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A REMARK ON SASAKIAN RECURRENT SPACES OF THE FIRST KIND

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1. Introduction. Let us consider an n-dimensional affinely connected sasakian space s_n with recurrent curvature R^i_{jkl} admitting an infinitesimal transformation :

$$\overline{x}^i = x^i + v^i(x)\,\delta t,$$

and we assume a basic condition $\underset{v}{\mathrm{L}}\lambda_m = 0$, where λ_m is a non-zero recurrence vector appeared in the equation

$$\nabla_m R^i_{jkl} = \lambda_m R^i_{jkl},$$

$\underset{v}{\mathrm{L}}$ denotes the Lie-derivative with respect to v^i and ∇ denotes the operator of covariant differentiation with respect to the Riemannian connection. In this case, the space has been called a Sasakian recurrent space of the first kind and is denoted in brief by an S^*_n – space. Under a condition $\alpha = \lambda_m v^m \neq$ constant, we have

$$Akl \overset{def}{=\!=} \nabla_l \lambda_k - \nabla_k \lambda_l \neq 0.$$

The present space with $\lambda \neq$ constant is called a Sasakian recurrent spaces of the first kind, or in brief an $-A\ S^*_n$–space.

Takano (1966) has studied the existence of an affine motion of recurrent form and give a remark there and it is read as follows

In this space satisfying $\underset{v}{\mathrm{L}} R^i_{jkl} = 0$, under a docomposition of curvature tensor of the form :

$\alpha R^i_{jkl} = Akl\,\nabla_j v^i$ in order to have $\underset{v}{\mathrm{L}} \Gamma^i_{jk} = 0$, we can suppose formally a parallel property of contravariant vector $\beta v^i - \alpha\,\eta^i$, but this supposition yields a contradiction.

However, generally speaking, the present space itself does not admit intrinsically such a vector field. In this short report, we shall prove this fact.

2. Some Formulae. The space with recurrent curvature is characterized by a basic condition

(2.1) $\nabla_m R^i_{jkl} = \lambda_m R^i_{jkl},$

and the useful fundamental formulae are the first and second Bianchi's identities given by

(2.2) $R^i_{klj} + R^i_{ljk} + R^i_{jkl} = 0,$

(2.3) $\lambda_m R^i_{jkl} + \lambda_k R^i_{jlm} + \lambda_l R^i_{jmk} = 0.$

An $A\ S^*_n$–space has not only (2.1), (2.2) and (2.3) but also the following formulae :

(2.4) $A_{kl}\, v^l = -\alpha.\alpha_k\,,$

(2.5) $\alpha_i v^i = 0$, *i.e.* , $\text{Ł}_v \alpha = 0,$

and

(2.6) $A_{kl} = \lambda_k \alpha_l - \lambda_l\, \alpha_k.$

The well-known Ricci's identity in this space may be derived from (2.1) with ease and it is written as

(2.7) $A_{mn}\, R^i_{jkl} = R^\alpha_{jkl}\ R^i_{\alpha mn} - R^i_{\alpha kl}\, R^i_{jmn} - R^i_{j\alpha l}\, R^\alpha_{kmn} - R^i_{jk\alpha}\, R^i_{lmn}.$

3. Main Subject. Being $\alpha_k \neq 0$, we can introduce a vector η^k so as to satisfy

(3.1) $\alpha_k \eta^k = 1.$

Then, multiplying (2.7) by $\eta^m\ v^n$ side by side and summing over m and n, we get

(3.2) $\alpha R^i_{jkl} + R^\alpha_{jkl}\ A^i_\alpha\ - R^i_{\alpha kl}\, A^\alpha_j - R^i_{j\alpha l}\, A^\alpha_k - R^i_{jk\alpha}\, A^\alpha_l\ = 0\,,$

where we have used $A_{mn}\, \eta^m\ v^n\ = -\alpha$, derived from (2.4) and (3.1), or (2.5), (2.6) and (3.1) and put $A^i_j = R^i_{jkl}\, \eta^k v^l$.

Next, multiplying (2.3) by v^m and summing over m owing to $R^i_{jmk} = -R^i_{jkm}$, we find

(3.3) $\alpha R^i_{jkl} + \lambda_k\, R^i_{jlm}\, v^m - \lambda_l\, R^i_{jkm}\, v^m = 0,$

from which, putting $\beta = \lambda_m \eta^m$, we are able to have

(3.4) $\lambda_k A^i_j = (\ \beta v^l - \ \alpha\eta^l)\ R^i_{jkl}.$

Eliminating $\alpha\, R^i_{jkl}$ from (3.2) and (3.3), we have

$R^\alpha_{jkl} A^i_\alpha - R^i_{\alpha kl}\, A^\alpha_j\ = R^i_{j\alpha l}\, (A^\alpha_k - \lambda_k\ v^\alpha) - R^i_{j\alpha k}\, (A^\alpha_l - \lambda_l v^\alpha)\ .$

At this moment, if we multiply the last formula by $\beta v^l - \alpha\, \eta^l$ and sum over the index l, then by virtue of (3.4), it follows that

(3.5) $A^i_j\ \lambda_\alpha\, (A^\alpha_k - \lambda_k\ v^\alpha) = R^i_{j\alpha k}\, (A^\alpha_k - \lambda_l v^\alpha)\ (\beta v^l - \alpha\eta^l).$

Now, from $R^i_{jkl} = -R^i_{jlk}$ and (2.2), we see that $R^i_{jkl} - R_{kjl} = -R^i_{ljk}$, so from (3.4), we can derive the relation

$$\lambda_k A^i_j - \lambda_j A^i_k = -\ (\beta v^l - \alpha\eta^l)\ R^i_{ljk}.$$

Hereupon, let us assume the parallelism of $\beta v^i - \alpha\eta^i$, then we have immediately $(\beta v^l - \alpha\eta^l)\ R^i_{ljk} = 0$, so under the present assumption, it is concluded that

$\lambda_k A^i_j = \lambda_j A^i_k$.

Being $\lambda_k \neq 0$, from the last formula, we may put

(3.6) $A^i_k = \lambda_k A^i$,

where A^i indicates a suitable vector. Making use of (3.6) into the right hand side of (3.5) and taking care of $(\beta \upsilon^l - \alpha n^l)\, \lambda_l = 0$, we have $A^i_j (\mu_k - \alpha\lambda_k) = 0$, where we have defined μ_k by $A^i_k \lambda_i$. It is easy to see that $A^i_j \neq 0$, and we have

(3.7) $\mu_k = \alpha\, \lambda_k$.

From (3.6), it follows that $\mu_k = \lambda_i A^i_k = \lambda_i A^i \lambda_k$ comparing the last result with (3.7), owing to, $\lambda_k \neq 0$ we get

(3.8) $\lambda_m A^m = \alpha$.

Substituting (3.6) into (3.2), we have

$\alpha\, R^i_{jkl} - (\nabla_l \nabla_k - \nabla_k \nabla_l)\, A^i_j - A^\alpha\, (\lambda_k R^i_{j\alpha l} - \lambda_l R^i_{jk\alpha}) = 0$

or by virtue of (2.3), we get

$\alpha\, R^i_{jkl} + A^\alpha\, \lambda_\alpha\, R^i_{jlk} = (\nabla_l \nabla_k - \nabla_k \nabla_l)\, A^i_j$.

Making use of $R^i_{jkl} = -R^i_{jlk}$ and (3.8), the above equation becomes

$$\nabla_l \nabla_k A^i_j - \nabla_k \nabla_l\; A^i_j = 0.$$

In this case, for a suitable vector δ_k, we can regard A^i_j to be a recurrent tensor given by

(3.9) $\nabla_k A^i_j = \delta_k\, A^i_j$.

Differentiating (3.4) covariantly and making use of parallelism of $\beta\upsilon^i - \alpha\eta^i$, we find

$$A^i_j \nabla_l \lambda_k + \lambda_k \delta_l A^i_j = \lambda_l \lambda_k A^i_j,$$

where we have used (2.1) and (3.9).

Being $A^i_j \neq 0$, from the last formula, we get

(3.10) $\nabla_l \lambda_k = \lambda_k\, \lambda_l - \lambda_k \delta_l$.

Now, remembering (3.6), (3.9) may be rewritten as

$$\lambda_j \nabla_k A^i + A^i\, A^i{}_k \lambda_j = \delta_k\, \lambda_j A^i.$$

Consequently, substituting (3.10) into the last relation, we get

(3.11) $\nabla_k A^i = (2\delta_k - \lambda_k) A^i$,

where we have neglected non-vanishing λ_j. Multiplying (3.11) by λ_i and summing over i, according to

$$\begin{aligned} \lambda_i \nabla_k A^i &= \nabla_k (\lambda_i A^i) - A^i \nabla_k \lambda_i \\ &= \alpha\, \alpha_k - A^i\, (\lambda_i \lambda_k - \lambda_i \delta_k) \\ &= \alpha\, \alpha_k - \alpha\lambda_k + \alpha\delta_k, \end{aligned}$$

and making an appeal to (3.8) and (3.10), we have

$$\alpha\, \alpha_k - \alpha\lambda_k + \alpha\delta_k = \alpha\, (2\delta_k - \lambda_k),$$

or $\alpha(\delta_k - \lambda_k) = 0,$

i.e. $\delta_k = \lambda_k.$

In this way, (3.10) becomes

$$\nabla_l \lambda_k = \lambda_k \lambda_l - \lambda_k \alpha_l .$$

hence we can see that

$$Akl = \nabla_l \lambda_k - \nabla_k \lambda_l = -\lambda_k \alpha_l + \lambda_l \lambda_k .$$

Comparing the last result with (2.6), we have here $Akl = 0$ and so, we have

$$\alpha = c \text{ (constant)}.$$

This conclusing contradicts clearly with our preliminary assumption on $\alpha = \lambda_m \upsilon^m$, for the AS^*_n– space has been introduced orginally by basic condition $\underset{\upsilon}{\mathrm{L}} \lambda_m = 0$ and $\alpha \neq$ constant.

This completes the proof of the fact emphasized at the beginning of this report. Thus, we can state here a definite conclusion as follows :

The AS^*_n– space does not admit a parallel vector field defined by $\beta\upsilon^i - \alpha\eta^i$

4. Appendix. In the preceding section, under $\nabla_l \nabla_k A^i_j - \nabla_k \nabla_l A^i_j = 0$ we have supposed formally a condition (3.9). We shall show here the possibility of (3.9).

If $\nabla_l \nabla_k A^i_j - \nabla_k \nabla_l A^i_j = 0$ will be the case, the formula (3.2) is, simplified as

$$\alpha R^i_{jkl} = R^i_{j\alpha l} A^\alpha_k + R^i_{jk\alpha} A^\alpha_l .$$

Contracting $\eta^k \upsilon^l$ and using (3.6), we get

$$\alpha A^i_j = R^i_{j\alpha l} \upsilon^l (\eta^k \lambda_k) A^\alpha + R^i_{jk\alpha} \eta^k (\upsilon^l \lambda_l) A^\alpha,$$

say

$$\alpha A^i_j = R^i_{j\alpha l} (\beta \upsilon^l - \alpha \eta^l) A^\alpha.$$

Differentiating the last formula covariantly and making use of (2.1) and the parallelism of $\beta\upsilon^l - \alpha\eta^l$, we have

$$\alpha \alpha_m A^i_j + \alpha \nabla_m A^i_j = \alpha \lambda_m A^i_j + R^i_{j\alpha l} (\beta \upsilon^l - \alpha \eta^l) \nabla_m A^\alpha,$$

i.e. $\alpha \alpha_m A^i_j + \alpha \nabla_m A^i_j = \alpha \lambda_m A^i_j + A^i_j (\alpha \alpha_m - A^\alpha \nabla_m \lambda_\alpha),$

That is to say, we have

$$\alpha \nabla_m A^i_j = (\alpha \lambda_m - A^\alpha \nabla_m \lambda_\alpha) A^i_j,$$

or, being $\alpha \neq 0$, we can see

$$\nabla_m A^i_j = \delta_m A^i_j, \qquad \delta_m \overset{def}{=\!=} \lambda_m - A^\alpha \nabla_m \lambda_\alpha / \alpha.$$

This completes the proof.

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