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NUMERICAL SOLUTION OF THE STEADY STATE NAVIER-STOKES EQUATIONS USING ELLIPTIC SOLVER

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ABSTRACT

The motion of a liquid under action of viscous forces is governed by Navier-Stokes equations. Exact solutions of *N-S* equations are available only under some ideal assumptions. Tremendous advances in the development and use of numerical methods, mainly due to availability of fast computing devices, have given rise to various types of numerical solutions of these equations. Driven cavity problem has worked as an ideal prototype, therefore present numerical scheme, which uses a fast direct elliptic equation solver is also tried on this problem. The results obtained compare well with those of previous authors.

1. Introduction. The Navier-Stokes equations for two dimensional steady flow of an incompressible fluid may be written in the vorticity *(w)* and streem function *(ψ)* formulation as :

$$\nabla^2\omega - Re\,(\psi_y\,\omega_x - \psi_x\omega_y) = 0 \qquad \dots (1)$$

$$\nabla^2\psi = -\,\omega \qquad \dots (2)$$

satisfying

$$u = \frac{\partial\psi}{\partial y},\ v = -\frac{\partial\psi}{\partial x} \qquad \dots (3)$$

and

$$w = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad \dots (4)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$$

The equation (1) and (2) can be formulated as follows.

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} - Re\left\{\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y}\right\} = 0 \qquad \dots (5)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -w \qquad \dots (6)$$

The equations (5) - (6) together with boudary conditions constitutes a non-linear elliptic boundary value problem. The degree of non-linearity increases with Reynolds number.

The numerical solution procedure generally consists of discretizing the domain over which equations (5) -(6) are defined. This gives two systems of algebraic equations. One can obtain the solution in terms of stream function and vorticity.

Burggraf [3] was first to employ central difference scheme for the solution of equations (5)-(6). The resulting systems of eqations were solved by using iterative procedures. But Iterative procedures fail to converge even at moderate Reynolds number. This difficulty originated the idea of upwind schemes. However, it is well known that upwind differncing introduces false diffusion effects and this will lead to additional error into the numerical solution as shown by Strikwerda [21]

Roache [16] gave the idea of what he calls Laplacian Driver Method. It is worth emphasizing that the superiority of Laplacian Driver Method over other methods is its simplicity to apply on computer, because one has to solve two Poisson's equations iteratively. We have applied an elliptic equation solver to solve equations (5) and Poisson's equation solver for equation (6). The system of equations (5) and (6) is solved iteratively until a desired convergence-criterion is satisfied.

Roache [16] has reported the solution of the driven cavity flow problem for Reynolds number $Re = 20$ with a 11×11 mesh. By using our technique, we have obtained the solutions upto Reynolds number $Re =$ 500 with a 21×21 mesh. The results obtained by us compare well with those of previous authors.

2. Derivation of Boundary Conditions. Since the values of vorticity on the boundary are not explicitly given, We compute these with the help of derivative boundary conditions on ψ Assuming equation (5) is valid near the wall and using n to represent the coordinate normal to the wall.

$$w(1) = -\left[\frac{\partial^2 w}{\partial n^2}\right]_1 \qquad \dots (7)$$

where the argument and subscript '1' denote the mesh point on the wall. Using a Taylor's series expansion for ψ with step length Δn about the mesh point on the wall.

$$\psi(2) = \psi(1) + \Delta n \left[\frac{\partial\psi}{\partial n}\right]_1 + \frac{\Delta n^2}{2}\left[\frac{\partial^2\psi}{\partial n^2}\right]_1 + \frac{\Delta n^3}{6}\left[\frac{\partial^3\psi}{\partial n^3}\right]_1 + O(\Delta n^4) \qquad \ldots (8)$$

$$\psi(3) = \psi(1) + 2\Delta n \left[\frac{\partial\psi}{\partial n}\right]_1 + \frac{4\Delta n^2}{2}\left[\frac{\partial^2\psi}{\partial n^2}\right]_1 + \frac{8\Delta n^3}{6}\left[\frac{\partial^3\psi}{\partial n^3}\right]_1 + O(\Delta n^4) \qquad \ldots (9)$$

From (8) and (9) and making use of (7), we get

$$w(1) = \left[\psi(3) - 8\psi(2) + 7\psi(1) + 6\Delta n\left\{\frac{\partial\psi}{\partial n}\right\}_1\right] / (2\Delta n^2).$$

Since ψ (1) is zero on all four boundaries, the above equation reduces to

$$w(1) = \left[\psi(3) - 8\psi(2) + 6\Delta n\left\{\frac{\partial\psi}{\partial n}\right\}_1\right] / (2\Delta n^2) \qquad \ldots (10)$$

This is second order approximation for vorticity and was used by Bozeman and Dalton [2]. Roache [16] calls it Jensen's fourmula. It is also referred as Briley's formula. Gupta and Manohar [9] have investigated the effects of boundary approximations on the solution and shown that Briley's formula gives more accurate results than any of the other formula.

3. Methematical Model. From a computational viwepoint, the cavity flow is an ideal prototype non-linear problem which is readily posed fo numerical solution. Because of its geometrical simplicity and comparatively minor singularities, it has served as a model problem for last twenty years for testing new numerical schemes and as a benchmark solution for making comparisons among various schemes. Fortunately, a number of numerical methods have been tried on this problem and results have been published. We have also applied our technique for the solution of this classical problem and obtained results which are comparable with the previous results.

The cavity, as shown in fig. (A) is rectangular in cross-section and filled with a Newtonian, viscous and incompressible fluid. The fluid is forced to make by the motion of the upper surface which travels with constant linear velocity in its own plane. The cavity is assumed to be long in the longitudinal (*Z*) direction so that the fluid motion is essentially two dimensional.

The formation of the governing equation for steady motion has been described in detail by Mills [13] and Burggraf [3]. The steady flow in the square cavity (fig. (B)) is governed by the equations (5) and (6) together with the under mentioned boundary conditions.

The boundary conditions for ψ may be derived from the conditions that each wall is impermeable and that the viscous boundary conditions implies that the fluid adjacent to the wall moves with the same velocity as the wall. The first condition implies that $\psi = 0$ on each boundary and second that the derivative of ψ in direction normal to the wall is equal to the tangential velocity of the wall. The boundary conditions thus obtained for driven cavity problem are as under :

$\psi = 0$ along all boundaries ... (11)

$\psi_x = 0$ along vertical walls OC and AB ... (12)

$\psi_y = 0$ along the bottom wall OA ... (13)

$\psi_y = -1$ along th sliding wall BC. ... (14)

4. Different Approaches Tried for This Problem. Burggraf [3] was first to employ central difference schemes for this problem using iterative procedures. Since the diagonal dominance property is not nacessarily satisfied, the standard iterative procedures such as Gauss-Seidel or SOR fail to converge rapidly even at moderate high Reynolds number. Later Spalding [20] discovered that stable solution could be obtained if upwind differencing methods were used for the convective terms in the vorticity equation. Further, Finite difference calculations using similar ideas have been presented by Nallaswamy and Krishna Prasad [14]. However, recently Strikwerda [21] and others have brought out clearly the drawbacks of upwind schemes.

The second approach to overcome the convergence problem is that at each outer iteration, the algebraic equations corresponding (5) - (6) are solved by direct fast solvers. Gupta [8] has applied this approach and used direct solver *MA 28* from the Harwell Package [5]. But These solvers do not take the benefit of sparsity into consideration and thus are not economical in computer-time and storage.

5. Difference Equations. The rectangular region over which the equations (1) and (2) are defined is divided into a uniform mesh by choosing mesh widths h along x-direction and k along y-direction. Applying central difference formulae to (1) and (2), relative to a uniform mesh system, we obtain

$$A_1 w_{i,j-1} + A_2 w_{i-1,j} + A_3 w_{i,j} + A_4 w_{i+1,j} + A_5 w_{i,j+1} = A_6 \quad \text{... (15)}$$

where

$$A_1 = \frac{1}{k^2} + \frac{Re}{4hk}(w_{i+1,j} - w_{i-1,j})$$

$$A_2 = \frac{1}{h^2} + \frac{Re}{4hk}(w_{i,j+1} - w_{i,j-1})$$

$$A_3 = -2\left[\frac{1}{h^2} + \frac{1}{k^2}\right]$$

$$A_4 = \frac{1}{h^2} + \frac{Re}{4hk}(w_{i,j+1} - w_{i,j-1})$$

$$A_5 = \frac{1}{k^2} + \frac{Re}{4hk}(w_{i+1,j} - w_{i-1,j})$$

$$A_6 = 0$$

and

$$B_1\psi_{i,j-1} + B_2\psi_{i-1,j} + B_3\psi_{i,j} + B_4\psi_{i+1,j} + B_5\psi_{i,j+1} = B_6 \qquad \ldots (16)$$

where

$$B_1 = \frac{1}{k^2},\quad B_2 = \frac{1}{h^2},\quad B_3 = -2\left[\frac{1}{h^2} + \frac{1}{k^2}\right],\ B_4 = \frac{1}{h^2},\quad B_5 = \frac{1}{k^2},\ B_6 = -w_{i\,j}.$$

6. Computational Procedure. Following are the steps involved in the computational procedure.

(a) Assign initial approximation $\psi^{(m)}$ with $m = 0$. Here approximation is taken as $\psi^{(0)} = 0$.

(b) Compute boundary values for ω by boundary conditions as in (10).

(c) Solve for Vorticity $\varpi^{(m+1)}$ from

$$A_1 w_{i,\,j-1}^{m+1} + A_2\, w_{i-1,j}^{m+1} + A_3\, w_{i,j}^{m+1} + A_4\, w_{i+1,j}^{m+1} + A_5 w_{i,j+1}^{m+1} = A_6$$

(d) Damp values of Vorticity in interior of domain as flollows

$$\omega^{(m+1)} = \delta\varpi^{(m+1)} + (1-\delta)\,\omega^{(m)}, \qquad 0< \delta <1.$$

(e) Solve stream function equation to obtain $\psi^{(m+1)}$ as

$$B_1\psi_{i,\,j-1}^{m+1} + B_2\,\psi_{i-1,j}^{m+1} + B_3\,\psi_{i,j}^{m+1} + B_4\,\psi_{i+1,j}^{m+1} + B_5\psi_{i,j+1}^{m+1} = B_6 \qquad \ldots (16)$$

(f) Damp values of stream function as

$$\psi^{(m+1)} = \delta\psi^{(m+1)} + (1-\delta)\,\psi^{(m)}$$

(g) Repeat steps (b) - (f) until following convergence criterion is satisfied :

$$max\ |\ \omega_{i,j}^{(m+1)} - \omega_{i,j}^{(m+1)}\ | < \epsilon.$$

(h) After above convergence criterion is satisfied, boundary values for ω are again computed as in (b) and solution is attained over whole domain.

7. Algorithm. The resulting structure of system of equation (15)-(16) can be rewritten as :

$$A\overline{U} = \overline{V} \qquad (17)$$

Coefficient matrix A is a block tri-diagonal system having n blocks where each block is of order (mxm). $\overline{U}$ is a column matrix of unknown and $\overline{V}$ is also a column matix containing right hand sides of equations. Since diagonal dominance property in such a system does not always hold, so iterative methods sometimes fail to converge. Recently some efficient direct methods have been employed to solve such a system. Linger [12] has applied a sem-direct method to solve Poisson's equation.

Elsner and Mehramann [6] have dealt in detail on the convergence condition of block iterative methods. The block tri-digonal system of linear equations is a sparse coefficient matrix system and it is possible to take the advantage of the sparsensess in order to reduce both computation time and storage requirements. Duff [4], Erisman and Reid [7] have dealt in detail with the direct methods for sparse matrices. Jennings [11] has also given methods like elimination using submatrices to deal with a sparse structure of such type.

Present algorithm also employs a direct elimination technique to solve block tri-diagonal system of linear equations. Imortant steps in brief are as follows. Detailed analysis of algorithm and storage etc. are given in Sharma and Agarwal [19].

The system (13) is written in matrix form as follows.

$$\begin{bmatrix} A_1 & B_1 & & & \\ C_2 & A_2 & B_2 & & \\ & \ddots & \ddots & \ddots & \\ & & C_{n-1} & A_{n-1} & B_{n-1} \\ & & & C_n & A_n \end{bmatrix}_{mn\times mn} \begin{bmatrix} \overline{U}_1 \\ | \\ | \\ | \\ \overline{U}_{n-1} \\ \overline{U}_n \end{bmatrix} = \begin{bmatrix} V_1 \\ | \\ | \\ | \\ V_{n-1} \\ V_n \end{bmatrix}_{mn\times 1} \qquad \dots (18)$$

where A_j's, $j = 1\ (1)\ n$ are tri-diagonal matrices, B_j's and C_j's, are diagonal matrices given by:

$$\begin{bmatrix} \beta_{1j} & \gamma_{1j} & & & \\ \alpha_{2j} & \beta_{2j} & \gamma_{2j} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \beta_{m-1,j} & \gamma_{m-1,j} \\ & & & \alpha_{mj} & \beta_{mj} \end{bmatrix}_{m\times m} \qquad \dots (19)$$

$$B_j = Diag\ [\delta_{1j}, \delta_{2j}, \dots \delta_{mj}]_{m\times m} \qquad \dots (20)$$

$$C_j = Diag\ [\eta_{1j}, \eta_{2j}, \dots \eta_{mj}]_{m\times m} \qquad \dots (21)$$

where $j = 1, 2, \dots, n$

$\overline{u}_j$, $j = 1\ (1)n$ is unkown column vector defined as

$\overline{u} = [u_{1j}, u_{2j}, \dots, u_{mj}]^T$.

V_j, $j = 1\ (1)\ n$ is right hand side modified after necessary adjustment due to given boundary conditions.

$V = [V_{1j}, V_{2j}, \dots V_{mj}]^T$.

Let R_{ij}, $i = 1(1)m$, $j = 1(1)n$ denotes i^{th} row of j^{th} block. Elimination algorithm is as follows :

Step 1.

For $j = 1$

Do $R_{i,1} = R_{i,1} - (\alpha_{i,1}/\beta_{i-1,1})^*(R_{i-1,1})$ for $i = 2$ to m.

Step 2.

For $j = 1$

(a) $i = 1$

(b) (i) $k = 1$

$R_{i,j} = R_{i,j} - (\eta_{\,i,j}/\beta_{k,\,j-1})^*(R_{k,\,j-1})$.

By this operation, first entry of first row of second block becomes zero and second entry of first row of second block becomes non-zero, say $\eta_{\,i,\,j}$.

(ii) For $k = k+1$ up m do

$\overline{R}_{i,j} = \overline{R}_{i,j} - (\eta_{\,i,j}/\overline{\beta}_{k,\,j-1})^*(\overline{R}_{k,\,j-1})$

Where "–" refers to modified entries.

(c) $i = i + 1$

(i) $R_{i,j} = R_{i,j} - (\eta_{\,i,j}/\overline{\beta}_{k,\,j-1})^*(\overline{R}_{k,\,j-1})$

(ii) for $k = i$ to m (just like 2(b))

$\overline{R}_{i,j} = \overline{R}_{i,j} - (\eta_{\,i,j}/\overline{\beta}_{k,\,j-1})^*(\overline{R}_{k,\,j-1})$

(iii) $\overline{R}_{i,j} = \overline{R}_{i,j} - (\alpha_{i,j}/\overline{\beta}_{i-1,j})^*(\overline{R}_{i-1,j})$.

(d) Again go to step 2(c) upto $i = m$

Step 3.

(i) Put $j = j+1$ and repeat steps 2(a) to 2 (d)

(ii) Continue this process upto $j = n$.

Step 4.

Now the system is reduced to upper triangular form and by back substitution process, we may obtain values of unknowns u_{ij}, $i = 1(1)m$, $j = 1(1)n$.

8. Result and Discussions. Computations are made taking uniform mesh-size $h = 1/20$ for various Reyonlds number. The point at which the value of ψ attains it's absolute maximum is called the center of primary vortex (vc). We denote the values of ψ and at the vortex center by ψ_{max} and ω_{vc} respectively. We also give the value of drag-coefficient on the sliding wall defined by-

$$cd = \frac{2}{Re}\int_0^1 \omega\ (s,1)\ ds = \frac{2}{Re}\,\overline{F}$$

where $\overline{F}$ is the value of shear force on sliding wall. The integral here is obtained by using Simpson's one-third rule over mesh points on the sliding wall.

It is evident from the table (I), (II) and (III) that resuts obtained compare well with those of previos authors [[2],[9], [14], [8], [10], [17], [18], [1]]. Streamlines and eqivorticities curves for different Reynolds number have been analysed. It is clear that there is no secondary vortex at $Re = 1$ and 10, but there exists two secondary vorticities at the downstream corners for Re =100 to Re = 500. Also the size of secondary vortices increases with the increase in Reynolds number as observed experimentally by Pan and Acrivos [15]. The equivorticity curves become more asymmetrical and recirculating eddies become more dominant with the increase in Reynolds number. The equivorticity curve at $Re = 500$ has a secondary eddy on the bottom wall at the level -1.0. The same nature of vorticity curve was observed by Ghia, Ghia and Shin [10] at $Re = 400$ using a much finer mesh.

TABLE -1

Values of Primary Vortex (ψ_{max}), Vorticity at the Vortex center (ω_{vc}) and Drag coefficient C_D for Re =1 and 10.

Re	ψ_{max}	ω_{vc}	(x,y)	C_D	Results	Reference	
1	0.0978	3.3553	(0.35,0.75)	21.1438	ψ_{max} = 0.0982	h=1/20	(9)
					= 0.0995	h=1/20	(9)
					= 0.0995	h=1/20	(8)
					= 0.1082	h=1/20	(9)
					ω_{vc} = 2.95	h=1/20	(9)
					= 3.02	h=1/20	(9)
					= 3.02	h=1/20	(8)
					= 3.33	h=1/20	(9)
10	0.1130	3.3496	(0.35,0.75)	2.1170	ψ_{max} = 0.1043	h=1/30	(17)
					= 0.1095	h=1/20	(9)
					ω_{vc} = 3.570	h=1/20	(9)
					= 3.155	h=1/50	(14)

TABLE -II

Values of Primary Vortex (ψ_{max}), Vorticity at the Vortex center (ω_{vc}) and Drag coefficient C_D for *Re =100* and *400*.

Re	ψ_{max}	ω_{vc}	(x,y)	C_D	Results	Reference
100	0.113	3.3496	(0.35,0.75)	0.2321	ψ_{max} = 0.1043	h=1/30 (17)
					= 0.1095	h=1/20 (9)
					ω_{vc} = 3.570	h=1/20 (9)
					= 3.155	h=1/50 (14)
400	0.1027	2.2363	(0.40,0.60)	0.0763	ψ_{max} = 0.1129	h=1/40 (18)
					= 0.1139	h=1/30 (10)
					= 0.0970	h=1/20 (1)
					ω_{vc} = 2.2810	h=1/40 (18)
					= 2.2947	h=1/30 (10)
					= 2.3600	h =1/20 (1)

TABLE -III

Values of Primary Vortex (ψ_{max}), Vorticity at the Vortex center (ω_{vc}) and Drag coefficient C_D for *Re* =500 and 1000.

Re	ψ_{max}	ω_{vc}	(x,y)	C_D	Results	Reference
500	0.1009	2.0132	(0.45,0.60)	0.0641	ψ_{max} = 0.1024	h=1/20 (8)
					ω_{vc} = 2.0504	h=1/20 (8)
					= 1.9048	h=1/20 (8)
1000	0.07914	1.6399	(0.45,0.60)	0.0367	ψ_{max} = 0.0812	h=1/50 (2)
					= 0.0972	h=1/20 (8)
					ω_{vc} = 1.7452	h=1/20 (8)

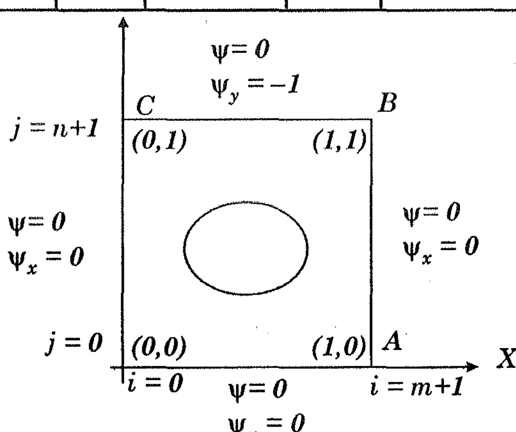

FIG. (A)- SQUARE DRIVEN CAVITY FLOW PROBLEM

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