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ON DISTRIBUTIONAL STRUVE TRANSFORMATION

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ABSTRACT

The classical Struve transformation has been extended to a class of generalized functions. The generalized Struve transformation, so defined is seen to be a smooth function.

1. Introduction. The integral transfrom

$$(H_\upsilon f)(x) = \int_0^\infty \sqrt{(xt)}\, H_\upsilon(xt) f(t)\, dt \qquad \text{... (1.1)}$$

where $H_\upsilon(x)$ is the Struve function, has been studied briefly in [5, 8.4]. Recently, Love, [2], Rooney, [4] and Heywood and Rooney [1] have studied it in details.

The aim of the present work is to extend the Struve transform (1.1) to a class of generalized functions. It will be shown that the generalized Struve transform is a smooth function.

R.S. Pathak and J.N. Pandey [3] have extended the Hardy transformation given by

$$f(x) = \int_0^\infty F_\upsilon(tx)\, t dt\, C_\upsilon(ty) y\, f(y) dy, \qquad \text{... (1.2)}$$

where $C_\upsilon(z) = \cos p\pi\, J_\upsilon(z) + \sin p\pi\, Y_\upsilon(z)$

and $$F_\upsilon(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\upsilon+2p+2m}}{\Gamma(p+m+1)\,\Gamma(p+m+\upsilon+1)}\,.$$

It has been shown there that the Struve transform (1.1) is a special case of (1.2) when $p = 1/2$. The present paper deals the transform (1.1) Indepenently and differently.

2. Definition and properties of Struve's function. In this section, we quote some results on the Struve's function which we need hereafter of and on.

The Struve's function $H_\upsilon(z)$, of oreder υ. is defined by the equations

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[6, p. 328].

$$H_\upsilon(z) = \frac{(z/2)^\upsilon}{\Gamma(\upsilon+1/2)\,\Gamma(1/2)} \int_0^1 (1-t^2)^{\upsilon-1/2} \sin zt.\, dt \qquad \dots (2.1)$$

$$= \frac{(z/2)^\upsilon}{\Gamma(\upsilon+1/2)\,\Gamma(1/2)} \int_0^{\pi/2} \sin(z\cos\theta)\, \sin^{2\upsilon}\theta\, d\theta, \qquad \dots (2.2)$$

provided that $Re(\upsilon) > -1/2$, or equivalently

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\upsilon+2m+1}}{\Gamma(m+3/2)\,\Gamma(\upsilon+m+3/2)}. \qquad \dots (2.3)$$

Also, [6,p. 328]

$$\frac{d}{dz}\{z^\upsilon H_\upsilon(z)\} = z^\upsilon H_{\upsilon-1}(z), \qquad \dots (2.4)$$

$$\frac{d}{dz}\{z^{-\upsilon} H_\upsilon(z)\} = \frac{1}{2^\upsilon\,\Gamma(\upsilon+3/2)\,\Gamma(1/2)} - z^{-\upsilon} H_{\upsilon+1}(z). \qquad \dots (2.5)$$

From [6, p. 337], we have

$$H_\upsilon(z) = o(z^{-1/2}) \quad (\upsilon \le 1/2)$$
$$= o(z^{\upsilon-1}) \quad (\upsilon \ge 1/2) \text{ as } z \to \infty \qquad \dots (2.6)$$

Also, $H_\upsilon(z) = o(z^{\upsilon+1})$ as $z \to 0$ .... (2.7)

3. The testing function space $H_a(I)$ and its dual $H'_a(I)$.

Let I be the interval $(0,\infty)$ and a, a fixed positive number. An infinitely differentiable function $\phi(x)$ defined over I is said to belong to $H_a(I)$ if

$$\gamma_k(\phi) = \sup_{0<t<\infty} |e^{-at}(tD)^k \phi(t)| < \infty \qquad \dots (3.1)$$

for $k = 0, 1, 2, \dots$ and $D = d/dt$.

The topology on $H_a(I)$ is defined by means of the separating collection of semi-norms $\{\gamma_k\}_{k=0}^{\infty}$. A sequence $\{\phi_\upsilon\}_{\upsilon=1}^{\infty}$ is said to converge in $H(I)$ to the limit ϕ if $\gamma_k(\phi_\upsilon - \phi) \to 0$ as $\upsilon \to \infty$ for each $k = 0, 1, 2, \dots$.

A sequence $\{\phi_\upsilon\}_{\upsilon=1}^{\infty}$ is said to be a Cauchy sequence in $H(I)$ if $\gamma_k(\phi_\upsilon - \phi_\mu)$ goes to zero as υ, μ both go to infinity independently of each other. It can be readily seen that $H_a(I)$ is locally convex, sequentially complete, Hausdorff topological vector space. The dual of $H_a(I)$ will be represented by $H'_a(I)$.

Lemma 3.1 . For $a > 0$, $\upsilon > -1/2$ and $t, x > 0$, then for a fixed $x > 0$, $h(xt) = \sqrt{(xt)}\, H_\upsilon(xt) \in H_\upsilon(I)$, $H_\upsilon(z)$ being Struve fucntion as defined in Section 2.

Proof. From differential properties of Struve functions, we have

$$|e^{-at}(tD_t)^k h(xt)|$$

$$= |e^{-at} \sum_{j=0}^{k} a_j(\upsilon)(xt)^{(1/2)+j} H_{\upsilon-j}(xt)| < \infty,$$

where $a_j(\upsilon)$ is a polynomial in υ.

Lemma 3.2. For $a > 0$, $\upsilon = -1/2$ and $t, x > 0$,

$$\frac{\partial^m}{\partial x^m}[\sqrt{(xt)}\, H_\upsilon(xt)] \in H_a(I).$$

Proof. We have

$$|e^{-at}(tD_t)^k \frac{\partial^m}{\partial x^m}[\sqrt{}^n (xt)^k H_\upsilon(xt)]|$$

$$= |e^{-at}(\sum_{j=0} \sum_{p=0} A_{j,p}(\upsilon).\, x^{-m}(xt)^{j+p+1/2} H_{\upsilon-j-p}(xt))| < \infty,$$

for a fixed $x > 0$, $\upsilon > -1/2$.

We now enlist some properties of the spaces $H_a(I)$ and its dual $H'_a(I)$.

Property 3.1. For $0 < a < b < 1/2$, $D(I) \subset H_b(I) \subset H_a(I) \subset E(I)$, all inclusions being continuous. Moreover as $D(I)$ is dense in $E(I)$, $H_a(I)$ is dense in $E(I)$.

Property 3.2. The dual $H'_a(I)$ equipped with the usual weak topology is Hausdorff locally convex, sequentially complete space of generalized functions.

For $f \in H'_a(I)$ there exists $C > 0$ and an integer r (C and r depending on f) such that

$$|<f, \phi>| \leq C \max_{0 \leq k \leq r} \gamma_k(\phi).$$

Property 3.3. A locally integrable function f in I such that $e^{at}f(t)$ is absolutely integrable on I, gives rise to a regular generalized function of $H'_a(I)$ through

$$|<f, \phi>| = \int_0^\infty f(t)\phi(t)\, dt, \ \forall \phi \in H_a(I).$$

Property 3.4. The differential operator (tD_t) is a continuous linear mapping from $H_a(I)$ into itself. $(t D_t)'$, the adjoint of $(t D_t)$ maps continuously $H'_a(I)$ into itself.

4. The generalized transform. For $f \in H'_a(I)$, the generalized Struve transform is defined by

$$s[f] = F(x) = <f(t), \sqrt{(xt)}\, H_\upsilon(xt)> \qquad \text{... (4.1)}$$

where x is a non-zero real number and $t > 0$. From Lemma 3.1 we know that for fixed $x > 0$,

$$\sqrt{(xt)}\, H_\upsilon(xt) \in H_a(I),$$

where $\upsilon > -1/2$, $a > 0$. The relation (4.1) is meaningful.

Theorem 4.1. For real $x \neq 0$, let $F(x)$ be defined by (4.1). Then $F(x)$ is infinitely differentiable and that

$$F^m(x) = <f(t), \frac{\partial^m}{\partial x^m}[\sqrt{(xt)}\, H_a(xt)] \qquad \text{... (4.2)}$$

for all real $x \neq 0$ and $m=1,2, \ldots$.

Proof. By Lemma 3.2, it follows that $\frac{\partial^m}{\partial x^m}[\sqrt{(xt)}\, H_a(xt)] \in H_a(I)$. Hence, we need merely to prove (4.2), what we do through an inductive argument. We assume that (4.2) is true for m replaced by $(m-1)$. It is true by definition for $m = 0$. Keeping x fixed and $\partial x \neq 0$, consider

$$\frac{1}{\partial x}[D_x^{m-1} F(x+\partial x) - D_x^{m-1} F(x) - < f(t), D_x^m \sqrt{(xt)}\, H_\upsilon(xt) >$$

$$= \frac{1}{\partial x}[D_x^{m-1} < f(t), \sqrt{[(x+\partial x)t]}\, H_\upsilon(\overline{x+\partial x}\, t)> - D_x^{m-1} < f(t), \sqrt{(xt)} H_\upsilon(xt) >]$$

$$- < f(t), D_x^m [\sqrt{(xt)}\, H_\upsilon(xt)]>$$

$$= < f(t), \frac{D_x^{m-1} h(\overline{x+\partial x}\, t)\, D_x^{m-1} h(xt)}{\partial x} - D_x^m h(xt) >$$

(writing $h(xt)$ for $\sqrt{(xt)}\, H_\upsilon(xt)$.)

$$= < f(t), A_{\partial x}(t) > (say) \qquad \ldots (4.3)$$

where,

$$A_{\partial x}(t) = \frac{D_x^{m-1} h(\overline{x+\partial x}\, t)\, D_x^{m-1} h(xt)}{\partial x} - D_x^m h(xt)$$

$$= \frac{1}{\partial x}\int_x^{x+\partial x} + D_u^m h(ut)\, du - \frac{1}{\partial x}\int_x^{x+\partial x} D_x^m h(xt)\, du$$

$$= \frac{1}{\partial x}\int_x^{x+\partial x} [D_u^m h(ut) - D_x^m h(xt)]\, du$$

$$= \frac{1}{\partial x}\int_x^{x+\partial x} du \int_x^u D_\eta^{m+1} h(\eta t)\, d\eta.$$

Now

$$| e^{-at} (tD_t)^k A_{\partial x}(t) |$$

$$= | e^{-at} \frac{1}{\partial x}\int_x^{x+\partial x} du \int_x^u D_\eta^{m+1} \{(tD_t)^k h(\eta t)\}\, d\eta |$$

$$= | e^{-at} \frac{1}{\partial x}\int_x^{x+\partial x} du \int_x^u D_\eta^{m+1} \{\Sigma a_j(\upsilon)(\eta t)^{1/2+j} H_{\upsilon-j}(\eta t)\, d\eta |$$

Let Δ denotes the interval $x - |\partial x| < n < x + |\partial x|$.

Then,

$$| e^{-at} (tD_t)^k A_{\partial x}(t) |$$

$$\leq \frac{|\partial x|}{2} e^{-at} \sup | D_\eta^{m+1} \sum_{j=0}^{k} a_j(\upsilon)(\eta t)^{1/2+j} H_{\upsilon-j}(\eta t) | \qquad \ldots (4.4)$$

Now, by Lemma 2.1,

$$e^{-at\ n\in\Delta}\ |D_{\eta}^{m+1} \sum_{j=0}^{k} a_j(\upsilon)(\eta t)^{1/2+j} H_{\upsilon-j}(\eta t)\ |$$

is bounded on $0 < t < \infty$, (taking $|\partial(x)| < 1$). Therefore, It follows from (4.4) that $A_{\partial x}(t)$ converges in $H(I)$ to zero as $\partial x \to 0$. Since $f \in H'(I)$, (4.3) converges to zero as $\partial x \to 0$. This completes our inductive proof of (4.2).

REFERENCES

[1] P. Heywood and P.G. Rooney, On the Struve Transformation, *SAIM J. Math. Anal.*, **25** (2) (1994) 450-461.

[2] E.R. Love, Inversion of the Struve Transform, *Fractional Calculus* (Glasgow, 1984), 75-86, *Res. Notes in Math.*, **138**, London, 1985.

[3] R.S. Pathak and J.N. Pandey, A distributional Hardy Transformation. *Proc. Camb. Phil. Soc.*, **76** (1974), 247-265.

[4] P.G. Rooney, On the Y_υ and H_υ transformations, *Canad. J. Math.*, **32** (5) (1980), 1021-1044.

[5] E.C. Titchmarsh, *Theory of Fourier Integrals*, Oxford, 1948.

[6] *G.N. Watson, Theory of Bessel Functions, Cambridge, 1962.*