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{m_3}\Gamma(f_j-F_j\eta)\prod_{j=1}^{n_3}\Gamma(1-e_j+E_j\eta)}{\prod_{J=m_3+1}^{q_3}\Gamma(1-f_j+F_j\eta)\prod_{j=n_3+1}^{p_3}\Gamma(e_j-E_j\eta)},$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, q_i, n_i and m_j are non-negative integers such that $p_i \geq n_i \geq 0$, $q_i \geq 0$, $q_j \geq m_j \geq 0$ $(i=1,2,3,\ldots;$ $j=2,3,\ldots)$. The contour L_1 is in the ξ-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j-\delta_j\xi)$ $(j=1,\ldots,m_2)$ lie to the right, and the poles of $\Gamma(1-c_j'+\gamma_j\xi)$ $(j=1,\ldots,n_2)$, $\Gamma(1-a_j+\alpha_j\xi+A_j\eta)$ $(j=1,\ldots,n_1)$ to the left of the contour.

The contourl L_2 is in the η-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary , to ensure that the poles of $\Gamma(f_j-F_j\eta)$ $(j=1,\ldots,n_2)$ lie to the right, and the poles of $\Gamma(1-e_j+E_j\eta)$ $(j=1,\ldots,n_3)$ and $\Gamma(1-a_j+\alpha_j\xi+A_j\eta)$ $(j=1,\ldots,n_1)$ to the left of the contour.

Also $$R=\sum_{j=1}^{p_1}\alpha_j+\sum_{j=1}^{p_2}\gamma_j-\sum_{j=1}^{q_1}\beta_j-\sum_{j=1}^{q_2}\delta_j<0,$$

$$S=\sum_{j=1}^{p_1}A_j+\sum_{j=1}^{p_3}E_j-\sum_{j=1}^{q_1}B_j-\sum_{j=1}^{q_3}F_j<0,$$

$$U=-\sum_{j=n_1+1}^{p_1}\alpha_j-\sum_{j=1}^{q_1}\beta_j+\sum_{j=1}^{m_2}\delta_j-\sum_{j=m_2+1}^{q_2}\delta_j+\sum_{j=1}^{n_2}\gamma_j-\sum_{j=n_2+1}^{p_2}\gamma_j<0,$$

$$V=-\sum_{j=n_1+1}^{p_1}A_j-\sum_{j=1}^{q_1}B_j+\sum_{j=1}^{m_3}F_j-\sum_{j=m_3+1}^{q_3}F_j+\sum_{j=1}^{n_3}E_j-\sum_{j=n_3+1}^{p_3}E_j<0,$$

and $|\arg x|<\tfrac{1}{2}\pi U$, $|\arg y|<\tfrac{1}{2}\pi V$.

Following results of Srivastava [5,p.426, (1.3);(1.4)] (with z replaced by iz) are required in the present paper:

$$z^\lambda F\left[\begin{matrix}(a),(a');(c);(c');\\(b),(b');(d);(d');\end{matrix}-x^2z^2,-y^2z^2\right]$$

$$=\sum_{n=0}^{\infty}\frac{(\lambda+2n)\Gamma(\lambda+n)}{n!}J_{\lambda+2n}(2z)\times F\left[\begin{matrix}-n,\lambda+n,(a),(a');(c);(c');\\(b),(b');(d);(d');\end{matrix}x^2,y^2\right] \tag{1.2}$$

and

$$z^\lambda F\left[\begin{matrix}(a),(a');(c);(c');\\(b),(b');(d);(d');\end{matrix}-x^2z^2,-y^2z^2\right]$$

$$=\Gamma(1+\lambda)\sum_{n=0}^{\infty}\frac{z^n}{2^n n!}J_{\lambda+n}(2z)\times F\left[\begin{matrix}-n,1+\lambda,(a),(a');(c);(c');\\(b),(b');(d);(d');\end{matrix}x^2,y^2\right], \tag{1.3}$$

where $A+A'+C\leq B+B'+D$, $A+A'+C'\leq B+B'+D'$, and for all values of λ with possible exception of zero and negative integers. (a) represents the sequence of A parameters a_1, a_2, $\ldots$, a_A and this convention will be retained throughout the paper. The notation for double hypergeometric function is due to Burchnall and Chaundy [4,p.112] in preference, for the sake of brevity, to an earlier one introduced by Kampé de Fériet [1,p.150].

2. The results to be established here are:

$$\int_0^\infty z^{\rho+\lambda-1}\sin 2z\, F\left[\begin{matrix}(a),(a');(c);(c');\\(b),(b');(d);(d');\end{matrix}\ -x^2z^2,-y^2z^2\right] H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}\beta z^{2m}\\ \gamma\end{matrix}\right]dz$$

$$=\sum_{n=0}^{\infty}\frac{(\lambda+2n)\Gamma(\lambda+n)}{2^{1+\rho-\lambda-2n}\,n!}F\left[\begin{matrix}-n,\lambda+n,(a),(a');(c);(c');\\(b),(b');(d);(d');\end{matrix}\ x^2,y^2\right]$$

$$\times H^{0,n_1:m_2+1,n_2+1:m_3,n_3}_{p_1,q_1:p_2+3,q_2+1:p_3,q_3}\left[\begin{matrix}\beta 2^{-2m}\\ \gamma\end{matrix}\middle|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}:(½-n-\lambda/2-\delta/2,m),\\(b_j;\beta_j,B_j)_{1,q_1}:\quad(½-\delta,2m),\end{matrix}\right.$$

$$\left.\begin{matrix}(c_j;\gamma_j)_{1,p_2},(1+2n+\lambda-\rho,2m),(1-n-\lambda/2-\rho/2,m):(e_j;E_j)_{1,p_3}\\(d_j;\delta_j,)_{1,q_2}\qquad\qquad\qquad :(f_j;F_j)_{1,q_3}\end{matrix}\right], \tag{2.1}$$

where $A+A'+C\leq B+B'+D$, $A+A'+C'\leq B+B'+D'$ and $|arg\ \beta|<½\pi U$, $|arg\ \gamma|<½\pi V$, where U and V are given in section 1.

$$\int_0^\infty z^{\rho+\lambda-1}\cos 2z\, F\left[\begin{matrix}(a),(a');(c);(c');\\(b),(b');(d);(d');\end{matrix}\ -x^2z^2,-y^2z^2\right] H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}\beta z^{2m}\\ \gamma\end{matrix}\right]dz$$

$$=\sum_{n=0}^{\infty}\frac{(\lambda+2n)\Gamma(\lambda+n)}{2^{1+\rho-\lambda-2n}\,n!}F\left[\begin{matrix}-n,\lambda+n,(a),(a'):(c);(c');\\(b),(b'):(d);(d');\end{matrix}\ x^2,y^2\right]$$

$$\times H^{0,n_1:m_2+1,n_2+1:m_3,n_3}_{p_1,q_1:p_2+3,q_2+1:p_3,q_3}\left[\begin{matrix}\beta z^{-2m}\\ \gamma\end{matrix}\middle|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}:(1-n-\lambda/2-\rho/2,m),\\(b_j;\beta_j,B_j)_{1,q_1}:\quad(½-\rho,2m),\end{matrix}\right.$$

$$\left.\begin{matrix}(c_j;\gamma_j)_{1,p_2},(1/2-n-\lambda/2-\rho/2,m),(1+2n+\lambda-\rho,2m)\\(d_j;\delta_j,)_{1,q_2}\end{matrix}\right], \tag{2.2}$$

which is valid under the conditions
$A+A'+C\leq B+B'+D$, $A+A'+C'\leq B+B'+D'$, and $|arg\ \beta|<½\pi U$, $|arg\ \gamma|<½\pi V$. where U and V are given in Section 1.

Proof. To prove (2.1) take the expansion (1.2), multiply both sides by $f(z)$, integrate with respect to z between the limits 0 to ∞ and interchange the order of integration and summation we get

$$\int_0^\infty z^\lambda F\left[\begin{matrix}(a),(a'):(c);(c');\\(b),(b'):(d);(d');\end{matrix} -x^2z^2, -y^2z^2\right] f(z)dz$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda+2n)\Gamma(\lambda+n)}{n!} \times F\left[\begin{matrix}-n, \lambda+n, (a),(a');(c),(c');\\(b),(b');(d),(d');\end{matrix} x^2, y^2\right]$$

$$\times \int_0^\infty J_{\lambda+2n}(2z) f(z)dz, \tag{2.3}$$

for $A+A'+C \leq B+B'+D$, $A+A'+C' \leq B+B'+D'$, $R(\lambda+\eta+1)>0$ and $R(\lambda+\xi+1)>0$ where $f(z)=o(|z|^{\eta})$, for small z and $f(z)=o(|z|^{\xi})$, for large z.

The change of integration and summation is justified [3,p.500] because

(i) the series

$$\sum_{n=0}^{\infty} \frac{(\lambda+2n)\Gamma(\lambda+n)}{n!} J_{\lambda+2n}(2z) \times F\left[\begin{matrix}-n, \lambda+n, (a),(a');(c),(c');\\(b),(b');(d),(d');\end{matrix} x^2, y^2\right]$$

is uniformaly convergent in $0 \leq z \leq N$, N being arbitrary;

(ii) $f(z)$ is a continuous function of z for all value of $z \geq z_0 > 0$;

(iii) the integral on the left of (2.3) converges absolutely under the stated conditions.

Now on taking

$$f(z)=z^{\rho-1}\sin 2z \; H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}\beta 2^{2m}\\ \gamma\end{matrix}\right] dz$$

in (2.3), replácing H-function on the right by its equivalent contour integral as given in (1.1), changin the order of integration which is justified due to the absolute convergence of the integrals, evaluating the inner integral with the help of [2,p.328(10)] and interpreting it with (1.1), we get (2.1).

If we take

$$f(z)=z^{\rho-1}\cos 2z \, H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}\beta 2^{2m}\\ \gamma\end{matrix}\right] dz$$

proceed on the parallel lines as mentioned above and then in the light of the result [2,p. 328 (11)], we obtain (2.2).

On considering the result (1.3), proceeding on the parallel lines as mentioned above and making use of the result [2,p.328(10);p.328(11)], we get the following different forms of the integral (2.1) and (2.2) as

$$\int_0^\infty z^{\rho+\lambda-1} \sin 2z \; F\left[\begin{matrix}(a),(a'):(c);(c');\\(b),(b'):(d);(d');\end{matrix} -x^2z^2, -y^2z^2\right] H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}\beta z^{2m}\\ \gamma\end{matrix}\right] dz$$

$$=\frac{\Gamma(\lambda+n)}{2^{1+\rho+\lambda}}\sum_{n=0}^{\infty}\frac{1}{2^n n!}F\left[\begin{matrix}-n,1+\lambda,(a),(a'):(c);(c');\\(b),(b'):(d);(d');\end{matrix}x^2,y^2\right]$$

$$\times H^{0,n_1:m_2+1,n_2+1:m_3,n_3}_{p_1,q_1:p_2+3,q_2+1:p_3,q_3}\left[\begin{matrix}\beta 2^{-2m}\\ \gamma\end{matrix}\left|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}:(\tfrac{1}{2}-n-\lambda/2-\rho/2,m),\\(b_j;\beta_j,B_j)_{1,q_1}:\quad(\tfrac{1}{2}-\rho-n,2m),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(c_j;\gamma_j)_{1,p_2},(1+\lambda-\rho,2m),(1-n-\lambda/2-\rho/2,m):(e_j;E_j)_{1,p_3}\\(d_j;\delta_j,)_{1,q_2}\qquad\qquad :(f_j;F_j,)_{1,q_3}\end{matrix}\right], \tag{2.4}$$

which is valid under the same conditions as (2.1) and

$$\int_0^{\infty} z^{\rho+\lambda-1}\cos 2z\, F\left[\begin{matrix}(a),(a'):(c);(c');\\(b),(b'):(d);(d');\end{matrix}-x^2z^2,-y^2z^2\right]H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}\beta z^{2m}\\ \gamma\end{matrix}\right]dz$$

$$=\frac{\Gamma(\lambda+1)}{2^{1+\lambda}}\sum_{n=0}^{\infty}\frac{1}{2^n n!}F\left[\begin{matrix}-n,1+\lambda,(a),(a'):(c);(c');\\(b),(b'):(d);(d');\end{matrix}x^2,y^2\right]$$

$$\times H^{0,n_1:m_2+1,n_2+1:m_3,n_3}_{p_1,q_1:p_2+3,q_2+1:p_3,q_3}\left[\begin{matrix}\beta 2^{-2m}\\ \gamma\end{matrix}\left|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}:(1-n-\lambda/2-\delta/2,m),\\(b_j;\beta_j,B_j)_{1,q_1}:\quad(\tfrac{1}{2}-\delta-n,2m),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(c_j;\gamma_j)_{1,p_2},(1+\lambda-\rho,2m),(\tfrac{1}{2}-n-\lambda/2-\rho/2,m):(e_j;E_j)_{1,p_3}\\(d_j;\delta_j,)_{1,q_2}\qquad\qquad :(f_j;F_j,)_{1,q_3}\end{matrix}\right], \tag{2.5}$$

The conditions of validity for (2.5) are the same as for (2.2).

3. Particular Case. For $a=b$ and $a'=b'$, the double hypergeometric function in the left breaks up into the product of two generalised hypergeometric functions and from (2.1), we thus get

$$\int_0^{\infty} z^{\delta+\lambda-1}\sin 2z\; {}_CF_D\left[\begin{matrix}(c);\\(d);\end{matrix}-x^2z^2\right]{}_{C'}F_{D'}\left[\begin{matrix}(c');\\(d');\end{matrix}-y^2z^2\right]H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}\beta z^{2m}\\ \gamma\end{matrix}\right]dz$$

$$=\sum_{n=0}^{\infty}\frac{(\lambda+2n)\Gamma(\lambda+n)}{2^{1+\delta-\lambda-2n}\,n!}F\left[\begin{matrix}-n,\lambda+n,(c);(c');\\ :\quad (d);(d');\end{matrix}x^2,y^2\right]$$

$$\times H^{0,n_1:m_2+1,n_2+1:m_3,n_3}_{p_1,q_1:p_2+3,q_2+1:p_3,q_3}\left[\begin{matrix}\beta 2^{-2m}\\ \gamma\end{matrix}\left|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}:(\tfrac{1}{2}-n-\lambda/2-\delta/2,m),\\(b_j;\beta_j,B_j)_{1,q_1}:\quad(\tfrac{1}{2}-\delta,2m),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(c_j;\gamma_j)_{1,p_2},(1+2n+\lambda-\delta,2m),(1-n-\lambda/2-\rho/2,m);(e_j;E_j)_{1,p_3}\\(d_j;\delta_j,)_{1,q_2} \qquad\qquad ;(f_j;F_j)_{1,q_3}\end{matrix}\right], \tag{3.1}$$

The conditions of validity for (3.1) are the same (with $A=B$, $A'=B'$) as given in (2.1).

On the other hand, since

$$F\left[\begin{matrix}(a),(a');(c),(c');\\(b),(b');(d),(d');\end{matrix} x,y\right] = {}_{A+A'+C}F_{B+B'+D}\left[\begin{matrix}(a),(a'),(c);\\(b),(b'),(d);\end{matrix} x\right]$$

when $y=0$,

The special case $A=A'=B=B'=0$ of (2.1) provides us

$$\int_0^\infty z^{\rho+\lambda-1}\sin 2z\;{}_CF_D\left[\begin{matrix}(c);\\(d);\end{matrix} -x^2z^2\right] H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}\beta z^{2m}\\ \gamma\end{matrix}\right]dz$$

$$=\sum_{n=0}^\infty \frac{(\lambda+2n)\Gamma(\lambda+n)}{2^{1+\rho-\lambda-2n}\,n!}\;{}_{C+2}F_D\left[\begin{matrix}-n,\lambda+n,(c);\\(d);\end{matrix} x^2\right]$$

$$\times H^{0,n_1:m_2+1,n_2+1:m_3,n_3}_{p_1,q_1:p_2+3,q_2+1:p_3,q_3}\left[\begin{matrix}\beta 2^{-2m}\\ \gamma\end{matrix}\left|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}:(\tfrac{1}{2}-n-\lambda/2-\rho/2,m),\\(b_j;\beta_j,B_j)_{1,q_1}: \qquad (\tfrac{1}{2}-\rho,2m),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(c_j;\gamma_j)_{1,p_2},(1+2n+\lambda-\rho,2m),(1-n-\lambda/2-\delta/2,m):(e_j;E_j)_{1,p_3}\\(d_j;\delta_j,)_{1,q_2} \qquad\qquad :(f_j;F_j)_{1,q_3}\end{matrix}\right], \tag{3.2}$$

which is valid under the same conditions as for (2.1) with $A=A'=B=B'=C'=D'=0$.

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