Jñānābha, Vol. 30, 2000

WAITING TIME DISTRIBUTION IN A REPAIR SYSTEM WITH MULTIDIMESIONAL STATE SPACE AND RANDOM BATCH ARRIVAL FOR EXCHANGABLE ITEMS

By

S.S. Mishra and S. Rizvi

Department of Mathematics and Statistics,

Dr. R.M.L. Avadh University, Faizabad-224001, U.P., India

(Received : July 5, 1999)

ABSTRACT

In the present paper, we consider multidimensional state space with random batch arrival and compute the probability distributions of the waiting times under steady state for exchangable items in the system.

We finally discuss the working and non-working states of the system and present the transitional probability distributions of it under these two states by making use of Markov Process and Kolmogorov forward Equations.

1. Introduction. Stochastic networks are models for flexible manufacturing system (*FMS*) computer networks, communication systems and complex repair system etc. From a customer's point of view, perhaps the most important performance measure of a such system is the sojourn time (or moment of it) in the network or in parts of it. These sojourn times represent (individual) production or repair times for an items, the time for a data packet to be transmitted by the communication system etc. In all these cases, we observe the individual possibly random route of a typical (tagged) customer from arrival at the system until departure. For a review of the results concerning the distribution of these individual sojourn times in stochastic networks, see for example Schassberger [5] and Diney and keing [4].

Breg and Posner [1] observe that these are important problems with sojourn times (delay times) in repair system or *FMS* that can not be modeled by standard individual sojourn times. These problems arise (in the example of Breg and Posner [1], p. 287) when customers bring failed items to the repair system and the items are exchangable in the sense that a customer does not necessarily want to back the particular item be brought into the system but rather any good item of the same kind. They distinguish a "Customer queue" under a first come first served

(FCFS) discipline where the customers wait for any repaired item and "item queue" of ***M/M/C-FCFS*** system where the items wait for repair at one of the repair channels. The multichannel structure of the repair system implies the possibility of overtaking for items such that customer does not necessarily obtain his own item back.

While delay times usually are computed for individual items at the item queue (which is easy to do for the *M/M/C* system under consideration) Breg and Posner [1] investigated delay times at the customer queue, a problem which turned out to be much more complicated. In addition, they introduce spares at the service centre which, if at hand, may be handed immediately to customers who brought in failed items.

Clearly, such problems arise in general complex repair system or in *FMS* (for an example see Schroder [6]) which leads to the questicns : what can be said about the delay distribution when customers are waiting in a queue (under some general discipline) for exchangable items which migrate through a general stochastic network.

A careful investigation of the proof of Berg and Posner [1] shows that it strongly depends on the fact that the state space of the system is one dimensioanl and it turns out that the necessity of more complicated state space to describe general (multiqueue) repair systems which lead to some apparently new problems. They considered the simplest system with two dimensional state space i.e. two different parallel sources each having its own queues under ***FCFS*** and computed the steady state probability distribution of the waiting times for exchangable items in that systems. They preferred two statespace case avoiding complexities of probabilty distribution, their generating functions and *LSTs* in the analysis of the multistate problem.

In the present work, we consider multidimensional state space with random batch arrival and compute the probability distribution of the waiting times under steady state for exchagable items in the system assuming that size of the batch of the customers be represented by random variable X such that $E(X) = a$.

We finally discuss the working and non-working states of the system and present the transitional probability distributions of it under these two states by making use of Markov Process and Kolmogorov Forward Equations.

2. The System. Customers arrive to the system (see figure) in Poisson stream of intensity a $\lambda > 0$ at a repair station deliver a failed item to the station and proceed to customer queue, which is organized by using ***FCFS*** rule. The items ae assumed to be exchangable that is customers accept any repaired item that is given to them. An item leaving the repair station is given to the customer at the head of queue which immediately departs from the queue.

The repair station consists of n-batch servers each having its own queue with queueing discipline ***FCFS***. An arriving customer joins the queue of server 1 with probability $p_1 \in (0,1)$, server 2 with probility $p_2 \in (0,1)$ ... server n-1 with probability $p_{n-1} \in (0,1)$ and the queue of server n with probability $p_n = 1 - \sum_{i=1}^{n-1} p_i$. The service times at server 1 is $exp\ (\mu_i)$, server 2 is $exp\ (\mu_2)$ ... server $n-1$ is $exp\ (\mu_{n-1})$, and at server n is $exp(\mu_n)$ distributed. We assume the family of inter arrival times, service times and routing decisions to be an independent family of random variables.

Because we are intrested in steady state results, we assume that $\lambda p_1 < \mu_1, \lambda p_2 < \mu_2, \dots, \lambda p_{n-1} < \mu_{n-1}$ *and* $\lambda p_n < \mu_n$.

A Markovian description of the systems development over time is given by recording the joint queue length process at server $1, 2, \dots, n-1$ and n. We assume that the associated Markov process is stationary with steady state distribution.

$$\Pi(m_1, m_2, \dots, m_{n-1}, m_n) = \prod_{i=1}^{n} \left(1 - \frac{a\lambda p_i}{\mu_i}\right)\left(\frac{a\lambda p_i}{\mu_i}\right)^{m_i} \prod_{i=1}^{n-1} \left(1 - \frac{a\lambda p_i}{\mu_i}\right)\left(\frac{a\lambda p_i}{\mu_i}\right)^{m_i}$$

$$(m_1, m_2, \dots, m_{n-1}, m_n) \in N^n$$

Theorem.

$$f(s) = \prod_{i=1}^{n}\left[\frac{\mu_i - a\lambda p_i}{\mu_i - a\lambda p_i\left[\frac{\mu_i}{\mu_i + \mu_{i+1}}\right]} \cdot \frac{\mu_{i+1} - a\lambda p_{i+1}}{\mu_{i+1} - a\lambda p_{i+1}\left[\frac{\mu_{i+1}}{\mu_i + \mu_{i+1}}\right]s}\left(1 + \frac{s}{\mu_i + \mu_{i+1}}\right)\right]$$

$$- \prod_{i=1}^{n}\left[\frac{s}{\mu_i + \mu_{i+1}}\left\{p_{i+1}\frac{\mu_i - a\lambda p_i}{\mu_{i+1} - a\lambda p_{i+1} + \left[\frac{\mu_n}{\mu_i + \mu_{i+1}}\right]s}\left(1 - s\frac{a\lambda p_{i+1}}{\mu_{i+1} - a\lambda p_{i+1}}\right)\right.\right.$$

$$\left(a\lambda p_i + \mu_i + \mu_{i+1} - \frac{\mu_{i+1}}{\mu_{i+1} + a\lambda p_{i+1}}s - \frac{1}{\theta(\mu_{i+1} p_{i+1} a\lambda p_i \mu_i)}\right)^{-1}$$

$$+ \prod_{i=1}^{n} p_i \frac{\mu_i - a\lambda p_{i+1}}{\mu_i - a\lambda p_i + \left[\frac{\mu_i}{\mu_i + \mu_{i+1}}\right]}\left(1 - s\frac{a\lambda p_{i+1}}{\mu_i - a\lambda p_i}\right)$$

$$\left.\left.\left(a\lambda p_{i+1} + \mu_i + \mu_{i+1} - \frac{\mu_i}{\mu_i + a\lambda p_i}s - \frac{1}{\theta(\mu_i, p_i, \mu_i, s)}\right)^{-1}\right\}\right]$$

where $\theta(\eta, r, w, s) = \frac{A}{2a\lambda r w}\eta > 0,\ r \in [0,1],\ w \geq 0,\ s \geq 0$

where $A = \eta \dfrac{(\lambda\,(1-2r)+\mu_i+\mu_{i+1}+s\,)}{-\sqrt{\eta^2\,(\lambda\,(1-2r)+\mu_i+\mu_{i+1}+s\,)^2-4a\lambda p_i p_{i+1}\,\mu_i\mu_{i+1}}}$.

Proof. We introduce a sequence of conditional *LSTs* associated with the departure proce ss from the system.

Suppose that the system jumps any transition into the state

$(m_1, m_2, \ldots, m_{n-1}, m_n) \in N^2 - [0,0]$

and a clock is started at this jump instant. For some $k \in [1, 2, \ldots m_1 + m_2, \ldots, m_{n-1} + m_n]$. Let the clock be stopped at the k^{th} departure instant after the clock is started and denoted by

$f\,(k\,;\, m_1, m_2, \ldots, m_{n-1}, m_n)\,(s)\,,\ s \geq 0.$

The *LST* of the distribution of the time recorded by that clock. Then

$f(s)=\Sigma\pi(m_1,m_2,\ldots,m_n)\{p_1 f(m_1+m_2+1,m_1+1,m_2)(s)+p_2\, f(m_1+m_2+1,m_2+1)(s)\},$

$\ldots, + p_n\, f(m_1, m_2, \ldots, m_{n-1}, m_n, m_{n+1} = m_1)(s)\},\ s \geq 0$ ...(1)

The key to our analysis is the following sequence of first entrance equation for the conditional *LSTs*. For

$1 \leq k \leq m_1 + m_2 + \ldots + m_n\,.$

$$f(k;\, m_1, m_2, \ldots, m_n)\,(s) = \frac{\pi(m_1, m_2, \ldots, m_n)}{\pi(m_1, m_2, \ldots, m_n)+s}\left[\frac{a\lambda p_1}{\pi(m_1, m_2)}\right.$$

$$f\,(k;m_1+1,\, m_2)(s) + \ldots + \ldots + \frac{a\lambda p_n}{\pi(m_1)} f(k;\, m_n+1,\, m_1)(s) + I_{m_{1>0}} \frac{\mu_1}{\lambda(m_1, m_2)}$$

$$f\,(k-1, m_2-1, m_2)(s) + I_{m_{2>0}} \frac{\mu_2}{\pi\ m_2, m_3} f\,(k-1,\, m_2-1,\, m_3)(s) + \ldots +$$

$$\left. I_{m_{n>0}}\ \frac{\mu_n}{\pi\ (m_n, m_1)} f\,(k-1,\, m_n,\, m_1-1\,)(s)\right]\,,\ s \geq 0 \qquad \text{...(2a)}$$

where $\lambda\ (m_1, m_2, \ldots, m_n) = a\lambda + I_{m_{1>0}}\,\mu_1 + \ldots + I_{(m_{n>0})}\,\mu_n$ and for

$m_1 + m_2 + m_3 + \ldots + m_n \geq 0,\ f(0,\, m_1, m_2, \ldots, m_n)(s) = 1,\ s \geq 0$ ... (2b)

We shall concentrate on cmputing in first step

$$F(0,s) = \left(\sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty}\ \prod_{i=1}^{m}\ \left(\frac{a\lambda p_i}{\mu_i}\right)^{m_i} . f\,(m_1 + m_2 + \ldots + m_n)(s),\ s \geq 0 \qquad \text{...3}$$

which by a symmetry argument will provide (1). It turns out that we have to compute a recursion for the sequence

$F(h,s)\ ;\ h = 0,1,\ldots,\ s \geq 0$

to eventually obtain $F(0,s)$, where

$$F(h,s) = \sum_{m_1=h_1}^{\infty} \ldots \sum_{m_n=h_n}^{\infty} \prod_{i=1}^{m} \left(\frac{a\lambda p_i}{\mu_i}\right)^{m_i} . f\,(\,m_1 + m_2 + \ldots + m_n + 1 - h_i, m_i + 1, \ldots, m_n\,)s,$$

$$s \geq 0,\ h_i = 0,1\ldots \qquad \text{...(4)}$$

For the given $h_i \in\ n,\ m_i \in\ h_i, \ldots,\ m_{n-1} \in h_i\,, m_n \geq 0$ and

$K = m_1 + m_2 +, \ldots, m_n - h_i$

we multiply (2a) *(with $m_1 + 1$, $m_2 + 1, \ldots$, instead of $m_1, m_2, \ldots, +m_n$) by*

$(\Lambda\ (m_1 + 1,\ m_2 + 1, \ldots, m_{n-1}, m_n) + s) \prod_{i=1}^{n} \left(\frac{\lambda p_i}{\mu_i}\right)^{m_i}$

and summing over $m = h_i, h_i + 1, \ldots, m_n = 0, 1, \ldots,$

we obtain, after some manipulations

$$F(h_1,s) = F(h_1 + 1, s)\frac{\mu_1 + \mu_2}{\mu_1 + \mu_2 + s} + \frac{\alpha\lambda p_i}{\mu_1}\frac{\mu_1}{(\mu_1 + \mu_2 + s)}$$

$$+ (D(h_1,s) + D(h_1 + 1; s)\left(\frac{\mu_2}{\alpha\lambda p_1}\right)\frac{\mu_1}{(\mu_1 + \mu_2 + s)}$$

$$+ (c(h_1,s) + c(h_1 + 1; s)\frac{\mu_1}{\mu_1 + \mu_2 + s}, s \ge 0,\ h_1 \in \mathrm{N}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$F(h_n,s) = F(h_n + 1; s)\frac{\mu_n + \mu_1}{\mu_n + \mu_1 + s} + \frac{\alpha\lambda p_n}{\mu_n}\frac{\mu_n}{(\mu_1 + \mu_n + s)}$$

$$+ (D(h_n,s) + D(h_n + 1; s)\left(\frac{\mu_n}{\alpha\lambda p_n}\right)\frac{\mu_n}{(\mu_n + \mu_1 + s)}$$

$$+ (c(h_n,s) + c(h_n + 1, s)\frac{\mu_n}{\mu_n + \mu_1 + s}, s \ge 0,\ h_1 \in \mathrm{N}.$$

We can write the above expression in general terms

$$F(h_i;s) = F(h_i + 1; s)\ \frac{\mu_i + \mu_{i+1}}{\mu_i + \mu_{i+1} + s}\ \frac{\alpha\lambda p_i}{\mu_i}\ \frac{\mu_i}{(\mu_i + \mu_{i+1} + s)}$$

$$+ (D(h_i,s) + D(h_i + 1; s)\left(\frac{\mu_i}{\alpha\lambda p_n}\right)\frac{\mu_i}{(\mu_i + \mu_{i+1} + s)}$$

$$+ (c(h_i,s) - c(h_i + 1; s),\frac{\mu_i}{\mu_i + \mu_{i+1} + s}, s \ge 0,\ h_1 \in \mathrm{N}$$

where

$$C(h_1;s) = \sum_{m_1 = h_1}^{\infty}\left(\frac{\alpha\lambda p_1}{\mu_1}\right)^{m_1} f(m_1 + 1 - h_1; m_1 + 1, 0)(s), s \ge 0, h_1 \in \mathrm{N}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$C(h_n;s) = \sum_{m_n = h_n}^{\infty}\left(\frac{\alpha\lambda p_n}{\mu_n}\right)^{m_n} f(m_n + 1 - h_n; m_n + 1, 0)(s), s \ge 0, h_n \in \mathrm{N}$$

$$C(h_i;s) = \sum_{m_1 = h_1}^{\infty}\left(\frac{\alpha\lambda p_i}{\mu_i}\right)^{m_n} f(m_i + 1 - h_i; m_i + 1, 0)(s), s \ge 0, h_i \in \mathrm{N}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$D(h_i;s) = \sum_{m_1 = 1}^{\infty}\left(\frac{\alpha\lambda p_1}{\mu_1}\right)^{h_1}\left(\frac{\alpha\lambda p_2}{\mu_2}\right)^{m_2} f(m_1, h_1; m_2)(s), s \ge 0, h_1 \in \mathrm{N}$$

$$D(h_i;s) = \sum_{m_n = 1}^{\infty}\left(\frac{\alpha\lambda p_n}{\mu_n}\right)\left(\frac{\alpha\lambda p_n}{\mu_n}\right)^{m_n} f(m_n, h_n; m_n)(s), s \ge 0, h_n \in \mathrm{N}$$

$$D(h_i;s) = \sum_{m_i = 1}^{\infty}\left(\frac{\alpha\lambda p_i}{\mu_i}\right)\left(\frac{\alpha\lambda p_i}{\mu_i}\right)^{m_i} f(m_i, h_i; m_i)(s), s \ge 0, h_i \in \mathrm{N}.$$

Evaluating the generating functions $\sum_{h_n=1}^{\infty}(h_n,s)z^{h_n}, |z|<1$, for any $s\geq 0$, at $z=\left(\frac{\mu_n+\mu_1}{\mu_n+\mu_1+s}\right)$

we eventually obtain

$$F(0,s)=\frac{\mu}{\mu_1(\mu_1+\mu_2+s)-\alpha\lambda p_1(\mu_1+\mu_2)}$$

$$+c(0,s)\frac{\mu_2}{\mu_1+\mu_2}-D(0,s)\frac{\mu_1\mu_2}{\alpha\lambda_1 p_1(\mu_1+\mu_2)}$$

$$-\sum_{h_1=0}^{\infty}\left(\frac{\mu_1+\mu_2}{\mu_1+\mu_2+s}\right)^{h_1}c(h,s)\frac{\mu_2}{(\mu_1+\mu_2)(\mu_1+\mu_2+s)}$$

$$+\sum_{h_1=0}^{\infty}\left(\frac{\mu_1+\mu_2}{\mu_1+\mu_2+s}\right)^{h_1}D(h_1,s)\frac{\mu_2(\mu_1+\mu_2+s)+\alpha\lambda p_1(\mu_1+\mu_2)}{\alpha\lambda p_1(\mu_1+\mu_2)(\mu_1+\mu_2+s)},\ s>0.$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$F(0,s)=\frac{\mu_n^2}{\mu_n^2(\mu_n+\mu_1+s)-\alpha\lambda p_1(\mu_n+\mu_1)}+C(0,s)\frac{\mu_n}{\mu_n+\mu_1}$$

$$-D(0,s)\frac{\mu_n\mu_1}{\alpha\lambda p_n(\mu_n+\mu_1)}-\sum_{h_n=0}^{\infty}\frac{\mu_n\mu_1}{\alpha\lambda p_n(\mu_1+\mu_2)}$$

$$-\sum_{h_n=0}^{\infty}\frac{\mu_n+\mu_1}{\mu_n+\mu_1+s}-D(h_n,s)\frac{\mu_2(\mu_2+\mu_1+s)+\alpha\lambda p_1(\mu_n+\mu_1)}{\alpha\lambda p_n(\mu_1+\mu_n)(\mu_1+\mu_n+s)}.$$

Finally, general expression can be established as

$$F(0,s)=\frac{\mu_i^2}{\mu_i(\mu_i+\mu_{i+1}+s)-\alpha\lambda p_1(\mu_i+\mu_{i+1})}-$$

$$+C(0,s)\frac{\mu_{i+1}}{\mu_i+\mu_{i+1}}-D(0,s)\frac{\mu_i\mu_{i+1}}{\alpha\lambda p_i(\mu_1+\mu_{i+1})}$$

$$-\sum_{h_i=0}^{\infty}\left(\frac{\mu_i+\mu_{i+1}}{\mu_i+\mu_{i+1}+s}\right)^{h_i}-D(h_i,s)\frac{\mu_i(\mu_i+\mu_{i+1}+s)+\alpha\lambda p_n(\mu_i+\mu_{i+1}+s)\mu_i}{\alpha\lambda p_i(\mu_i+\mu_{i+1})(\mu_i+\mu_{i+1}+s)} \quad ...(5)$$

It remains to compute the boundary terms. This will be done using the observations that the following equations hold

$$f(k;k+m_1,m_2)(s)=f(k;k,m_2)(s),\ k,m_1,m_2\in N \quad ...(6a)$$
$$f(k;m_1,k+m_2)(s)=f(k;m_1,k)(s),\ k,m_1,m_2\in N \quad ...(6b)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$f(k;k+m_{n-1},m_n)(s)=f(k;k,m_n)(s),\ k,m_{n-1},m_n\in N \quad ...(6c)$$
$$f(k;m_{n-1},k+m_n)(s)=f(k;m_{n-1},k)(s),\ k,m_{n-1},m_n\in N \quad ...(6d)$$
$$f(k;k+m_n,m)(s)=f(k;k,m_{n-1})(s),\ m_n,m_n\in N \quad ...(6e)$$
$$f(k;m_n,k+m_1)(s)=f(k;m_n,k)(s),\ k,m_n,m_1\in N \quad ...(6f)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$f(k;m_i,k+m_{i+1})(s)=f(k;m_i,k)(s),\ k,m_i,m_{i+1}\in N \quad ...(6g)$$

These equations are easily interpreted. In the time for the next departures from system that to be computed, we can neglect queuing customers that set atleast k customers in front of themselves (in their own queue) because after the departure of these k customers the requested time has already passed.

We introduce,

$$B(j,s) = \sum_{m_1=0}^{\infty}\left(\frac{a\lambda p_1}{\mu_n}\right)^{m_1} f(m_1+1,m_1+1,j)(s); s \geq 0,\ j \in N$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$B(j_n,s) = \sum_{m_n=0}^{\infty}\left(\frac{a\lambda p_n}{\mu_n}\right)^{m_n} f(m_n+1,m_n+1,j_n)(s); s \geq 0,\ j \in N$$

We can write the general expression of the above equations

$$B(j_i,s) = \sum_{m_i=0}^{\infty}\left(\frac{a\lambda p_i}{\mu_i}\right)^{m_i} f(m_i+1,m_i+1,j_i)(s_i); s \geq 0,\ j_i \in N$$

Now,

$$C(h_1,s) = \sum_{m_1=h_1}^{\infty}\left(\frac{a\lambda p_1}{\mu_1}\right)^{m_1} f(m_1+1-h_1,m_1+1-h_1,0)(s); s \geq 0,\ h_1 \in N$$

$$= B(0,s)\left(\frac{a\lambda p_i}{\mu_i}\right), s \geq 0\ h_1 \in N.$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$C(h_n,s) = \sum_{m_1=0}^{\infty}\left(\frac{a\lambda p_n}{\mu_n}\right)^{m_n} f(m_n+1-h_n,m_n+1-h_n,0)(s); s \geq 0,\ h_n \in N.$$

Thus general expression from equations (6a) (6g) may be obtained as

$$C(h_i,s) = \sum_{m_i=h_i}^{\infty}\left(\frac{a\lambda p_i}{\mu_i}\right)^{m_i} f(m_i+1-h_i,m_i+1-h_i,0)(s); s \geq 0,\ h_i \in N.$$

To obtain $B(0,s)$ note that for the conditional *LSTs* introduces (2a) reduces

$$f(m_1+1,m_1+1,j_1)(s)\,[a\lambda p_2+\mu+I_{(j>0)}\mu_2+s]$$
$$= f(m_1+1,m_1+1,j_1+1)(s)a\lambda p_2 + f(m_1,m_1,j)(s)(\mu_1) + f(m_1,m_1,j-1)(a\lambda)I_{(j_1>0)}$$
$$j \in N,\ m_1 \in N,\ s \geq 0. \qquad \ldots(7)$$

$$f(m_n+1,m_n+1,j_n+1)(s)\,[a\lambda(p_1)+\mu_n+I_{(j_n>0)}\mu_1+s]$$
$$= f(m_n+1,m_n+1)(s)\,a\lambda(p_n+1)$$
$$+ f(m_n,m_n,j_n)(s)(\mu_n) + f(m_n,m_n,j_n-1)(\lambda)((\mu_{n+1}z\mu_1)I_{(j_n>0)}$$
$$j_n \in N,\ m_n \in N,\ s \geq 0 \qquad \ldots(8)$$

Multiplication of (7) and (8) by $\left(\frac{a\lambda p_1}{\mu_1}\right)^{m_1}$ and $\left(\frac{a\lambda p_n}{\mu_n}\right)^{m_n}$ respectively and summation over m_1 and $m_n \in N$ yeilds the $B(j_n,s)$ respectively.

Evaluating the generating functions

$$\sum_{j_1=0} z^{j_1} B(j_1,s), |z| < 1,\ s \geq 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\sum_{j_n=0} z^{j_n} B(j_n,s), |z| < 1,\ s \geq 0$$

at $\theta(\mu_1, p_1, \mu_2, s)$ and $\theta_1(\mu_n, p_n, \mu_1, s)$ the smaller roots of the equation

$$a\lambda p_1\mu_2 z \ldots \mu_1(\lambda(1-2p_1) + \mu_1 + \mu_2 + s)z + a\lambda p_2\mu_1 = 0$$

$$a\lambda p_{n-1}\mu_n z \ldots \mu_n(\lambda(1-2p_n) + \mu_n, \mu_1, s)z + a\lambda p_1\mu_n = 0$$

We obtain general expression as under

$$B(0,s) = \frac{\theta(\mu_i + p_i + \mu_{i+1}, s)}{1-\theta(\mu_i, p_i, \mu_{i+1}, s)} \quad \frac{\mu_i + \mu_{i+1}\,.\,\theta(\mu_i, p_i, \mu_{i+1}, s)}{\alpha\lambda p_{i+1} - \mu_{i+1}\,.\,\theta(\mu_i, p_i, \mu_{i+1} + s)}$$

and

$$C(0,s)\frac{\mu_{i+1}}{\mu_i + \mu_{i+1}} - \sum_{h_i=0}^{\infty}\left(\frac{\mu_i + \mu_{i+1}}{\mu_1 + \mu_{i+1} + s}\right)^h C(h,s)\frac{s\mu_{i+1}}{(\mu_i + \mu_{i+1})(\mu_i + \mu_{i+1} + s)}$$

$$= \frac{\theta(\mu_i, p_i, \mu_{i+1}, s)}{1-\theta(\mu_i, \mu_{i+1}, s)} \cdot \frac{\mu_i + \mu_{i+1}\,.\,\theta(\mu_i, p_i, \mu_{i+1}, s)}{\lambda p_{i+1} - \mu_{i+1}\,\theta(\mu_i, p_i, \mu_{i+1}, s)}$$

$$\mu_{i+1} \quad \frac{\mu_{i+1} + a\lambda p_i}{\mu_i(\mu_i + \mu_{i+1} + s) - a\lambda(\mu_i + \mu_{i+1})} \qquad \ldots(9)$$

To obtain $D(h,s)$, $h \in N$ note that for the conditinoal *LSTs* envolved, because of (6), (2a) reduces to

$$f(m_n,h,m_n)(s)\,[\lambda p_i + I_{(h>0)}\,\mu_i + \mu_i + s]$$
$$= f(m_i,h+1,m_i)(s)[a\lambda p_i + I_{(h>0)}\,\mu_i f(m_n-1,h-1,m_n-1]$$
$$+\mu_n f(m_n-1,h,m_n-1)\,(s)\ s \geq 0,\ h \in N,\ m_n \geq 1. \qquad \ldots(10)$$

Multiplication of (6) by $\left(\frac{a\lambda p_i}{\mu_i}\right)^h \left(\frac{a\lambda p_{i+1}}{\mu_{i+1}}\right)^{m_n}$ and summation over

$m_n = 1, 2 \ldots$ *yeilds*

$$D(h,s)[a\lambda\,(1-2\,p_{i+1}) + 1_{(h>0)}\,\mu_i + \mu_{i+1} + s]$$

$$= \left(\frac{a\lambda p_i}{\mu_i}\right)^h a\lambda p_{i+1}\,(1 + I_{(h_i>0)}\,\frac{\mu_i}{\mu_{i+1}})$$

$$+ D\,(h_{i+1}, s)\,\mu_i$$

$$+ D(h-1,s)\,\frac{a\lambda^n p_i p_{i+1} I_{(h_i>0)}}{\mu_i},\ s \geq 0,\ h \in N.$$

For generating function, we therfore have

$$\sum_{h=0}^{\infty} Z^h D(h,s)\left[(a\lambda(1-2p_{i+1}) + \mu_i + \mu_{i+1} + s) - \frac{\mu_i}{2} - \frac{a\lambda^2 p_i p_{i+1} z}{\mu_{i+1}}\right.$$

$$= D(0,s)\mu_i\ \left(\frac{z-1}{2}\right) + \frac{a\lambda^2 p_{i+1}\mu_i}{\mu_{i+1}} \cdot \frac{\mu_{i+1} + a\lambda p_i z}{\mu_i - a\lambda p_i z},$$

$$|z| < 1,\ s \geq 0 \qquad \ldots(11)$$

With $\theta(\mu_{i+1}, p_i, a\lambda p_i, s)$ the smaller root of the equation

$$p_i p_{i+1} - z^n - \mu_{i+1}(a\lambda(1-2p_{i+1}) + \mu_i + \mu_{i+1} + s)z + \mu_{i+1}\mu_i = 0,$$

we obtain from (11)

$$F(0,s) = \frac{a\lambda p_{i+1}}{\mu_{i+1}} \frac{\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}$$

$$\frac{\mu_{i+1} + a\lambda p_i}{\mu_i -} \frac{\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_i, p_{i+1}, a\lambda p_i, s)}$$

and

$$\sum_{h=0}^{\infty} \left(\frac{\mu_i + \mu_{i+1}}{\mu_i + \mu_{i+1} + s}\right)^h D(h,s) \frac{\mu_i(\mu_i + \mu_{i+1} + s) + a\lambda p_i(\mu_i + \mu_{i+1})}{a\lambda p_i(\mu_i + \mu_{i+1})(\mu_i + \mu_{i+1} + s)}$$

$$\mu_i - D(0,s)\frac{\mu_i + \mu_{i+1}}{a\lambda p_i,(\mu_i + \mu_{i+1})}$$

$$= -\frac{\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)} \frac{\mu_i + a\lambda p_i \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{\mu_i - a\lambda p_i \cdot \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}$$

$$\frac{p_{i+1}\mu_i}{p_i(\mu_i + \mu_{i+1})}\left[1 + \frac{s\mu_i}{\mu_i s + (\mu_i + \mu_{i+1})(\mu_{i+1} - a\lambda p_{i+1})}\right] + \frac{p_{i+1}\mu^2}{p_i}$$

$$\frac{\mu_i + (\mu_i + \mu_{i+1})(\mu_i + a\lambda p_i)}{(\mu_i s + \mu_i + \mu_{i+1})(\mu_{i+1} - a\lambda p_i)(\mu_i s + \mu_i + \mu_{i+1})(\mu_{i+1} - a\lambda p_{i+1})} \quad \text{...(12)}$$

Inserting (7) and (12) into (5) we finally obtain

$$F(0,s) = \frac{\mu_i}{\mu_i s + (\mu_i + \mu_{i+1})(\mu_{i+1} - a\lambda p_i)}$$

$$\left(1 + \frac{p_{i+1}}{\mu_i} \cdot \frac{\mu_i s + (\mu_i + \mu_{i+1})(\mu_{i+1} + a\lambda p_i)}{\mu_i s + (\mu_i + \mu_{i+1})(\mu_{i+1} + a\lambda p_i)}\right)$$

$$\frac{-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)} \frac{\mu_i + a\lambda p_i \cdot \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{\mu_i - a\lambda p_i \cdot \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}$$

$$\frac{p_{i+1}\mu_i}{p_i} \frac{\mu_{i+1} - a\lambda p_{i+1} + s}{\mu_n s + (\mu_i + \mu_{i+1})(\mu_{i+1} - a\lambda p_{i+1})}$$

$$+ \frac{\theta(\mu_i, p_i, \mu_i, s)}{1-\theta(\mu_i, p_i, \mu_i, s)} \frac{\mu_i + \mu_{i+1} \cdot \theta(\mu_i, p_i, \mu_i, s)}{a\lambda p_{i+1} - \mu_{i+1} \cdot \theta(\mu_i, p_i, \mu_i, s)}$$

$$\mu_{i+1} \frac{\mu_i - a\lambda p_i}{\mu_i s + (\mu_i + \mu_{i+1})(\mu_i - a\lambda p_i)} \quad \text{...(13)}$$

The transformation

$$\begin{pmatrix} \mu_i \\ \mu_{i+1} \\ p_i \end{pmatrix} \xrightarrow{T} \begin{pmatrix} \mu_i \\ \mu_{i+1} \\ p_i \end{pmatrix}$$

$$\frac{\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)} \; \frac{\mu_i + a\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{\mu_i - a\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}$$

$$\xrightarrow{T} \frac{\theta(\mu_i, p_i, \mu_i, s)}{1-\theta(\mu_i, p_i, \mu_i, s)} \; \frac{\mu_i + \mu_{i+1} . \theta(\mu_i, p_i, \mu_{i+1}, s)}{\alpha\lambda p_{i+1} - \mu_{i+1} . \theta(\mu_i, p_{i+1}, a\lambda p_i, s)}$$

$$\xrightarrow{T} \frac{\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)} \; \frac{\mu_i + \alpha\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{\mu_i - \alpha\lambda p_i - \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}$$

Therefore, applying T to $F(0,s)$ and inserting the results and $F(0,s)$ into (1) yeilds

$$F(0,s) = \frac{(\mu_i + \mu_{i+1})(\mu_i + \mu_{i+1} + s)(\mu_i - a\lambda p_i)(\mu_{i+1} - a\lambda p_{i+1})}{\mu_i s + (\mu_i + \mu_{i+1})(\mu_i - a\lambda p_i)(\mu_{i+1} s + \mu_i + \mu_{i+1})(\mu_{i+1} - a\lambda p_{i+1})}$$

$$-s\left\{\frac{\theta(\mu_i, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)} \; \frac{\mu_i + a\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{\mu_i - a\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}\right\}$$

$$\frac{1}{\mu_i} \; \frac{p_{i+1}}{\mu_{i+1} s} \; \frac{(\mu_i - a\lambda p_i)}{+ (\mu_i + \mu_{i+1})} \; \frac{(\mu_{i+1} - a\lambda p_{i+1})}{(\mu_i - a\lambda p_{i+1})}$$

$$+ \frac{\theta(\mu_i, p_i, \mu_i, s)}{1-\theta(\mu_i, p_i, \mu_i, s)} \; \frac{\mu_i + \mu_{i+1} . \theta(\mu_i, p_i, \mu_{i+1}, s)}{\alpha\lambda p_{i+1} - \mu_{i+1} . \theta(\mu_i, p_i, \mu_i, s)}$$

$$\frac{1}{\mu_i} \; \frac{p_i \; (\mu_i - \alpha\lambda p_i)(\mu_{i+1} - \alpha\lambda p_{i+1})}{\mu_i s + (\mu_i + \mu_{i+1})(\mu_i - \alpha\lambda p_i)}.$$

After sum direct input this yield the theroem.

3. Probability Distribution of Working and Non Working States of the System. Assumptions

We analyse the probability distirbution of the system when it is in working and non-working state as under:

Following assumptions are laid down before we compute the transition probabilities from working state of the system to non-working, vice versa and when system remains at itself.

(i) Servers are parallelly arranged.

(ii) Servers are simulteneously busy and working as one single system.

(iii) System may go in non-working state at any epoch of the time and assumed to come back in normal state i.e.; working after certain time period. Reasons of non-working may be such as idleness, failure or any type of distribution etc.

(iv) Let working state of the system be represented by 1 and non working state be represented by *0*.

3.1 Transitional Probability distribution. Let the lengths of working and non-working periods be independent random variables having negative exponential distribution with means $1/b$ and $1/c$ respectively $(b,c > 0)$.

Now

$q_{01}(\Delta t)$ = Pr {change of state from 0 to 1 in time Δt }
$= b\,\Delta t + o\,(\Delta t)$

$q_{10}(\Delta t)$ = Pr {change of state from 1 to 0 in time Δt }
$= c\,\Delta t + o\,(\Delta t)$.

Transition probabilities are

$$b_{01} = b, \quad b_{00} = -b \text{ and } b_{10} = c, \quad b_{11} = -c$$

Now the transition probabilities martix may be written as

$$A = \begin{bmatrix} -b & b \\ c & -c \end{bmatrix}$$

The Kolmogrov forward equations for $i = 0, 1$ are

$$q'_{i_0}(t) = -b\,q_{i_0}(t) + cq_{i_0}(t)$$
$$q'_{i_1}(t) = b\,q_{i_0}(t) - cq_{i_1}(t)$$

We use the following equations while finding the transition probabilities $q_{i_1}(t)$:

$$q_{00}(t) - q_{01}(t) = 1$$
$$q_{10}(t) + q_{11}(t) = 1.$$

We have

$$q'_{00}(t) + (b-c)\,q_{00}(t) = c$$

and

$$q'_{11}(t) + (b+c)\,q_{11}(t) = b$$

The solution of the frist of these differential equations is

$$q_{00}(t) = \frac{c}{b+c} + Ce^{-(b+c)t}$$

with $q_{00}(0) = 1$, we find $C = \frac{c}{b+c}$ so that

$$q_{00}(t) = \frac{c}{b+c} + \frac{c}{b+c}\,e^{-(b+c)t}$$

Hence

$$q_{01}(t) = 1 - q_{00}\frac{b}{b+c} + \frac{b}{b+c}\;{}^{-(b+c)t}$$

Proceeding exactly in the same way, the solution of the second differential equation with the initial condition $q_{11}(t) = 1$, yelids

$$q_{11}(t) = \frac{b}{b+c} + \frac{b}{b+c}\,e^{-(b+c)t}$$

and therefore

$$q_{11}(t) = 1 - q_{11}(t)$$
$$= \frac{c}{b+c} - \frac{c}{b+c} e^{-(b+c)t}$$

We can finally conclude that the present work not only envisages the multistate space with working and non-working states of the system but also it respeakes more general and elegant unification of the results including Daduna [3] and others. The results are obtainable in turn, after specializing various parameters such as $i = 1,2$; $m_i = m_1$, $m_2 = n$, $E(x) = 1$ and non-working state is assumed not to occur with the system.

REFERENCES

[1] M. Berg and M.J.M. Posner, Customer Delay in *M/M/C* Repair System with spares, *Naval Reslogist Quart.* 32 (1985), 287-299.

[2] M. Berg and M.J.M. Posner, On the Regulation of Queues of Lelt 4 (1986) 221-224.

[3] H. Daduna, Exchangeble items in Repair systems; *Delay Times, Operation Research* (1989) 349-354.

[4] R.L. Disney and D. Konig, Queucing Networks; A Survwy of their Random Process, *Siam Rev.* 27 (1985), 335-403.

[5] R. Schass Berger, *Exact Results on Response time Distribution in Networks of Queues.* In Beilher *H.* (ed). Messung, Modellerung and Bewerluing Von Rechensystemen, 1985.

[6] H. Schroder, Eine Spexielle Methods Zir Fertifufssouring beider Reparates Von Flugzengu Leufteien, *In Oshe Det al. Operation Research Proceeding,* DGOR, Springer, Berlin, (1985), *P. 239.*