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CLASS OF CERTAIN UNIVALENT FUNCTIONS

By

K.K. Dixit and A.L. Pathak

Department of Mathematics,

Janta College, Bakewar (Etawah)-206124, U.P.

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ABSTRACT

Let $D(A,B,\alpha)$ be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, regular in unit disc $U = \{z : |z| < 1\}$, and satisfying

$$\frac{f'(z)}{g'(z)} = \frac{1 + \{(A-B)(1-\alpha) + B\} w(z)}{1+B\, w(z)},$$

$$-1 \le B < A \le 1,\ z \in U,\ 0 \le \alpha < 1,$$

where $w(z)$ is regular in U and satisfies $w(0) = 0$, $|w(z)| < 1$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is regular and univalent in U. In this paper we obtain coefficient estimates, radius of convexity for $f(z) \in D(A,B,\alpha)$ and some more interesting results for the subclasses of $D(A,B,\alpha)$.

1. Introduction. Let D denote the class of function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, regular in unit dist $U = \{z : |z| < 1\}$, and satisfying the condition

$$\text{(1.1)} \qquad \left| \frac{f'(z)}{g'(z)} - 1 \right| < 1, \qquad z \in U,$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is regular and univalent in U.

Let $D(\delta)$ denote the class of function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, regular in U and satisfying the condition

$$\text{(1.2)} \qquad \left| \frac{\{f'(z)/g'(z)\} - 1}{\{f'(z)/g'(z)\} + 1} \right| < \delta, \qquad z \in U,$$

where $0 < \delta \le 1$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is regular and univalent in U.

In this paper we introduce the class $D(A, B, \alpha)$ of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ regular in U and satisfying the condition

$$\text{(1.3)} \qquad \frac{f'(z)}{g'(z)} = \frac{1 + \{(A-B)(1-\alpha) + B\} w(z)}{1+B\, w(z)},$$

$-1 \leq B < A \leq 1,\ z \in U,\ 0 \leq \alpha < 1,$

where $w(z)$ is regular in U and satisfies the conditions $w(0) = 0$, $|w(z)| < 1$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is regular and univalent in U.

With appropriate choices of A and B and α, the class $D(A, B,\alpha)$ reduces to following known important subclasses :

(i) For $A = 1$ and $B = 0$ and $\alpha = 0$, the class $D(A,B,\alpha)$ reduces to the class D.

(ii) For $A = \delta$ and $B = -\delta$ and $\alpha = 0$, the class $D(A,B,\alpha)$ reduces to the class $D(\delta)$.

In this paper we obtain sharp coefficient estimates and sharp radius of convexity for $f(z) \in D(A,B,\alpha)$. Also, we obtain the sharp result concerning the radius of convexity for the subclasses of $D(A,B,\alpha)$ associated with each of the cases :

(i) $g(z)$ is starlike in U,

(ii) $g(z)$ is convex in U,

(iii) $Re\, g'(z) > 0$ in U.

Recently, Lakshminarashimhan [3] have studied the class $D(\delta)$ and Ratti [4] have studied the class D. Thus our results naturally generalize the corresponding results of Lakshminarasimhan [3] and Ratti [4].

In fact, an attempt has been made to have a unified and detailed study of these classes of univalent functions.

Throughout this paper we assume that $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$ and R_1, R_2, K_1 and L_1 have the same values as in Lemma 2.1.

2. To establish the results of subsequent section we require the following lemmas:

Lemma 2.1. If $p(z) \in P(A,B,\alpha)$, then on $|z| = r < 1$,

$$Re\left\{\frac{zp'(z)}{p(z)}\right\} \geq \begin{cases} -\dfrac{(A-B)(1-\alpha)r}{[1+\{(A-B)(1-\alpha)+B\}r](1-Br)}, & R_1 \leq R_2 \\[2ex] \dfrac{(A-B)(1-\alpha)+2B}{(A-B)(1-\alpha)} + \dfrac{2[(L_1K_1)^{1/2} - [1-\{(A-B)(1-\alpha)+B\}Br^2]}{(A-B)(1-\alpha)(1-r^2)} & R_1 \leq R_2 \end{cases}$$

where $R_1 = (L_1/K_1)^{1/2}$, $R_2 = \dfrac{[1-\{(A-B)(1-\alpha)+B\}r]}{(1-Br)}$,

$$L_1 = [1-\{(A-B)(1-\alpha)+B\}][1+\{(A-B)(1-\alpha)+B\}r^2]$$

and $K_1 = (1-B)(1+Br^2)$.

The result is sharp. The above lemma follows from Theorem 1 of Anh

$$Re\left[\frac{1+\{(A-B)(1-\alpha)+B\}\,w_1(z)}{1+B\,w_1(z)}\right]=R_1 \qquad \text{at } z=-r.$$

Remark. From the above theorem the following result can be obtained immediately from which result of Ratti ([4], Theorem 3) turn out to be special case by taking $A=1$, $B=0$ and $\alpha=0$.

Corollary. If $f(z)\in D(A,B,\alpha)$, $g(z)$ is regular and convex in U, then $f(z)$ is convex in

$$|z| < \begin{cases} r_3 & \text{for } R_1 \le R_2 \\ r_4 & \text{for } R_2 \le R_1 \end{cases}$$

where r_3 and r_4 respectively the smallest positive roots of the following equations:

$$(1-r)\,[1-\{(A-B)(1-\alpha)+B\}\,r]\,(1-Br)-(A-B)(1+r)\,r=0,$$

and

$$(1-r^2)\ (A-B)(1-\alpha)+\{(A-B)(1-\alpha)+2B\}(1-r^2)+2\,[\,(L_1K_1)^{1/2}-\{1-((A-B)(1-\alpha)+B)\,Br^2\}]=0\,.$$

The resutl is sharp.

Theorem 3.5. If $f(z)\in D(A,B,\alpha)$, $Re\,g'(z)>0$ in U, then

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} \ge \begin{cases} M_5(r) \text{ if } R_1 \le R_2 \\ M_6(r) \text{ if } R_2 \le R_1 \end{cases}$$

where

$$M_5(r)=\frac{1-2r+r^2}{1-r^2}-\frac{(A-B)(1-\alpha)\,r}{[1-\{(A-B)(1-\alpha)+B\}r]\,(1-Br)},$$

and

$$M_6(r)=\frac{1-2r+r^2}{1-r^2}+\frac{(A-B)(1-\alpha)+2B}{(A-B)(1-\alpha)}+\frac{2[(L_1K_1)^{1/2}-\{1-(A-B)(1-\alpha)+B\}Br^2\}]}{(A-B)(1-\alpha)\,(1-r^2)}$$

The result is sharp.

Proof. Since $Re\,g'(z)>0$ in U, we have by Lemma 2.3

$$Re\,\left\{1+\frac{zg''(z)}{g'(z)}\right\}\ge\frac{1-2r+r^2}{1-r^2}\,.$$

Using this estimate in (3.4) we get the required result on the lines of proof of Theorem 3.2.

To see the bounds are sharp we consider the following two cases:

(i) $$\frac{f'(z)}{g'(z)}=\frac{1+\{(A-B)(1-\alpha)+B\}\,z}{1+B\,z}$$

with $g(z)=-z-2\log(1-z)$, when $R_1\le R_2$

and

(ii) $$\frac{f'(z)}{g'(z)}=\frac{1+\{(A-B)(1-\alpha)+B\}\,w_1(z)}{1+B\,w_1(z)}$$

and Tuan [1] by putting $\alpha = 0$ and $\beta = 1$.

Lemma 2.2. If $h(z) = 1+c_1z+ \dots$ is regular in U and $Re\ h(z) > 0$ for $z \in U$, then

$$Re\ h(z) \geq \frac{1-r}{1+r}, \ |z| = r.$$

This is a well known result due to Caratheodory.

Lemma 2.3. If $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is regular $Re\ g'(z) > 0$ in U, then

$$Re\left\{1 + \frac{z g''(z)}{g'(z)}\right\} \geq \frac{1-2r+r^2}{1-r^2}, \ |z| = r.$$

The result is sharp.

The above result is contained in the proof of Theorem 4 of Ratti [4].

3. In this section we obtain main results.

Theorem 3.1. If $f(z) \in D(A,B,\alpha)$, then

$$|a_n - b_n| \leq \frac{(A-B)(1-\alpha)}{n}, \ n = 2, 3, \dots \text{ and } 0 \leq \alpha < 1.$$

If $Re\ (a_k \overline{b_k}) \leq 0$, $k = 2, 3, \dots$.

The result is sharp.

Proof. Since $f(z) \in D(A,B,\alpha)$, we have

$$\frac{f'(z)}{g'(z)} = \frac{1 + \{(A-B)(1-\alpha) + B\}\, w(z)}{1+B\, w(z)},$$

or

$$f'(z) - g'(z) = [(A-B)(1-\alpha)\, g'(z) + B\, \{g'(z) - f'(z)\}]\, w(z).$$

That is

$$(3.1) \quad \sum_{n=2}^{\infty} n\, (a_n - b_n)\, z^{n-1} = \{(A-B)(1-\alpha)(1+ \sum_{n=2}^{\infty} n\, b_n z^{n-1}) + B \sum_{n=2}^{\infty} n\, (a_n - b_n)\, z^{n-1}\} \sum_{n=1}^{\infty} c_n z^n$$

For $n \geq 2$, we have from (3.1)

$$\sum_{k=2}^{n} k\, (a_k - b_k)\, z^{k-1} + \sum_{k=n+2}^{\infty} d_k z^{k-1} = [(A-B)(1-\alpha) + (A-B)(1-\alpha) \sum_{k=2}^{n-1} k\, b_k z^{k-1} + B \sum_{k=2}^{n-1} k\, (b_k - a_k)\, z^{k-1}]\, w(z)$$

which gives

$$\left| \sum_{k=2}^{n} k\, (a_k - b_k)\, z^{k-1} + \sum_{k=n+1}^{\infty} d_k z^{k-1} \right|^2 \leq \left| (A-B)(1-\alpha) + (A-B)(1-\alpha) \sum_{k=2}^{n-1} k\, b_k z^{k-1} + B \sum_{k=2}^{n-1} k\, (b_k - a_k)\, z^{k-1} \right|^2.$$

On integrating over $|z| = r$, $0 < r < 1$, we get

$$\sum_{k=2}^{n} k\,(a_k - b_k)\, z^{k-1} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2k-2} \le (A-B)^2 (1-\alpha)^2 + (A-B)^2 (1-\alpha)^2$$

$$+ (A-B)^2 (1-\alpha)^2 \sum_{k=2}^{n-1} k^2\, |b_k|^2 r^{2k-2} + B^2 \sum_{k=2}^{n-1} k^2\, |b_k - a_k|^2 r^{2k-2}.$$

If we take the limit as $r \to 1$, then

$$\sum_{k=2}^{n} k^2\, |a_k - b_k|^2 \le (A-B)^2(1-\alpha)^2 \sum_{k=2}^{n-1} k^2\, |b_k|^2 + B^2 \sum_{k=2}^{n-1} k^2\, |b_k - a_k|^2$$

or

$$n^2 |a_n - b_n|^2 \le (A-B)^2(1-\alpha)^2 + (A-B)^2 (1-\alpha)^2 \sum_{k=2}^{n} k^2 |b_k|^2 + B^2 \sum_{k=2}^{n-1}$$

$$k^2\, [\,|b_k|^2 + |a_k|^2 - 2|a_k|\,|b_k|\,] - \sum_{k=2}^{n-1} k^2\, [\,|b_k|^2 + |a_k|^2 - 2|a_k|\,|b_k|\,]$$

$$= (A-B)^2 (1-\alpha)^2 + \sum_{k=2}^{n-1} k^2\, [\{(A-B)^2 (1-\alpha)^2 + B^2 - 1\}\, |b_k|^2 + (B^2-1)\,|k|_2$$

$$+ 2 \sum_{k=2}^{n-1} k^2 (1-B^2)\, Re\,(a_k\, \overline{b_k})] \le (A-B)^2 (1-\alpha)^2,$$

$$\{A-1 \le B < A \le 1,\ 0 \le \alpha < 1,\ Re\,(a_k\, \overline{b_k}) \le 0\}$$

that is,

$$|a_n - b_n| \le \frac{(A-B)\,(1-\alpha)}{n},\ n = 2, 3, \ldots\ .$$

To establish the sharpness of the result we consider the following funcitons

$$f(z) = \int_0^z \frac{dt}{1 + \{(A-B)(1-\alpha)+B\}\, t^{n-1}}\ ,$$

and $$g(z) = \int_0^z \frac{(1+B\, t^{n-1})\, dt}{[1 + \{(A-B)(1-\alpha)+B\}\, t^{n-1}]^2}\ .$$

It is obvious that $f(z) \in D(A,B,\alpha)$.

Now

$$f(z) = \int_0^z \frac{dt}{1 + \{(A-B)(1-\alpha)+B\}\, t^{n-1}}\ ,$$

$$= \int_0^z [1 + \{(A-B)(1-\alpha) + B\} t^{n-1}]^{-1}\, dt$$

$$= z - \frac{\{(A-B)(1-\alpha) + B\}}{n}\, z^n + \ldots$$

and

$$g(z) = \int_0^z \frac{(1+B\, t^{n-1})\, dt}{[1 + \{(A-B)(1-\alpha)+B\}\, t^{n-1}]^2}\ ,$$

$$= \int_0^z [(1+B\, t^{n-1})\, [1 + \{(A-B)(1-\alpha) + B\} t^{n-1}]^{-2}\, dt$$

$$= z - \frac{[2\{(A-B)(1-\alpha)+B\} - B]}{n} z^n + \ldots$$

Clearly, $a_n = -\dfrac{\{(A-B)(1-\alpha)+B\}}{n}$ and $b_n = -\dfrac{[2\{(A-B)(1-\alpha)+B\} - B]}{n}$.

Hence

$$|a_n - b_n| = \frac{(A-B)(1-\alpha)}{n}.$$

This shows the sharpness of the result.

Remark. On putting $A = \delta$ and $B = \delta$ and $\alpha = 0$ in Theorem 3.1 the result of Lakshminarasimhan([3], Theorem 5) follows.

Theorem 3.2. If $f(z) \in D(A,B,\alpha)$, then

$$Re\left\{1 + \frac{z f''(z)}{f'(z)}\right\} \le \begin{cases} M_1(r) & \text{for } R_1 \le R_2 \\ M_2(r) & \text{for } R_2 \le R_1 \end{cases}$$

where

$$M_1(r) = \frac{1-4r+r^2}{1-r^2} - \frac{(A-B)(1-\alpha)\, r}{[1-\{(A-B)(1-\alpha)+B\}r]\,(1-Br)}.$$

$$M_2(r) = \frac{1-4r+r^2}{1-r^2} + \frac{\{(A-B)(1-\alpha)+2B\}}{(A-B)(1-\alpha)}.$$

$$+ \frac{2\,[\,(L_1K_1)^{1/2} - \{1-(A-B)(1-\alpha)+B\}\, Br^2\}]}{(A-B)(1-\alpha)\,(1-r^2)}$$

Proof. Since $f(z) \in D(A,B,\alpha)$, we have

$$\frac{f'(z)}{g'(z)} = \frac{1 + \{(A-B)\,(1-\alpha) + B\}\, w(z)}{1+B\, w(z)},$$

(3.2) Let $\dfrac{f'(z)}{g'(z)} = p(z)$, where $p(z) = \dfrac{1 + \{(A-B)\,(1-\alpha) + B\}\, w(z)}{1+B\, w(z)}$.

On differentiating (3.2) logarithmically, we get

$$\frac{z f''(z)}{f'(z)} - \frac{z g''(z)}{g'(z)} = \frac{z p'(z)}{p(z)}.$$

or $Re\left\{1+\dfrac{z f''(z)}{f'(z)}\right\} - Re\left\{1+\dfrac{z g''(z)}{g'(z)}\right\} = Re\left(\dfrac{z p'(z)}{p(z)}\right)$.

Using Lemma 2.1 in (3.3), we have

(3.4) $$Re\left\{1+\frac{z f''(z)}{f'(z)}\right\} - Re\left\{1+\frac{z g''(z)}{g'(z)}\right\}$$

$$\ge \begin{cases} -\dfrac{(A-B)\,(1-\alpha)\, r}{[1 - \{(A-B)\,(1-\alpha) + B\}r]\,(1-Br)}, & \text{if } R_1 \le R_2 \\[2ex] \dfrac{(A-B)\,(1-\alpha) + 2B}{(A-B)\,(1-\alpha)} + \dfrac{2[(L_1K_1)^{1/2} - [1-\{(A-B)(1-\alpha)+B\}Br^2]}{(A-B)\,(1-\alpha)\,(1-r^2)} & \text{if } R_2 \le R_1 \end{cases}$$

Since $g(z)$ is univalent, we have [2]

$$(3.5)\ Re\left\{1+\frac{z\,g''(z)}{g'(z)}\right\} \geq \frac{1-4r+r^2}{1-r^2},\ |z|=r.$$

Combining (3.4) and (3.5) we get the required result.
Sharpness of the bounds follows if we choose

(i) $\dfrac{f'(z)}{g'(z)} = \dfrac{1+\{(A-B)(1-\alpha)+B\}z}{1+Bz}$ with $g(z)=\dfrac{z}{(1-z)^2}$, when $R_1 \leq R_2$

and

(ii) $\dfrac{f'(z)}{g'(z)} = \dfrac{1+\{(A-B)(1-\alpha)+B\}w_1(z)}{1+Bw_1(z)}$ with $g(z)=\dfrac{z}{(1-z)^2}$,

when $R_2 \leq R_1$

where $w_1(z) = \dfrac{z(z-c)}{1-cz}$ with c defined by the condition

$$Re\left[\frac{1+\{(A-B)(1-\alpha)+B\}w_1(z)}{1+Bw_1(z)}\right] = R_1 \qquad \text{at } z=-r.$$

Remark. From the above theorem the following result can be obtained immediately from which result of Ratti ([4], Theorem 1) turn out to be special case by taking $A = 1$, $B = 0$ and $\alpha = 0$.

Corollary. If $f(z) \in D(A,B,\alpha)$, then $f(z)$ is convex in

$$|z| < \begin{cases} r_1 & \text{for } R_1 \leq R_2 \\ r_2 & \text{for } R_2 \leq R_1 \end{cases}$$

where r_1 and r_2 are respectively the smallest positive roots of the following equations :

(3.6) $(1-4r+r^2)[1-\{(A-B)(1-\alpha)+B\}r](1-Br)-(A-B)(1-\alpha)(1-r^2)r=0$

(3.7) $(1-4r+r^2)\{(A-B)(1-\alpha)\}+\{(A-B)(1-\alpha)+2B\}(1-r^2)$
$+2[(L_1K_1)^{1/2}-\{1-((A-B)(1-\alpha)+B)Br^2\}]=0.$

The result is sharp.

Theorem 3.3. If $f(z) \in D(A,B,\alpha)$, $g(z)$ is regular and starlike in U, then

$$Re\left\{1+\frac{z f''(z)}{f'(z)}\right\} \geq \begin{cases} M_1(r) & \text{if } R_1 \leq R_2 \\ M_2(r) & \text{if } R_2 \leq R_1 \end{cases}$$

where $M_1(r)$ and $M_2(r)$ are same as in Theorem 3.2.

The result is sharp.

Proof. Since $g(z)$ is starlike in D implies $g(z)$ is univalent there, the proof of this theorem follows on the steps Theorem 3.2.

Remark. From the above theorem the following result can be obtained immediately from which result of Ratti ([4], Theorem 2) turn out to be special case by taking $A = 1$ and $B = 0$ and $\alpha = 0$.

Corollary. If $f(z) \in D(A,B,\alpha)$, $g(z)$ is regular and starlike in U, then $f(z)$ is convex in

$$|z| < \begin{cases} r_1 & \text{for } R_1 \le R_2 \\ r_2 & \text{for } R_2 \le R_1 \end{cases}$$

where r_1 and r_2 are respectively the smallest positve roots of the equations (3.6) and (3.7).

Theorem 3.4. If $f(z) \in D(A,B,\alpha)$, $g(z)$ is regular and convex in U, then

$$Re\left\{1 + \frac{z f''(z)}{f'(z)}\right\} \ge \begin{cases} M_3(r) \text{ if } R_1 \le R_2 \\ M_4(r) \text{ if } R_2 \le R_1 \end{cases}$$

where

$$M_3(r) = \frac{1-r}{1+r} - \frac{(A-B)(1-\alpha)\, r}{[1-\{(A-B)(1-\alpha)+B\}r]\,(1-Br)},$$

and

$$M_4(r) = \frac{1-r}{1+r} - \frac{\{(A-B)(1-\alpha)+2B\}}{(A-B)(1-\alpha)} + \frac{2[(L_1K_1)^{1/2} - \{1-(A-B)(1-\alpha)+B\}Br^2\}]}{(A-B)(1-\alpha)\,(1-r^2)}$$

The result is sharp.

Proof. Since $g(z)$ is convex in U, therefore $g'(z) \neq 0$ in U and

$$Re\left\{1 + \frac{z g''(z)}{g'(z)}\right\} > 0 \text{ in } U.$$

The function

$$1 + \frac{z g''(z)}{g'(z)} = 1 + c_1 z + \ldots$$

is regular in U and has positive real part, therefore by Lemma 2.2

$$Re\left\{1 + \frac{z g''(z)}{g'(z)}\right\} \ge \frac{1-r}{1+r}, \quad |z| = r.$$

Using this estimate in Theorem 3.4, we get the required result on the lines of proof of Theorem 3.2.

To see that the bounds are sharp, we consider the following two cases.

(i) $\dfrac{f'(z)}{g'(z)} = \dfrac{1 + \{(A-B)(1-\alpha) + B\} z}{1+Bz}$ with $g(z) = \dfrac{z}{1-z}$, when $R_1 \le R_2$

and

(ii) $\dfrac{f'(z)}{g'(z)} = \dfrac{1 + \{(A-B)(1-\alpha) + B\} w_1(z)}{1+B w_1(z)}$ with $g(z) = \dfrac{z}{1-z}$, when $R_2 \le R_1$

where $w_1(z) = \dfrac{z(z-c)}{1-cz}$ with c defined by the condition where $w_1(z) = \dfrac{z\,(z-c)}{1-cz}$ with c defined by the condition

$$Re\left[\frac{1+\{(A-B)\,(1-\alpha)+B\}\,w_1(z)}{1+B\,w_1(z)}\right]=R_1 \qquad \text{at } z=-r.$$

Remark. From the above theorem the following result can be obtained immediately from which result of Ratti ([4], Theorem 4) turn out to be special case by taking $A=1,\ \ B=0$ and $\alpha=0$.

Corollary. If $f(z)\in D(A,B,\alpha)$, $Re\, g'(z)>0$ in U, then $f(z)$ is convex in

$$|z|<\begin{cases} r_5 & \text{for } R_1\le R_2 \\ r_6 & \text{for } R_2\le R_1 \end{cases}$$

where r_5 and r_6 respectively the smallest positive roots of the following equations:

$$(1-2r-r^2)\,[1-\{\,(A-B)\,(1-\alpha)+B\}\,r]\,(1-Br)-(A-B)(1-\alpha)\,(1-r^2)\,r=0$$

and

$$(1-2r-r^2)\,\{\,(A-B)\,(1-\alpha)\}+\{\,(A-B)(1-\alpha)+2B\}(1-r^2)+2\,[\,(L_1K_1)^{1/2}-$$
$$\{1-((A-B)\,(1-\alpha)+B)\,Br^2\}]=0.$$

The result is sharp.

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