SOME EXPECTATIONS ASSOCIATED WITH MULTIVARIATE GAMMA AND BETA DISTRIBUTIONS INVOLVING THE MULTIPLE HYPERGEOMETRIC FUNCTION OF SRIVASTAVA AND DAOUST

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ABSTRACT

In the present paper, we obtain some density functions associated with the multivariate gamma and beta distributions and make their applications to obtain the expectations involving multiple hypergeometric function of Srivastava and Daoust [48] (see also Srivastava and Manocha [51], p.64). Finally, we also derive the moments for these multivariate beta and gamma distributions and discuss their special cases.

1. Introduction. Different distributions have discussed by various authors Block and Rao [1], Carlson [2], Daley [5] Datt [6], Kabe [8], Kaufman, Mathai and Saxena [9], Kendall [10], Khatri and Pillai [11,12], Khatri and Srivastava [13], Littler and Fackerell [15], Lukacs and Naha [16], Lukacs [17], Mathai ([18] to [29]), Mathai and Rathie ([30] to [35]), Mathai and Saxena ([36] to [42]), Miller [43], Pillai, Al-Ani and Jouris [44], Pillai and Jouris [45], Pillai and Nagarsenker [46], Robbins and Pitman [47], Strawderman [52], Thaung [53], and Wilks [54]

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In the present paper, we extend the above work and establish some probability density functions associated with the multivariate beta and gamma distributions and make their applications to obtain some expectations involving the most generalized multiple hypergeometric function of Srivastava and Daoust [48] (see also Srivastava and Manocha ([51],p.64). Finally, we also drive the moments for these multivariate beta and gamma distributions and discuss their special cases.

2. Formulae Required. For ready stock, in this section we write the following results which will be used in our investigations:

The Liouville’s Theorem (Also see Chandel [3,p.83 (3.1)]

\[ \int_{0}^{\infty} \cdots \int_{0}^{\infty} f(x_{1} + \cdots + x_{n}) x_{1}^{\mu_{1} - 1} \cdots x_{n}^{\mu_{n} - 1} \, dx_{1} \cdots dx_{n} = \frac{\Gamma(\mu_{1}) \cdots \Gamma(\mu_{n})}{\Gamma(\mu_{1} + \cdots + \mu_{n})} \int_{0}^{\infty} f(t) \, t^{\mu_{1} + \cdots + \mu_{n} - 1} \, dt, \]

provided that \( \Re(\mu_{i}) > 0, \) \( i = 1, \ldots, n. \)

Euler’s definition for gamma function

\[ \Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} \, dt, \quad \Re(z) > 0. \]

The definition of beta function (see, Srivastava and Manocha [51, p.26 eq.(4.6)])

\[ B(\alpha, \beta) = \int_{0}^{\infty} \frac{\mu^{\alpha-1}}{(1+\mu)^{\alpha+\beta}} \, d\mu, \quad \Re(\alpha) > 0, \Re(\beta) > 0. \]

3. Multivariate Gamma Distribution. Consider the function

\[ f(x_{1}, \ldots, x_{n}) = \frac{\Gamma(\mu_{1} + \cdots + \mu_{n}) \lambda^{\mu_{1} + \cdots + \mu_{n}}}{\Gamma(\mu_{1}) \cdots \Gamma(\mu_{n}) \Gamma(\mu_{1} + \cdots + \mu_{n})} \exp\{-x_{1} + \cdots + x_{n}\lambda\} \]

\[ (x_{1} + \cdots + x_{n})^{\mu_{1} + \cdots + \mu_{n}} \]

provided that \( \Re(\lambda) > 0, x_{i} \geq 0, \Re(\mu_{i}) > 0, i = 1, \ldots, n \)

and \( f(x_{1}, \ldots, x_{n}) = 0 \) elsewhere.

Making an appeal to (2.1) and (2.2) the value of multiple integral of \( f(x_{1}, \ldots, x_{n}) \) over the region defined above in (3.1) becomes unity.

Hence \( f(x_{1}, \ldots, x_{n}) \) is a probability density function for multivariate gamma distribution.

4. Expectation Associated with Multivariate Gamma Distribution.

The expectation value of the function \( g(x_{1}, \ldots, x_{n}) \) is defined as

\[ \langle g(x_{1}, \ldots, x_{n}) \rangle = \int_{0}^{\infty} \cdots \int_{0}^{\infty} f(x_{1}, \ldots, x_{n}) \, g(x_{1}, \ldots, x_{n}) \, dx_{1} \cdots dx_{n} \]

corresponding to density function \( f(x_{1}, \ldots, x_{n}) \) defined by (3.1).
Consider the function

\[
(4.2) \quad g_1(x_1, \ldots, x_n) = F^{A:B', \ldots; B^{(n)}_{C:D', \ldots; D^{(n)}}} \left[ [(\alpha): \theta', \ldots, \theta^{(n)}]; \frac{[(b') \phi']}{[(d') \delta']}; \frac{z_1(x_1 + \cdots + x_n)^{\nu_1}}{\lambda_1^{\nu_1}} \cdots \frac{z_n(x_1 + \cdots + x_n)^{\nu_n}}{\lambda_n^{\nu_n}} \right]
\]

where \( F^{A:B', \ldots; B^{(n)}}_{C:D', \ldots; D^{(n)}} \) is most generalized multiple hypergeometric function of Srivastava and Daoust [48] (also see Srivastava and Manocha [51, (18),(19),(20),p.64]).

Now making an appeal to (2.1) and (2.2) the expectation of \( g_1(x_1, \ldots, x_n) \) having density function \( f(x_1, \ldots, x_n) \) is given by

\[
(4.3) \quad \mathbb{E}[g_1(x_1, \ldots, x_n)] = F^{A+1:B', \ldots; B^{(n)}_{C:D', \ldots; D^{(n)}}} \left[ \frac{[(\alpha): \theta', \ldots, \theta^{(n)}]}{[(\alpha)+1: \theta', \ldots, \theta^{(n)}]}; \frac{[(b') \phi']}{[(d') \delta']}; \frac{z_1(x_1 + \cdots + x_n)^{\nu_1}}{\lambda_1^{\nu_1}} \cdots \frac{z_n(x_1 + \cdots + x_n)^{\nu_n}}{\lambda_n^{\nu_n}} \right]
\]

provided that

\[
1 + \sum_{j=1}^{C} \sum_{j=1}^{D} \delta^{(j)} - \sum_{j=1}^{A} \theta^{(j)} + \sum_{j=1}^{B} \phi^{(j)} - \nu_i > 0, \quad i = 1, \ldots, n.
\]

Corresponding to density function \( f(x_1, \ldots, x_n) \) defined by (3.1), if we consider the function

\[
(4.4) \quad g_2(x_1, \ldots, x_n) = F^{A+1:B', \ldots; B^{(n)}_{C:D', \ldots; D^{(n)}}} \left[ [(\alpha): \theta', \ldots, \theta^{(n)}]; \frac{[(b') \phi']}{[(d') \delta']}; \frac{z_1(x_1 + \cdots + x_n)^{\nu_1} x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{\lambda_1^{\nu_1} \cdots \lambda_n^{\nu_n}} \right]
\]

Then expectation of \( g_2 \) is given by

\[
(4.5) \quad \mathbb{E}[g_2(x_1, \ldots, x_n)] = F^{A+1:B', \ldots; B^{(n)}_{C:D', \ldots; D^{(n)}}+1} \left[ [(\alpha): \theta', \ldots, \theta^{(n)}]; \frac{[(b') \phi']}{[(d') \delta']}; \frac{z_1(x_1 + \cdots + x_n)^{\nu_1}}{\lambda_1^{\nu_1}} \cdots \frac{z_n(x_1 + \cdots + x_n)^{\nu_n}}{\lambda_n^{\nu_n}} \right]
\]

valid if

\[
1 + \sum_{j=1}^{C} \sum_{j=1}^{D} \delta^{(j)} - \sum_{j=1}^{A} \theta^{(j)} + \sum_{j=1}^{B} \phi^{(j)} - \nu_i > 0, \quad i = 1, \ldots, n.
\]

5. The Multivariate Beta Distribution.

Consider the function \( F(x_1, \ldots, x_n) \) defined by
(5.1) $F(x_1, ..., x_n) = \frac{\Gamma(\mu_1 + \ldots + \mu_n) \Gamma(\alpha + \lambda + \mu_1 + \ldots + \mu_n)}{\Gamma(\mu_1) \Gamma(\mu_2) \Gamma(\alpha + \lambda + \mu_1 + \ldots + \mu_n)} (x_1 + \ldots + x_n)^{\alpha - \lambda + \mu_1 + \ldots + \mu_n}$

$Re(\alpha) > 0, Re(\mu_j) > 0, x_i > 0 \ i = 1, ..., n$ and $F(x_1, ..., x_n) = 0$ elsewhere. Now making an appeal to (2.1) and (2.3) the value of multiple integral of $F(x_1, ..., x_n)$ over the region defined above in (5.1), becomes unity. Hence $F(x_1, ..., x_n)$ is probability density function for multivariate beta distribution.


Corresponding to density function $f(x_1, ..., x_n)$ defined by (5.1), consider the function

\[
G(x_1, ..., x_n) = F^{A : B', ..., B^{(a)} / C : D', ..., D^{(a)} / (\gamma) : \gamma', ..., \gamma^{(a)} / (\delta') : \delta', ...} [ (a) : \phi', ..., \phi^{(a)} / (\psi) : \psi', ..., \psi^{(a)} / (\xi) : \xi', ..., \xi^{(a)} / (\phi) : \phi', ..., \phi^{(a)} / (\xi) : \xi', ..., \xi^{(a)} ]
\]

\[
= F^{A : B', ..., B^{(a)} / C : D', ..., D^{(a)} / (\gamma) : \gamma', ..., \gamma^{(a)} / (\delta') : \delta', ...} [ (a) : \phi', ..., \phi^{(a)} / (\psi) : \psi', ..., \psi^{(a)} / (\xi) : \xi', ..., \xi^{(a)} / (\phi) : \phi', ..., \phi^{(a)} / (\xi) : \xi', ..., \xi^{(a)} ]
\]

Then expectation of $G(x_1, ..., x_n)$ is given by

\[
\langle G(x_1, ..., x_n) \rangle = F^{A : B', ..., B^{(a)} / C : D', ..., D^{(a)} / (\gamma) : \gamma', ..., \gamma^{(a)} / (\delta') : \delta', ...} [ (a) : \phi', ..., \phi^{(a)} / (\psi) : \psi', ..., \psi^{(a)} / (\xi) : \xi', ..., \xi^{(a)} / (\phi) : \phi', ..., \phi^{(a)} / (\xi) : \xi', ..., \xi^{(a)} ]
\]

where

\[
i = 1, ..., n.
\]

7. Moment Generating Function (m.g.f.) for Gamma Distribution.

The m.g.f. is defined as

\[
M(t_1, ..., t_n) = \int_0^\infty \ldots \int_0^\infty e^{t_1 x_1 + \ldots + t_n x_n} f(x_1, ..., x_n) \, dx_1 \ldots dx_n,
\]

provided that the integral is a function of the parameters $t_1, ..., t_n$ only. Thus m.g.f. for multivariate gamma distribution (3.1) is given by

\[
M(t_1, ..., t_n) = \frac{\Gamma(\mu_1 + \ldots + \mu_n) \lambda^{\mu_1 + \ldots + \mu_n}}{\Gamma(\mu_1) \ldots \Gamma(\mu_n) \Gamma(\mu_1 + \ldots + \mu_n)}
\]

\[
\int_0^\infty \ldots \int_0^\infty e^{t_1 x_1 + \ldots + t_n x_n} e^{-(x_1 + \ldots + x_n)} (x_1 + \ldots + x_n)^{\mu_1 - 1} \ldots x_n^{\mu_n - 1} \, dx_1 \ldots dx_n
\]
Now making an appeal to (2.1) and the result due to Srivastava[49,p.4.(12.1)],
\[
\sum_{m_1=0}^{\infty} f(m_1,\ldots+m_n) \frac{x_1^{m_1}}{m_1!}\cdots \frac{x_n^{m_n}}{m_n!} = \sum_{M=0}^{\infty} f(M) (x_1,\ldots+x_n)^M, n \geq 1,
\]
we finally derive
\[
M(t_p,\ldots,t_n) = F_D(n) (\mu_1,\ldots+\mu_n, \mu_1,\ldots, \mu_n; \mu_1,\ldots, \mu_n; \frac{t_1}{\lambda},\ldots, \frac{t_n}{\lambda}),
\]
where \( F_D(n) \) is Lauricella's fourth multiple hypergeometric function of several variables [14].

As a special case for \( \mu = 0 \), (7.3) gives
\[
M(t_p,\ldots,t_n) = \prod_{i=1}^{n} \left( 1 - \frac{t_i}{\lambda} \right)^{-\mu_i}.
\]

8. Moments for Gamma Distribution. The moment \( \mu'_{r_1,\ldots,r_n} \) for
gamma distribution about \((0,\ldots,0)\) of order \( r_1,\ldots,r_n \), is defined as the
coefficient of \( \frac{t_1^{r_1}}{r_1!}\cdots \frac{t_n^{r_n}}{r_n!} \) in \( M(t_p,\ldots,t_n) \) when it is expended in
powers of \( t_1,\ldots,t_n \). Thus an appeal to (2.1) gives
\[
\mu'_{r_1,\ldots,r_n} = \frac{(\mu_1)^{r_1}\cdots(\mu_n)^{r_n} (\mu_1,\ldots+\mu_n)^{r_1+\cdots+r_n}}{(\mu_1,\ldots+\mu_n)^{r_1+\cdots+r_n}}.
\]

9. Moment for Beta Distribution. The moment \( \mu'_{r_1,\ldots,r_n} \) of density
function \( F(x_1,\ldots,x_n) \) about \((0,\ldots,0)\) for beta distribution is defined as
\[
\mu'_{r_1,\ldots,r_n} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^{r_1}\cdots x_n^{r_n} F(x_1,\ldots,x_n) \, dx_1\cdots dx_n.
\]
Now substituting the value of \( F(x_1,\ldots,x_n) \) from (5.1) in (9.1) and making
an appeal to (2.1) and (2.3), we finally derive
\[
\mu'_{r_1,\ldots,r_n} = \frac{\Gamma(\alpha+\lambda-(r_1,\ldots,\lambda+r_n)) (\mu_1)^{r_1}\cdots(\mu_n)^{r_n}}{\Gamma(\alpha+\mu_1,\ldots+\mu_n) (\mu_1,\ldots+\mu_n)^{r_1+\cdots+r_n}}.
\]

10. Special Cases. For \( n=1 \), from (7.3), we derive the following
m.g.f. for gamma distribution:
\[
M(t_1) = (1 - t_1/\lambda)^{-\mu_1}.
\]
For \( n=1 \), from (8.1), we obtain the following moment of \( r_1 \)th order
about origin for gamma distribution:
\[
\mu'_{r_1} = \frac{(\mu_1)^{r_1}}{\lambda^{r_1}}.
\]
Also for \( n=1 \), (9.2) gives the following moment of \( r_1 \)th order for beta
distribution:
\[
\mu'_{r_1} = \frac{\Gamma(\alpha+\lambda-r_1)}{\Gamma(\lambda) \Gamma(\alpha+\mu_1)}.
\]
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