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SOME EXPECTATIONS ASSOCIATED WITH MULTIVARIATE GAMMA AND BETA DISTRIBUTIONS INVOLVING THE MULTIPLE HYPERGEOMETRIC FUNCTION OF SRIVASTAVA AND DAOUST

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ABSTRACT

In the present paper, we obtain some density functions associated with the multivariate gamma and beta distributions and make their applications to obtain the expectations involving multiple hypergeometric function of Srivastava and Daoust [48] (see also Srivastava and Manocha [51], p.64). Finally. we also derive the moments for these multivariate beta and gamma distributions and discuss their special cases.

1. Introduction. Different distributions have discussed by various authors Block and Rao [1], Carlson [2], Daley [5] Datt [6]. Kabe [8], Kaufman, Mathai and Saxena [9], Kendall [10], Khatri and Pillai [11,12]. Khatri and Srivastava [13], Littler and Fackerell [15], Lukacs and Naha [16], Lukacs [17], Mathai ([18] to [29]), Mathai and Rathie ([30] to [35]), Mathai and Saxena ([36] to [42]), Miller [43], Pillai, A1-Ani and Jouris [44], Pillai and Jouris [45], Pillai and Nagarsenker [46], Robbins and Pitman [47], Strawderman [52], Thaung [53], and Wilks [54]. Srivastava and Singhal [50], studied many of the classical statistical distributions, which were associated with the beta and gamma distributions. Further Exton [7], discussed generalized beta and gamma distributions with other special multivariate distributions,like Dirichlet distributions and multivariate normal distributions. He also discussed the expectations of some functions invoiving Lauricalla's multiple hypergeometric functions [14]. Recently the authors Chandel and Vishwakarma [4] discussed some multivariate beta and gamma distributions and made their applications to derive their expectations in terms of different multiple hypergermetric functions of several veriables.

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In the present paper, we extend the above work and establish some probability density functions associated with the multivariate beta and gamma distributions and make their applications to obtain some expectations involving the most generalized multiple hypergeometric function of Srivastava and Daoust [48] (see also Srivastava and Manocha ([51],p.64). Finally,we also drive the moments for these multivariate beta and gamma distributions and discuss their special cases.

2. Formulae Required. For ready stock, in this section we write the following results which will be used in our investigations:

The Liouville's Theorem (Also see Chandel [3,p.83 (3.1)]

$$(2.1)\ \int_0^\infty \cdots \int_0^\infty f(x_1+\ldots+x_n)\quad x_1^{\mu_1-1}\ldots x_n^{\mu_n-1}\,dx_1\ldots dx_n$$

$$=\frac{\Gamma(\mu_1)\ldots\Gamma(\mu_n)}{\Gamma(\mu_1+\ldots+\mu_n)}\int_0^\infty f(t)\quad t^{\mu_1+\ldots+\mu_n-1}\,dt,$$

provided that $Re\,(\mu_i) > 0,\ i=1,\ldots,n.$

Euler's definition for gamma function

$$(2.2)\quad \Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt, \qquad\qquad Re\,(z)>0.$$

The definition of beta function (see, Srivastava and Manocha [51,p.26 eq.(4.6)])

$$(2.3)\quad B(\alpha,\beta)=\int_0^\infty \frac{\mu^{\alpha-1}}{(1+\mu)^{\alpha+\beta}}\,d\mu\,,\ Re\,(\alpha)>0,\ Re\,(\beta)>0.$$

3. Multivariate Gamma Distribution. Consider the function

$$(3.1)\ f(x_1,\ldots,x_n)=\frac{\Gamma(\mu_1+\ldots+\mu_n)\,\lambda^{\mu+\mu_1+\ldots+\mu_n}}{\Gamma(\mu_1)\ldots\Gamma(\mu_n)\quad\Gamma(\mu+\mu_1+\ldots+\mu_n)}\qquad exp\{-(x_1+\ldots+x_n)\lambda\}$$

$$(x_1+\ldots+x_n)^{\mu}$$

provided that $Re(\lambda)>0,\ x_i\geq 0,\ Re\,(\mu_i)>0,\ i=1,\ldots,n$ and $f(x_1,\ldots,x_n)=0$ elsewhere.

Making an appeal to (2.1) and (2.2) the value of multiple integral of $f(x_1,\ldots,x_n)$ over the region defined above in (3.1) becomes unity. Hence $f(x_1,\ldots,x_n)$ is a probability density function for multivariate gamma distribution.

4. Expectation Associated with Multivariate Gamma Distribution.

The expectation value of the function $g(x_1,\ldots,x_n)$ is defined as

$$(4.1)\ \langle g(x_1,\ldots,x_n)\rangle=\int_{\infty}^{\infty}\ldots\int_{\infty}^{\infty} f(x_1,\ldots,x_n)\,g(x_1,\ldots,x_n)\,dx_1\ldots dx_n$$

corresponding to density function $f(x_1,\ldots,x_n)$ defined by (3.1).

Consider the function

$$(4.2)\ g_1(x_1,\dots,x_n) = F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}];[(b'),\phi'];\dots;\\ [(c):\psi',\dots,\psi^{(n)}];[(d'),\delta'];\dots;\end{matrix}\right.$$

$$\left.\begin{matrix}[(b^{(n)}):\phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\ z_1(x_1+\dots+x_n)^{\nu_1},\dots,z_n(x_1+\dots+x_n)^{\nu_n}\right]$$

where $F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}$ is most generalized multiple hypergeometric function of Srivastava and Daoust [48] (also see Srivastava and Manocha [51, (18),(19),(20),p.64)]).

Now making an appeal to (2.1) and (2.2)the expectation of $g_1(x_1,\dots,x_n)$having density function $f(x_1,\dots,x_n)$ is given by

$$(4.3)\ \langle g_1(x_1,\dots,x_n)\rangle = F^{A+1:B';\dots;B^{(n)}}_{C\ \ :D';\dots;D^{(n)}}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}];\\ [(c):\Psi',\dots,\Psi^{(n)}];\end{matrix}\right.$$

$$\left.\begin{matrix}[\mu+\mu_1+\dots+\mu_n:\nu_1,\dots,\nu_n]:[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}];\\ [(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ \frac{z_1}{\lambda_1^{\nu_1}},\dots,\frac{z_n}{\lambda_n^{\nu_n}}\right]$$

provided that

$$1+\sum_{j=1}^{C}\Psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}-\nu_i>0,\ i=1,\dots,n.$$

Corresponding to density fuction $f(x_1,\dots,x_n)$ defined by (3.1), if we consider the function

$$(4.4)\ \langle g_2(x_1,\dots,x_n) = F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]:[(b'):\phi'];\\ [(c):\Psi',\dots,\Psi^{(n)}]:[(d'):\delta'];\end{matrix}\right.$$

$$\left.\begin{matrix}\dots;[(b^n):\phi^n];\\ \dots;[(d^n):\delta^n];\end{matrix}\ z_1x_1^{\alpha_1}(x_1+\dots+x_n)^{\nu_1},\dots,z_nx_n^{\alpha_n}(x_1+\dots+x_n)^{\nu_n}\right]$$

Then expectation of g_2 is given by

$$(4.5)\ \ g_2(x_1,\dots,x_n)\rangle = F^{A+1:B';\dots;B^{(n)}+1}_{C+1:\ D';\dots;D^{(n)}}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}],\\ [(c):\Psi',\dots,\Psi^{(n)}],\end{matrix}\right.$$

$$\begin{matrix}[\mu+\mu_1+\dots+\mu_n:\alpha_1+\nu_1,\dots,\alpha_n+\nu_n]:[(b'):\phi'],[\mu_1:\alpha_1];\dots;[(b^n):\phi^{(n)}],[\mu_n:\alpha_n];\\ [\mu+\mu_1+\dots+\mu_n:\alpha_1,\dots,\alpha_n]:[(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}$$

$$\left.z_1/\lambda^{\alpha_1+\nu_1},\dots,z_n/\lambda^{\alpha_n+\nu_n}\right],$$

valid if

$$1+\sum_{j=1}^{C}\Psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}-\alpha_i-\nu_i>0,\ i=1,\dots,n.$$

5. The Multivariate Beta Distribution.

Consider the function $F(x_1,\dots,x_n)$defined by

(5.1) $F(x_1,\dots,x_n) = \dfrac{\Gamma(\mu_1+\dots+\mu_n)\,\Gamma\alpha+\lambda+\mu_1+\dots+\mu_n)\,(x_1+\dots+x_n)^{\alpha}}{\Gamma(\mu_1)\dots\Gamma(\mu_n)\,\Gamma(\lambda)\,\Gamma(\alpha+\mu_1+\dots+\mu_n)\,(1+x_1+\dots+x_n)^{\alpha+\lambda+\mu_1+\dots+\mu_n}}$

$Re(\alpha)>0, Re(\mu_i)>0, x_i>0\ i=1,\dots,n$ and $F(x_1,\dots,x_n)=0$ elsewhere.

Now making an appeal to (2.1) and (2.3) the value of multiple integral of $F(x_1,\dots,x_n)$ over the region defined above in (5.1), becomes unity.

Hence $F(x_1,\dots,x_n)$ is probability density function for multivariate beta distribution.

6. Expectation Associated with Multivariate Beta Distribution.

Correspondiing to density function $f(x_1,\dots,x_n)$ defined by (5.1), consider the function

$$(6.1)\ G(x_1,\dots,x_n) = F^{A':B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]:[(b'):\phi'];\dots;\\ [(c):\psi',\dots,\psi^{(n)}]:[(d'):\delta'];\dots;\end{matrix}$$

$$\begin{matrix}[(b^{(n)}):\phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\ \frac{z_1 x_1^{\xi_1}(x_1+\dots+x_n)^{\eta_1}}{(1+x_1+\dots+x_n)^{\zeta_1+n_1}},\dots,\frac{z_n x_n^{\xi_n}(x_1+\dots+x_n)^{\eta_n}}{(1+x_1+\dots+x_n)^{\zeta_n+\eta_n}}$$

Then expectation of $G(x_1,\dots,x_n)$ is given by

$$(6.2)\ \langle G(x_1,\dots,x_n)\rangle$$

$$= F^{A+1:B'+;\dots;B^{(n)}+1}_{C+2:D';\dots;D^{(n)}}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}],[\alpha+\mu_1+\dots+\mu_n:\eta_1+\xi_1,\dots,\eta_n+\xi_n]:\\ [(c):\psi',\dots,\psi^{(n)}],[\mu_1+\dots+\mu_n:\xi_1,\dots,\xi_n],\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'],[\mu_1:\xi_1],\dots,[(b^{(n)}):\phi^{(n)}],[\mu_n:\xi_n];\\ [\alpha+\lambda+\mu_1+\dots+\mu_n:\eta_1+\zeta_1,\dots,\eta_n+\zeta_n]:[(d'):\delta'];\dots;[(d^n):\delta^n];\end{matrix}\ z_1,\dots,z_n\right]$$

where

$$1+\sum_{j=1}^{C}\Psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}+\zeta_i-\xi_i>0,$$

$i=1,\dots,n.$

7. Moment Generating Function (*m.g.f.*)for Gamma Distribution.

The *m.g.f.* is defined as

$$(7.1)\quad M(t_1,\dots,t_n)\quad \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} e^{x_1t_1+\dots+x_nt_n} f(x_1,\dots,x_n)\,dx_1\dots dx_n,$$

provided that the integral is a function of the parameters $t_1,\dots,t_n$ only. Thus *m.g.f.* for multivariate gamma distribution (3.1) is given by

$$(7.2)\ M(t_1,\dots,t_n) = \frac{\Gamma(\mu_1+\dots+\mu_n)\ \lambda^{\mu+\mu_1+\dots+\mu_n}}{\Gamma(\mu_1)\dots\Gamma(\mu_n)\ \Gamma(\mu+\mu_1+\dots+\mu_n)}$$

$$\int_0^{\infty}\dots\int_0^{\infty} e^{\,x_1t_1+\dots+x_nt_n}\, e^{-(x_1+\dots+x_n)\lambda}\,(x_1+\dots+x_n)^{\mu}\, x_1^{\mu_1-1}\dots x_n^{\mu_n-1}\,dx_1\dots dx_n$$

Now making an appeal to (2.1) and the result due to Srivastava[49,p,4.(12).]

$$\sum_{m_1,\dots,m_n=0}^{\infty} f(m_1+\dots+m_n)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}=\sum_{M=0}^{\infty} f(M)\ (x_1+\dots+x_n)^M, n\geq 1,$$

we finally derive

(7.3) $M(t_1,..,t_n) = F_D^{(n)}(\mu+\mu_1+...+\mu_n\,,\,\mu_1,...,\mu_n\,;\,\mu_1+...+\mu_n\,;\,\frac{t_1}{\lambda},...,\frac{t_n}{\lambda}),$

where $F_D^{(n)}$ is Lauricella's fourth multiple hypergeometric function of several variables [14].

As a special case for $\mu = 0$, (7.3) gives

(7.4) $M(t_1,...,t_n) = \prod_{i=1}^{n} (1-t_i/\lambda)^{-\mu_i}.$

8. Moments for Gamma Distribution. The moment $\mu'_{r_1,...,r_n}$ for gamma distribution about $(0,...,0)$ of order $r_1,...,r_n$, is defined as the coefficient of $\frac{t_1^{r_1}}{r_1!}\cdots\frac{t_n^{r_n}}{r_n!}$ in $M(t_1,...,t_n)$ when it is expended in powers of $t_1,...,t_n$. Thus an appeal to (2.1) gives

(8.1) $\mu'_{r_1,...,r_n}\ \frac{(\mu_1)_{r_1}\cdots(\mu_n)_{r_n}\quad(\mu+\mu_1+...+\mu_n)_{r_1+...+r_n}}{(\mu_1+...+\mu_n)_{r_1+...+r_n}\quad\lambda^{r_1+...+r_n}}.$

9. Moment for Beta Distribution. The moment $\mu'_{r_1,...,r_n}$ of density function $F(x_1,...,x_n)$ about $(0,...,0)$ for beta distribution is defined as

(9.1) $\mu'_{r_1,...,r_n} = \int_{\infty}^{\infty}\cdots\int_{\infty}^{\infty} x_1^{r_1}...x_n^{r_n}F(x_1,..,x_n)\ dx_1...dx_n.$

Now substituting the value of $F(x_1,...,x_n)$ from (5.1) in (9.1) and making an appeal to (2.1) and (2.3), we finally derive

(9.2) $\mu'_{r_1,...,r_n} = \frac{\Gamma(\alpha+\lambda-(r_1+,...,+r_n))\,(\mu_1)_{r_1}...(\mu_n)_{r_n}}{\Gamma(\lambda)\ \Gamma(\alpha+\mu_1+...+\mu_n)\,(\mu_1+...+\mu_n)_{r_1+...+r_n}}.$

10. Special Cases. For $n=1$, from (7.3), we derive the following *m.g.f.* for gamma distribution:

(10.1) $M(t_1) = (1-t_1/\lambda)^{-\mu+\mu_1}.$

For $n=1$, from (8.1), we obtain the following moment of r_1th order about origin for gamma distribution:

(10.2) $\mu'_{r_1} = \frac{(\mu+\mu_1)_{r_1}}{\lambda^{r_1}}.$

Also for $n=1$, (9.2) gives the following moment of r_1th order for beta distribution :

(10.3) $\mu'_{r_1} = \frac{\Gamma(\alpha+\lambda-r_1)}{\Gamma(\lambda)\,\Gamma(\alpha+\mu_1)}.$

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