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SOME OPERATION- TRANSFORM FORMULAE FOR S_μ- TRANSFORM

By

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1. Introduction. In an earlier Paper [1], S_μ- transform has been defined by

$$(1.1)\quad S_\mu [f(t)] = F(s) = \int_0^\infty f(t) \frac{e^{-\mu(s/t)}}{s+t} dt, \ (\mu \geq 0),$$

where $f(t)$ is a suitably restricted conventional function defined on the real line $0 < t < \infty$ and $0 < Re(S) < \infty$. It has been generalised in the case of generalised functions as

$$(1.2)\quad S_\mu [f(t)] = F(s) = \left< f(t), \frac{e^{-\mu(s/t)}}{s+t} \right>, \ (\mu \geq 0).$$

Its inversion formula has been also derived. Here it is proposed to discuss some operation-transform formulae of the transform given by (1.1).

2. Operation- Transform Formulae for S_μ- Tranform

Differentation : If $\phi \in B_\mu$, where B_μ is the space of all complex valued smooth functions $\phi(t)$ such that for each $\phi(t) \in B_\mu$, we have

$$\rho_n(\phi) = \sup_{0<t<\infty} |D^n \phi(t)| \qquad (n = 0,1,2,...)$$

bounded.

We shall prove that

$$(2.1)\quad \rho_n[-D\phi] = \rho_{n+1}[\phi].$$

Since,

$$\rho_n[-D\phi] = \sup_{0<t<\infty} |D^n(-D\phi)|$$
$$= \sup_{0<t<\infty} |D^{n+1}\phi|$$
$$= \rho_{n+1} |\phi|.$$

Therefore, we get

$$\rho_n[-D\phi] = \rho_{n+1}[\phi]$$

From (2.1) it follows that $\phi \to -D\phi$ is a continuous and linear mapping of B_μ on to itself. Therefore, from Theorem 1. 10–1 due to Zemanien [2.

p. 29], the adjoint mapping $f \to Df$ is also a continuous and linear mapping of β'_μ on to itself where B'_μ is the dual of B_μ and therefore we get

(2.2) $< Df(t), \phi(t) > = < f(t), -D\phi(t) >$

Now, we prove that the following operation-transform formula

(2.3) $S_\mu [D^n f] \leq K.S_\mu [|f(t)|]$

Proof. Using, the generalised definition of S_μ-transform and the relation (2.2), we get

$$S_\mu [D^n f] = < D^n f(t), \frac{e^{-\mu(s/t)}}{s+t} >$$

$$= < f(t), (-D)^n \frac{e^{-\mu(s/t)}}{s+t} >$$

$$= < f(t), \sum_{\upsilon=0}^{n} {}^n c_\upsilon (-D)^{n-\upsilon} e^{-\mu s/t} (-D)^\nu \frac{1}{s+t} >$$

Therefore, we get

(2.4) $$S_\mu [D^n f(t)] = < f(t), \frac{e^{-\mu(s/t)}}{s+t} . \frac{P_n(t)}{Q_n(t)} >$$

where $P_n(t)$ and $Q_n(t)$ are the polynomials in t such that order of $Q_n(t) \geq$ order of $P_n(t)$.

Let us suppose that f is a regular generalised function of B'_μ. Therefore, for, $\phi \in \beta_\mu$, we have

$$< f, \phi > = \int_0^\infty f(t)\,\phi(t)\,dt$$

and

$$|< f, \phi >| \leq \int_0^\infty |f(t)|\ |\phi(t)|\,dt$$

Consequently, we get

(2.5) $|< f, \phi >| \leq < |f|, |\phi| >$.

An appeal to (2.4) and (2.5) gives

$$|S_\mu [D^n f]| \leq < |f(t), |\frac{e^{-\mu s/t}}{s+t}|.|\frac{P_n(t)}{Q_n(t)}| >$$

$$\leq < |f(t), |\frac{e^{-\mu s/t}}{s+t}|. K >,$$

where $|\frac{P_n(t)}{Q_n(t)}| \leq K$ (const); $0 < t < \infty$; $0 < \mu.s < \infty$ and $\mu \geq 0$.

Therefore, we get

$$|S_\mu [D^n f]| \leq K < |f(t), |\frac{e^{-\mu s/t}}{s+t}| >$$

$$\leq K.\ S_\mu [|f(t)|].$$

This completes the proof.

Multiplication by an Exponential Function. Let μ be a real number such that $\mu \geq 0$. Now we prove that $\phi(t) \to e^{-\mu t}\phi(t)$ is a continuous and linear mapping from B_μ on to itself.

Proof. Let $\phi \in B_\mu$, We have

$$D^n[e^{-\mu t}\phi(t)] = \sum_{\upsilon=0}^{n} {}^nc_\upsilon D^{n-\upsilon}e^{-\mu t} D^\upsilon \phi(t)$$

$$= \sum_{\upsilon=0}^{n} {}^nc_\upsilon (-\mu)^{n-\upsilon} e^{-\mu t} D^\upsilon \phi(t).$$

Therefore, we get

$$|D^n[e^{-\mu t}\phi(t)]| \leq \sum_{\upsilon=0}^{n} K\, |D^\upsilon \phi(t)|$$

where $|\, {}^nc_\upsilon (-\mu)^{n-\upsilon} e^{-\mu t}| \leq K$ for $\mu \geq 0$ and $0 < t < \infty$.

Thus we get

(2.6) $\rho_n[e^{-\mu t}\phi(t)]\, K \leq \sum_{\upsilon=0}^{n} |\rho_\upsilon[\phi(t)]| \quad (n = 0,1,2, \ldots ; \upsilon = 0,1,2, \ldots).$

From (2.6), it follow that $\phi(t) \to e^{-\mu t}\phi(t)$ is a continuous and lineir mapping of B_μ on to itself, Therefore, from Theorem 1.10-1 due to Zemanian [2,p.29] the adjoint mapping $f \to e^{-\mu t} f$ is also a continuous and linear mapping of B'_μ on to itself and we get

(2.7) $< e^{-\mu t} f(t), \phi(t) > = < f(t), e^{-\mu t}\phi(t) >.$

An appeal to (2.7) and the generalised definition of S_μ-transform, we get

$$S_\mu[e^{-\mu t} f(t)] = < e^{-\mu t} f(t), \frac{e^{-\mu s/t}}{s+t} >$$

$$= < f(t), e^{-\mu t} \frac{e^{-\mu s/t}}{s+t} >.$$

Therefore,

$$|S_\mu[e^{-\mu t} f(t)]| \leq < |f(t)|, |e^{-\mu t}|.|\frac{e^{-\mu s/t}}{s+t}| >$$

by (2.5) if f is a regular generalised funciton

$$\leq \quad M < |f(t)|, \frac{e^{-\mu s/t}}{s+t} >$$

$$\leq \quad M S_\mu[\,|f(t)|\,]$$

where $|e^{-\mu t}| \leq M$

Thus we get an operation-transform formula

(2.8) $|S_\mu[e^{-\mu t} f(t)]| \leq M S_\mu[\,|f(t)|\,].$

Multiplication by $(s+t)^{-\lambda}$ where $\lambda > 0$; $0 < t < \infty$ and $0 < s < \infty$.

We prove that $\phi(t) \to (s+t)^{-\lambda}\phi(t)$ is a continuous and linear mapping of B_μ on to itself, where $\lambda > 0$; $0 < t < \infty$ and $0 < s < \infty$.

Proof. Let $\phi \in B_\mu$, we have

$$D_n[(s+t)^{-\lambda}\,\phi(t)] = \sum_{\upsilon=0}^{n} {}^n c_\upsilon\, D^{n-\upsilon}(s+t)^{-\lambda}\, D^\upsilon \phi(t)$$

$$= \sum_{\upsilon=0}^{n} {}^n c_\upsilon\,(-\lambda)(-\lambda-1)\ldots(-\lambda-(n-\upsilon-1)(s+t)^{\lambda-\upsilon+\upsilon}\, D^\upsilon \phi(t).$$

Therefore, we get

$$|D_n[(s+t)^{-\lambda}\,\phi(t)]| \leq M \sum_{\upsilon=0}^{n} |D^\upsilon \phi(t)|$$

where $|{}^n c_\upsilon\,(-\lambda)(-\lambda-1)\ldots(-\lambda-(n-\upsilon-1)(s+t)^{\lambda-\upsilon+\upsilon}| \leq M.$

$(n = 0,1,2, \ldots. ; \upsilon = 0,1,2,\ldots)$

Thus, we get

(2.9) $\rho_n\,[(s+t)^{\lambda}\,\phi(t)] \leq M \sum_{\upsilon=0}^{n} \rho_\upsilon\,[\phi(t)].$

From (2.9) it follows that $\phi(t) \to (s+t)^{-\lambda}\,\phi(t)$ is a continuous and linear mapping of B_μ on to itself. Therefore, from the theorem 1.10–1 due to Zemanian [2.p 29]. the adjoint mapping $f \to (s+t)^{-\lambda} f$ of $\phi \to (s+t)^{-\lambda}\phi$ is also a continuous and linear mapping of B'_μ on itself and we get

(2.10) $< (s+t)^{-\lambda} f(t).\ \phi(t), \quad f(t).(s+t)^{-\lambda}\,\phi(t)>.$

An appeal to (2.10) and the generalised definition of S_μ-tranform gives

$$S_\mu[(s+t)^{-\lambda} f(t)| = <(s+t)^{-\lambda} f(t), e^{-\mu s/t}>$$

$$= < f(t), (s+t)^{-\lambda} \frac{e^{-\mu s/t}}{s+t} > .$$

If be a regular generalised function then by using (2.5), *we get*

$$|S_\mu[(s+t)^{-\lambda} f(t)]| \leq < |f(t)|, |(s+t)^{-\lambda}|\, |\frac{e^{-\mu s/t}}{s+t}| >$$

$$\leq N < |f(t)|, \frac{e^{-\mu s/t}}{s+t} > \;\leq N\, S_\mu\,[\,|f(t)|\,]$$

where $\quad 0 \leq |(s+t)^{-\lambda}| \leq N.$

Thus we get an operation-transform formula

(2.11) $|S_\mu[(s+t)^{-\lambda} f(t)]| \leq N\, S_\mu\,[\,|f(t)|\,].$

Shifting. Let T be a fixed real number such that $0 < t + T < \infty$ and $0 < t < \infty$. Let $\phi(t) \in B_\mu$ Now we shall prove that $(t+T)$ is a continuous and linear mapping of B_μ on to itself.

Proof. Let us consider

$$D^n\,[\phi(t+T)] = (d/dt)^n\; |\phi\,(t+T)|$$

$$= [\frac{d}{d(1+T)}\;\frac{d(t+T)}{dt}]^n\; |\phi\,(t+T)|$$

$$= \left(\frac{d}{d(t+T)}\right)^n [\phi\,(t+T)]$$
$$= D^n_{t+T}\,\phi\,(t+T)$$
$$= D^n_{t_1}\,[\phi\, t_1],\ [t_1 = t+T]$$

where $0 < t + T < \infty$ and $0 < t < \infty$.

Therefore, we get

(2.12) $D^n_t\,[\,\phi(t+T)] = D^n_t\,[\,\phi(t)],\ [t_1 = t]$

i.e. $\rho_n\,[\,\phi(t+T)] = \rho_n\,[\,\phi(t)]$.

Thus from (2.12), it follows that $\phi\,(t) \to \phi(t+T)$ is a continuous and linear mapping of B_μ on to itself. Its inners sapping $\phi\,(t) \to \phi(t+T)$ is also a continuous and linear mapping of B_μ to on itself. Therefore $\phi\,(t) \to (t+T)$ is an isomorphiam of B_μ onto itself, the adjoint mapping of $\phi\,(t) \to \phi(t+T)$ is $f(t) \to f\,(t+T)$ which is also a continuous and linear mapping of B'_μ onto itself due to Theorem 1.10-1 of Zemanian [2,p. 29] and we get

(2.13) $< f\,(t+T), \phi\,(t) > = < f(t), \phi\,(t+T) >$

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