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THE INTEGRATION OF CERTAIN PRODUCTS OF THE MULTIVARIABLE *H*-FUNCTION WITH GENERAL POLYNOMIALS AND EXTENDED JACOBI POLYNOMIALS

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ABSTRACT

In this paper the authors present four integral formulas for the *H*-function of several complex variables, which was introduced and studied by H.M. Srivastava and R. Panda ([16, 17]; see also [11]). Each of these integral formulas involves a product of the multivariable *H*-function, extended Jacobi polynomials and general polynomials with essentially arbitrary coefficients, which were considered else where by H.M. Srivastava [13]. By assigning suitable special values to these coefficients, the main results (contained in the theorem 1 and 2) can be reduced to integrals' involving the classical orthogonal polynomials including, for example, Hermite, Jacobi, [and, of course, Gegenbauer (or ultraspherical), Legendre, and Tchebycheff], and Laguerre polynomials, the Bessel polynomials considered by H.L. Krall and O. Frink [8], and such other classes of generalized hypergeometric polynomials as those studied earlier by F. Brafman [2] an by H.W. Gould and A.T. Hopper [7]. On the other hand, the multivariables *H*-function occuring in each of our main results can be reduced, under various special cases, to such simpler functions as the generalized Lauricella hypergeometric functions of several complex variables [due to H.M. Srivastava and M.C. Daoust (cf. [14] and [15])], which indeed include a great many of the useful functions (or the products of several such functions) of hypergeometric type (in one or more variables) as their particular cases (see, e.g., [1],[9]). We record here only two special cases of our main integrals, the first one involves the extended Jacobi

Polynomials, general polynomials and multivariable H-function, and the other one involves the extended Jacobi polynomials, general polynomials and Lauricella function of several complex variables. Out of several known results which follow as special cases of our integrals we refer here only to the results of Srivastava and Singh [18], Srivastava and Panda [17] and Sharma [10].

1 Introduction. The H-Function of several complex variables defined by Srivastava and Panda ([16], and [17]) by means of the multiple Mellin-Barnes type integral (see also [11], p. 251, eqn. (C–1) et. seq.)

$$H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = H^{0,\lambda\,:\,(u',\,v');\,\dots\,;\,(u^{(r)},\,v^{(r)})}_{A,\,C:\,(B',D');\dots;(B^{(r)},D^{(r)})}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(r)}]:[(b'):\varphi'];\dots;[b^{(r)}:\varphi^{(r)}];\\ [(c):\psi',\dots,\psi^{(r)}]:[(d'):\delta'];\dots;[d^{(r)}:\delta^{(r)}];\end{matrix} z_1,\dots,z_r\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r} U_1(\xi_1)\dots U_r(\xi_r)\, V(\xi_1,\dots,\xi_r)\, z_1^{\xi_1}\dots x_R^{\xi_r}\, d\xi_1\dots d\xi_r,\ \omega=\sqrt{-1} \qquad \dots(1.1)$$

where

$$U_i(\xi_i) = \frac{\prod_{j=1}^{u^{(i)}}\Gamma[d_j^{(i)}-\delta_j^{(i)}\xi_i]\ \prod_{j=1}^{v^{(i)}}\Gamma[1-b_j^{(i)}+\phi_j^{(i)}\xi_i]}{\prod_{j=u^{(i)}+1}^{D^{(i)}}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}\xi_j)\ \prod_{j=v^{(i)}+1}^{B^{(i)}}\Gamma(b_j^{(i)}-\phi_j^{(i)}\xi_i)},\qquad \forall i\in\{1,\dots,r\} \qquad \dots(1.2)$$

$$V(\xi_1,\dots,\xi_r) = \frac{\prod_{j=0}^{\lambda}[1-a_j+\sum_{i=1}^{r}\theta_j^{(i)}\xi_i]}{\prod_{j=\lambda+1}^{A}[a_j-\sum_{i=1}^{r}\theta_j^{(i)}\xi_i]\ \prod_{j=1}^{C}\Gamma[1-c_j+\sum_{i=1}^{r}\psi_j^{(i)}\xi_i]} \qquad \dots(1.3)$$

The multiple integral in (1.1) converges absolutely, if

$$T_i>0 \text{ and } |\arg z_i| < T_i\pi/2 \quad \forall i\ \{1,\dots r\} \qquad \dots(1.4)$$

where

$$T_i = \sum_{j=1}^{\lambda}\theta_j^{(i)} - \sum_{j=\lambda+1}^{A}\theta_j^{(i)} + \sum_{j=1}^{v^{(i)}}\phi_j^{(i)} - \sum_{j=v^{(i)}+1}^{B^{(i)}}\phi_j^{(i)} - \sum_{j=1}^{C}\psi_j^{(i)} + \sum_{j=1}^{u^{(i)}}\delta_j^{(i)}$$
$$-\sum_{j=u^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0,\ \forall i\ \{1,\dots r\} \qquad \dots(1.5)$$

These conditions are assumed to be satisfied by the various H-function of several variables occuring in this paper.

The general polynomials (multivariable) defined by Srivastava [12] represented in the following manner :

$$S^{p_1,\dots,p_S}_{q_1,\dots,q_S}\begin{bmatrix} x_1 \\ \vdots \\ x_S \end{bmatrix} = S\,[x_1,\dots,x_S]$$

$$= \sum_{k_1=0}^{[q_1/p_1]} \cdots \sum_{k_s=0}^{[q_s/p_s]} \frac{(-q_1)_{p_1k_1}}{k_1!} \cdots \frac{(-q_s)_{p_sk_s}}{k_s!} . A\, [q_1,k_1; \ldots ;q_s,k_s]$$

$$x_1^{k_1} \ldots x_s^{k_s} \qquad \ldots (1.6)$$

where $q_{i'} = 0,1,2,\ldots; p_{i'} \neq 0\ (i' = 1,\ldots,s)$ is an arbitrary positive integer and the coefficients $A\,[q_1,k_1; \ldots ;q_s,k_s]$ are arbitrary constants real or complex.

If we take $s = 1$ in the equation (1.6) and denote $A\,[q,k]$ thus obtained by $A_{q,k}$, we arrive at the general class of polynomials $S_q^p(x)$ intorducted by Srivastava ([13], p.1, eqn. (1)).

Fujiwara [6] defined a generalized classical polynomials $R_n(x)$ on an interval (p,q) by means of the following Rodrigues' type formula:

$$R_n(x) = \frac{(-1)^n}{n!\,W(x)} \frac{d^n}{dx^n} [W(x)\{\upsilon(x)\}^n] \qquad \ldots(1.7)$$

where $\upsilon(x) = \mu\,(x-p)(q-x)$

$$W(x) = \frac{(x-p)^{\beta}(q-x)^{\alpha}}{(q-p)^{\alpha+\beta+1}\,B\,(\alpha+1,\,\beta+1)}\ \alpha > -1,\ \beta > -1 \qquad \ldots (1.8)$$

The class of polynomials $R_n(x)$ provides a unification of the classical orthogonal polynomials such as Jacobi, Laguerre and Hermite etc. The polynomials denoted by $F_n(\beta;\alpha;x)$ are called extended Jacobi Polynomials.

Thakare [20] studied the Fujiwara's polynomials extensively and obtained the following hypergeometric form of $F_n(\beta;\alpha;x)$.

$$F_n(\beta;\alpha;x) = \frac{(-\mu)^n\,(1+\beta)_n}{n!}\,(q-x)^n\ {}_2F_1\left(\begin{matrix}-n,-n-\alpha \\ 1+\beta\end{matrix};\frac{p-x}{q-x}\right),\ p < x < q \qquad \ldots(1.9)$$

When $p = -1$, $q = 1$ and $\mu = 1/2$ in (1.9), $F_n(\beta;\alpha;x)$ reduces to the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$.

Lemma. If $Re\,(\rho) > -1$, $Re\,(\sigma) > -1$ and $p \neq q$, then

$$\int_p^q (x-p)^{\rho}(q-x)^{\sigma} F_n(\beta;\alpha;x)\,dx = \frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n!\,\Gamma\,(\rho+\sigma+n+2)}\,(q-p)^{\rho+\sigma+n+1}$$

$$\sum_{m=0}^{n} \frac{(-1)^m\,(-n)_m\,(-n-\alpha)_m\,(\rho+m+1)\,\Gamma\,(\sigma+n+1-m)}{m!\,(1+\beta+m)} \qquad \ldots (1.10)$$

Proof. The above lemma follows at once by applying the definition (1.9) in conjunction with the following form of the well known Eulerian integral for the Beta function

$$\int_p^q (x-p)^{\rho-1}(q-x)^{\sigma-1}\,dx = (q-p)^{\rho+\sigma+n-1}\,\frac{\Gamma(\rho)\,\Gamma(\sigma)}{\Gamma\,(\rho+\sigma)},\ p \neq q \qquad \ldots (1.11)$$

where for convergence, $min\,\{Re\,(\rho),\,Re(\sigma)\} > 0$

2. The main integral formulas. Our main reults of the present paper are the integral formulas contained in the following theorems:

Theorem 1. With t_i defined by (1.5) let $|arg\, z_i| < T_i\pi/2\ \forall i\ \{1,...r\}$, where each of the equali Ties holds for suitably restricted values of the complex variables $z_1,...z_r$. Also let the polynomials $F_n(\beta;\alpha;x)$ be defined by (1.9) for every postive integer n.

Then

$$\int_p^q (x-p)^\rho (q-x)^\sigma F_n(\beta;\alpha;x)\ S_{q_1,...,q_s}^{p_1,...,p_s}\begin{bmatrix} y_1(x-p)^{g_i}(q-x)^{w_i} \\ \vdots \\ y_s(x-p)^{g_s}(q-x)^{w_s}\end{bmatrix} H\begin{bmatrix} z_1(x-p)^{\rho_1}(q-x)^{\sigma_1} \\ \vdots \\ z_r(x-p)^{\rho_r}(q-x)^{\sigma_r}\end{bmatrix} dx$$

$$= \frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n!}(q-p)^{\rho+\sigma+n+1}\sum_{m=0}^{n}\frac{(-1)^m(-n)_m(-n-\alpha)_m}{m!\,\Gamma(1+\beta+m)}L(y_1,...,y_s)$$

$$H^{0,\lambda+2:(u',v');...;(u^{(r)},v^{(r)})}_{A+2,C+1:(B',D');...;(B^{(r)},D^{(r)})}\left(\begin{matrix}[-m-\rho-g_1k_1-...-g_sk_s:\rho_1,...,\rho_r], \\ [-1-n-\rho-\sigma-(g_1+w_1)k_1-...-(g_s+w_s)g_s:\rho_1+\sigma_1,...,\rho_r+\sigma_r]\end{matrix}\right.$$

$$[m-\sigma-n-w_1k_1-...-w_sk_s:\sigma_1,...,\sigma_r],[(a):\theta',...,\theta^{(r)}]:[(b'):\varphi'];...;[b^{(r)}:\varphi^{(r)}];$$
$$[(c):\psi',...,\psi^{(r)}]:[(d'):\delta'];[(d''):\delta''];...;[d^{(r)}:\delta^{(r)}];$$

$$\left. z_1(q-p)^{\sigma_1-\epsilon}, z_2(q-p)^{\sigma_2},..., z_r(q-p)^{\sigma_r}\right) \qquad ...(2.1)$$

where

$$L(y_1,...,y_s) = \sum_{k_1=0}^{[q_1/p_1]}\cdots\sum_{k_s=0}^{[q_s/p_s]}\prod_{j'=1}^{s}\left[\frac{(-q_{j'})p_{j'}k_{j'}}{k_{j'}!}y_{j'}^{k_{j'}}(q-p)^{(g_{j'}+w_{j'})k_{j'}}\right]$$

$$A[q_1,k_1;...;q_s,k_s] \qquad ...(2.2)$$

provided that $p\neq q$, $\rho_i,\sigma_i > 0\ \forall i\in\{1,...,s\}$; $g_j, w_j > 0$, $\forall j\in\{1,...,s\}$; $p_{i'}$ $(i'=1,...,s)$ is an arbitrary positive integer and the coefficients $A[q_1,k_1;...;q_s,k_s]$ are arbitrary constant, real or complex.

$$Re\left(\rho+\sum_{i=1}^{r}\rho_i\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1,\ Re\left(\sigma+\sum_{i=1}^{r}\sigma_i\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1, \qquad ...(2.3)$$

and of course $T_i > 0$, $|arg\, z_i| < T_i\pi/2$ $i=1,...r$, $j=1,...,u^{(r)}$.

Theorem 2. Under hypothesis preceding the assertion (2.1)of theorem 1,

$$\int_p^q (x-p)^\rho (q-x)^\sigma F_n(\beta;\alpha;x)\ S_{q_1,...,q_s}^{p_1,...,p_s}\begin{bmatrix} y_1(x-p)^{g_i}(q-x)^{w_i} \\ \vdots \\ y_s(x-p)^{g_s}(q-x)^{w_s}\end{bmatrix} H\begin{bmatrix} z_1(x-p)^{-\epsilon}(q-x)^{\sigma_1} \\ z_2(q-x)^{\sigma_2} \\ \vdots \\ z_r(q-x)^{\sigma_r}\end{bmatrix} dx$$

$$= \frac{(-\mu)^n \Gamma(1+\beta+n)}{n!}(q-p)^{\rho+\sigma+n+1} \sum_{m=0}^{n} \frac{(-1)^m (-n)_m (-n-\alpha)_m}{m!\,\Gamma(1+\beta+m)} L(y_1,\ldots,y_s)$$

$$H^{0,\lambda+1:(u',+1,v');(u'',v'');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1:(B',D'+1);(B'',D'');\ldots;(B^{(r)},D^{(r)})}\left(\begin{matrix}[-m-\sigma-w_1k_1-\ldots-w_sk_s:\sigma_1,\ldots,\sigma_r],\\ [-1-n-\rho-\sigma-(g_1+w_1)k_1-\ldots-(g_s+w_s)k_s:\end{matrix}\right.$$

$$[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\varphi'];[(b''):\varphi''];\ldots;[b^{(r)}:\varphi^{(r)}];$$
$$\sigma_1-\epsilon,\sigma_2,\ldots,\sigma_r],[(c):\psi',\ldots,\psi^{(r)}]:[\rho+1+m+g_1k_1+\ldots+g_sk_s:\epsilon],[(d'):\delta'];\ldots;[d^{(r)}:\delta^{(r)}];$$
$$\left. z_1(q-p)^{\sigma_1-\epsilon}, z_2(q-p)^{\sigma_2},\ldots,z_r(q-p)^{\sigma_r}\right) \qquad \ldots(2.4)$$

The Integral (2.4) holds provided that $p\neq q$, $\sigma_1>\epsilon>0$, $\sigma_j>0$ $j=2,\ldots r;$ $g_{j'},w_{j'}>0$ $\forall j'\in\{1,\ldots,s\}$; $p_{i'}$ $(i'=1,\ldots,s)$ are arbitrary positive integers and the coefficients $A\,[q_1,k_1;\ldots;q_s,k_s]$ are arbitrary constants, real or complex.

$$Re\left(\rho-\epsilon\frac{d_j'}{\delta_j'}\right)>-1,\; Re\left(\sigma+\sum_{i=1}^{r}\sigma_i\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1 \qquad \ldots(2.5)$$

and of course $T_i>0$, $|arg\, z_i|<T_i\pi/2$, $i=1,\ldots r$, $j=1,\ldots,u^{(r)}$.

$$\int_p^q (x-p)^\rho (q-x)^\sigma F_n(\beta;\alpha;x)\; S^{p_1,\ldots,p_s}_{q_1,\ldots,q_s}\begin{bmatrix} y_1(x-p)^{g_i}(q-x)^{w_i}\\ \vdots\\ y_s(x-p)^{g_s}(q-x)^{w_s}\end{bmatrix} H\begin{bmatrix} z_1(x-p)^{\rho_1}(q-x)^{-\gamma}\\ z_2(x-p)^{\rho_2}\\ \vdots\\ z_r(x-p)^{\rho_r}\end{bmatrix}dx$$

$$= \frac{(-\mu)^n \Gamma(1+\beta+n)}{n!}(q-p)^{\rho+\sigma+n+1} \sum_{m=0}^{n} \frac{(-1)^m (-n)_m (-n-\alpha)_m}{m!\,\Gamma(1+\beta+m)} L(y_1,\ldots,y^s)$$

$$H^{0,\lambda+2:(u',+1,v');(u'',v'');\ldots;(u^{(r)},v^{(r)})}_{A+1,C+1:(B',D'+1);(B'',D'');\ldots;(B^{(r)},D^{(r)})}\left(\begin{matrix}[-\rho-m-g_1k_1-\ldots-g_sk_s:\rho_1,\ldots,\rho_r],\\ [-1-n-\rho-\sigma-(g_1+w_1)k_1-\ldots-(g_s+w_s)k_s:\end{matrix}\right.$$

$$[(a):\theta',\ldots,\theta^{(r)}]:$$
$$\rho_1-\gamma,\rho_2,\ldots,\rho_r],[(c):\psi_1,\ldots,\psi^{(r)}]:[1+\sigma+n-m+w_1k_1+\ldots+w_sk_s:\gamma];$$

$$\left.\begin{matrix}[(b'):\varphi'];[(b''):\varphi''];\ldots;[b^{(r)}:\varphi^{(r)}];\\ [(d'):\delta'];[(d''):\delta''];\ldots;[d^{(r)}:\delta^{(r)}];\end{matrix}\quad z_1(q-p)^{\rho_1+\sigma_1},\ldots,z_r(q-p)^{\rho_r+\sigma_r}\right) \qquad \ldots(2.6)$$

The Integral (2.6) holds provided that $p\neq q$, $\rho_1>\gamma>0$, $\rho_j>0$ $j=2,\ldots r;$ $g_{j'},w_{j'}>0$ $\forall j'\in\{1,\ldots,s\}$; $p_{j'}$ $(j'=1,\ldots,s)$ is an arbitrary positive integer and the coefficients $A\,[q_1,k_1;\ldots;q_s,k_s]$ are arbitrary constants, real or complex.

$$Re\left(\rho+\sum_{i=1}^{r}\rho_i\frac{d_j'}{\delta_j'}\right)>-1,\ Re\left(\sigma-\gamma\frac{d'_j}{\delta'_j}\right)>-1 \qquad ...(2.7)$$

and of course $T_i>0, |arg\ z_i|<T_i\pi/2$, $i=1,\dots r, j=1,\dots,u^{(r)}$.

$$\int_p^q (x-p)^\rho (q-x)^\sigma F_n(\beta;\alpha;x)\ S_{q_1,\dots,q_s}^{p_1,\dots,p_s}\begin{bmatrix} y_1(x-p)^{g_1}(q-x)^{w_1} \\ \vdots \\ y_s(x-p)^{g_s}(q-x)^{w_s}\end{bmatrix} H\begin{bmatrix} z_1(x-p)^{-\epsilon}(q-x)^{-\gamma} \\ z_2 \\ \vdots \\ z_r\end{bmatrix} dx$$

$$=\frac{(-\mu)^n\Gamma(1+\beta+n)}{n!}(q-p)^{\rho+\sigma+n+1}\sum_{m=0}^{n}\frac{(-1)^m(-n)_m(-n-\alpha)_m}{m!\,\Gamma(1+\beta+m)}L(y_1,\dots,y_s)$$

$$H^{0,\lambda:(u'+2,v');(u'',v'');\dots;(u^{(r)},v^{(r)})}_{A,C:(B'+1,D'+2);(B'',D'');\dots;(B^{(r)},D^{(r)})}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(r)}]: \\ [(c):\psi 1,\dots,\psi^{(r)}]:[(d'):\delta'];\end{matrix}\right.$$

$$[\rho+\sigma+n+2+(g_1+w_1)k_1+\dots+(g_s+w_s)k_s:\gamma+\epsilon];[(b'):\varphi'];\dots;[b^{(r)}:\varphi^{(r)}];$$
$$[\rho+m+1+g_1k_1+\dots g_sk_s:\epsilon],[\sigma+n-m+1+w_1k_1+\dots+w_sk_s:\gamma];[(d'):\delta'];\dots;[d^{(r)}:\delta^{(r)}];$$
$$\left.z_1(q-p)^{-\gamma-\epsilon},z_2\dots,z_r\right) \qquad ...(2.8)$$

The Integral (2.8) holds provided that $p\neq q$, $\gamma>0$, $\epsilon>0$, $g_{j'},w_{j'}>0\ \forall j'\in\{1,\dots,s\}$; $p_{j'}$ $(j'=1,\dots,s)$ is an arbitrary positive integer and the coefficients $A[q_1,k_1;\dots;q_s,k_s]$ are arbitrary constants, real or complex.

$$Re\left(\rho-\epsilon\frac{d_j'}{\delta_j'}\right)>-1,\ Re\left(\sigma-\gamma\frac{d'_j}{\delta'_j}\right)>-1 \qquad ...(2.9)$$

and of course $T_i>0, |arg\ z_i|<T_i\pi/2$, $i=1,\dots r, j=1,\dots,u^{(r)}$,
where $L(y_1,\dots,y_s)$ in (2.4), (2.6) and (2.8) is same as given in (2.2)

Proof. To establish (2.1) express the generalized polynomials with the help of equation (1.6) and the multivariable H-function in terms of Mellin-Barnes type Contour integral by virtue of (1.1), interchanging the order of intergration and summation (which is easily seen to be permissible under the conditions stated) and then evaluate the x-integral with the help of (1.10). On interpreting the result thus obtained by virtue of (1.1), we arrive at the right hand side of (2.1).

The remaining integrals (2.4), (2.6) and (2.8) can be evaluated in a similar manner.

3. Special Cases:

(i) Letting $s=1$, in eqn. (2.1), (2.4), (2.6) and (2.8), we get interesting integrals involving extended Jacobi polynomials, general class of polynomials and multivariable H-function as follows :

$$\int_p^q (x-p)^{\rho}(q-x)^{\sigma} F_n(\beta;\alpha;x)\ S_q^p[\,y\,(x-p)^g\,(q-x)^w\,]\ H\begin{bmatrix} z_1(x-p)^{\rho_1}(q-x)^{\sigma_1} \\ \vdots \\ z_r(x-p)^{\rho_r}(q-x)^{\sigma_r} \end{bmatrix} dx$$

$$= \frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n\,!}(q-p)^{\rho+\sigma+n+1} \sum_{m=0}^{n} \sum_{k=0}^{[q/p]} \frac{(-1)^m\,(-n)_m\,(-n-\alpha)_m\,(-q)_{pk}}{m!\,\Gamma\,(1+\beta+m)\,k!}.$$

$$A_{q,k}\,[y\,(q-p)^{g+w}\,]^k\quad H^{0,\lambda+2\,:\,(u',\,v');\,\ldots\,;\,(u^{(r)},\,v^{(r)})}_{A+2,C+1:\,(B',D');\ldots;(B^{(r)},D^{(r)})}$$

$$\left(\begin{matrix}[-m-\rho-gk:\rho_1,\ldots,\rho_r],\ [m-\sigma-n-wk:\sigma_1,\ldots,\sigma_r],\ [(a):\theta',\ldots,\theta^{(r)}] \\ [1-n-\rho-\sigma-(g+w)k:\rho_1+\sigma_1,\ldots,\rho_r+\sigma_r]\ [(c):\psi',\ldots,\psi^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}:[(b'):\varphi'];\ldots;[b^{(r)}:\varphi^{(r)}]; \\ [(d'):\delta'];\ldots;[d^{(r)}:\delta^{(r)}];\end{matrix}\ z_1(q-p)^{\rho_1+\sigma_1},\ldots,\ z_r(q-p)^{\rho_r+\sigma_r}\right) \qquad \ldots(3.1)$$

$$\int_p^q (x-p)^{\rho}(q-x)^{\sigma} F_n(\beta;\alpha;x)\ S_q^p[\,y\,(x-p)^g\,(q-x)^w\,]\ H\begin{bmatrix} z_1(x-p)^{-\epsilon}(q-x)^{\sigma_1} \\ z_2(q-x)^{\sigma_2} \\ \vdots \\ z_r\,(q-x)^{\sigma_r} \end{bmatrix} dx$$

$$= \frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n\,!}(q-p)^{\rho+\sigma+n+1} \sum_{m=0}^{n} \sum_{k=0}^{[q/p]} \frac{(-1)^m\,(-n)_m\,(-n-\alpha)_m\,(-q)_{pk}}{m!\,\Gamma\,(1+\beta+m)\,k!}$$

$$A_{q,k}\,[y\,(q-p)^{g+w}\,]^k\quad H^{0,\lambda+1\,:\,(u',+1,v');\,(u'',\,v'');\ldots\,;\,(u^{(r)},\,v^{(r)})}_{A+1,C+1:\,(B',D'+1);(B'',D'');\ldots;(B^{(r)},D^{(r)})}$$

$$\left(\begin{matrix}[m-\sigma-n-wk:\sigma_1,\ldots,\sigma_r],\ [(a):\theta',\ldots,\theta^{(r)}]: \\ [-1-n-\rho-\sigma-(g+w)k:\sigma_1-\epsilon,\ \sigma_2,\ldots,\sigma_r],[(c):\psi',\ldots,\psi^{(r)}]:\ [\rho+1+m+gk:\epsilon];\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\varphi'];\ldots;[b^{(r)}:\varphi^{(r)}]; \\ [(d'):\delta'];\ldots;[d^{(r)}:\delta^{(r)}];\end{matrix}\ z_1(q-p)^{\sigma_1-\epsilon}\,z_2(q-p)^{\sigma_2},\ldots,\ z_r(q-p)^{\sigma_r}\right) \qquad \ldots(3.2)$$

$$\int_p^q (x-p)^{\rho}(q-x)^{\sigma} F_n(\beta;\alpha;x)\ S_q^p[\,y\,(x-p)^g\,(q-x)^w\,]\ H\begin{bmatrix} z_1(x-p)^{\rho_1}(q-x)^{-\gamma} \\ z_2(x-p)^{\rho_2} \\ \vdots \\ z_r\,(x-p)^{\rho_r} \end{bmatrix} dx$$

$$= \frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n\,!}(q-p)^{\rho+\sigma+n+1} \sum_{m=0}^{n} \sum_{k=0}^{[q/p]} \frac{(-1)^m\,(-n)_m\,(-n-\alpha)_m\,(-q)_{pk}}{m!\,\Gamma\,(1+\beta+m)\,k!}$$

$$A_{q,k}\,[y\,(q-p)^{g+w}]^k\; H^{0,\lambda+1\,:\,(u'+1,v');\,(u'',v'');\dots;\,(u^{(r)},v^{(r)})}_{A+1,C+1:\,(B',D'+1);(B'',D'');\dots;(B^{(r)},D^{(r)})}$$

$$\left(\begin{matrix}[-\rho-m-gk:\rho_1,\dots,\rho_r],\;[(a):\theta',\dots,\theta^{(r)}]: \\ [-1-n-\rho-\sigma-(g+w)k:\rho_1-\gamma,\rho_2,\dots,\rho_r],[(c):\psi',\dots,\psi^{(r)}]:[1+\sigma+n-m+wk:\gamma],\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\varphi'];\dots;[b^{(r)}:\varphi^{(r)}]; \\ [(d'):\delta'];\dots;[d^{(r)}:\delta^{(r)}];\end{matrix}\; z_1(q-p)^{\rho_1-\gamma}, z_2(q-p)^{\rho_2},\dots, z_r(q-p)^{\rho_r}\right) \qquad \dots(3.3)$$

$$\int_p^q (x-p)^\rho(q-x)^\sigma F_n(\beta;\alpha;x)\;\; S_q^p[\,y\,(x-p)^g\,(q-x)^w\,]\; H\begin{bmatrix} z_1(x-p)^{-\epsilon}(q-x)^{-\gamma} \\ z_2 \\ \vdots \\ z_r\end{bmatrix} dx$$

$$= \frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n\,!}(q-p)^{\rho+\sigma+n+1} \sum_{m=0}^{n}\sum_{k=0}^{[q/p]} \frac{(-1)^m\,(-n)_m\,(-n-\alpha)_m\,(-q)_{pk}}{m!\,\Gamma\,(1+\beta+m)\,k!}$$

$$A_{q,k}\,[y\,(q-p)^{g+w}]^k\; H^{0,\lambda\,:\,(u'+2,v');\,(u'',v'');\dots;\,(u^{(r)},v^{(r)})}_{A,C:\,(B'+1,D'+2);(B'',D'');\dots;(B^{(r)},D^{(r)})}$$

$$\left(\begin{matrix}[(a):\theta',\dots,\theta^{(r)}]:\,[\rho+\sigma+n+2+(g+w)k:\,\gamma+\epsilon]; \\ [(c):\,\psi',\dots,\psi^{(r)}]:\,[\rho+m+1+gk:\,\epsilon],\,[\sigma+n-m+1+wk:\gamma]\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\varphi'];\dots;[b^{(r)}:\varphi^{(r)}]; \\ [(d'):\delta'];\dots;[d^{(r)}:\delta^{(r)}];\end{matrix}\qquad z_1(q-p)^{-\gamma-\epsilon},\, z_2,\dots, z_r\right) \qquad \dots(3.4)$$

which are valid under essentially the same conditions as those of their parent formulas (2.1), (2.4), (2.6), and (2.8), respectively.

(ii) Giving suitable values to parameters and using a relation [17] in (2.1), (2.4), (2.6) and (2.8) we obtain the integrals involving generalized Lauricella function [15], but for the sake of brevity, they all are not presented here. (for example, putting $\lambda=A$, $u^{(i)}=1$, $v^{(i)}=B^{(i)}$, replecing $D^{(i)}$ by $D^{(i)}+1$ $\forall i=1,\dots,r$ in (2.1), we have

$$\int_p^q (x-p)^\rho(q-x)^\sigma F_n(\beta;\alpha;x)\;\; S^{p_1,\dots,p_s}_{q_1,\dots,q_s}\begin{bmatrix} y_1(x-p)^{g_1}\,(q-x)^{w_1} \\ \vdots \\ y_s\,(x-p)^{g_s}(q-x)^{w_s}\end{bmatrix}$$

$$F^{\,A:B',\dots,B^{(r)}}_{\,C:D',\dots,D^{(r)}}\begin{bmatrix} -z_1(x-p)^{\rho_1}(q-x)^{\sigma_1} \\ \vdots \\ -z_r(x-p)^{\rho_r}(q-x)^{\sigma_r}\end{bmatrix} dx \;=\; \frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n\,!}(q-p)^{\rho+\sigma+n+1}$$

$$\sum_{m=0}^{n} \frac{(-1)^m\,(-n)_m\,(-n-\alpha)_m}{m!\,\Gamma\,(1+\beta+m)}\, L\,(y_1,\dots,y_s)$$

$$\frac{\Gamma(1+\rho+m+g_1k_1+\dots g_sk_s)\Gamma(1+\sigma+n-m+w_1k_1+\dots+w_sk_s)}{\Gamma(2+n+\rho+\sigma+(g_1+w_1)k_1+\dots+(g_s+w_s)k_s)} F^{A+2:B';\dots;B^{(r)}}_{C+1:D';\dots;D^{(r)}}$$

$$\left(\begin{matrix}[1+m+\rho+g_1k_1+\dots+g_sk_s:\rho_1,\dots,\rho_r],[1-m+\sigma+n+w_1k_1+\dots+w_sk_s:\sigma_1,\dots,\sigma_r],\\ [2+n+\rho+\sigma+(g_1+w_1)k_1+\dots+(g_s+w_s)k_s:\rho_1+\sigma_1,\dots,\rho_s+\sigma_r]:[1-(c):\psi',\dots,\psi^{(r)}]:\\ [1-(a):\theta',\dots,\theta^{(r)}]:[1-(b'):\varphi'];\dots;[1-b^{(r)}:\varphi^{(r)}];\\ [1-(d'):\delta'];\dots;[1-d^{(r)}:\delta^{(r)}];\end{matrix}\; -z_1(q-p)^{\rho_1+\sigma_1},\dots,-z_r(q-p)^{\rho_s+\sigma_s}\right) \quad \dots(3.5)$$

valid within the domain of convergence of the resulting series where n is non-negative integer, $Re\ (\rho) > -1$, $Re\ (\sigma) > -1, \rho_i, \sigma_i > 0$ $\forall i = \{1,..,r\}$, $g_{j'}, w_{j'} > 0\ \forall j' \in \{1,\dots,s\}$

(iii) Taking $g = 0, w=1, \alpha=\beta=n=0$ the results in eqn. (3.1), (3.2), (3.3) and (3.4) reduce to the known results given by Srivastava and Singh ([18],p. 166, eqn. (2.2), (2.4), (2.6) and (2.8).

(iv) Letting $p = -1, q = 1, \beta = \rho, \rho_1=\rho_2,\dots,\rho_r=0, \mu=1$ the result (3.1) reduces to a result obtained by Srivastava and Panda ([17]).

(v) Letting $\rho = \beta$, q_j $(j = 1,\dots,s)=0$ in (2.1) we obtained a result givin by Sharma ([10],p. 57, eqn. (2.2.3)).

(vi) Letting $\sigma_1=\sigma_2=\dots=\sigma_r=0$ in case (v) above we obtained a result given by Sharma ([10], p. 56, eqn. (2.2.2)).

(vii) Letting $\rho_1=\rho_2=\dots=\rho_r=0$ in case (v) above we obtained a result given by Sharma ([10], p. 55, eqn. (2.2.1)

Several other special case of the integral established here can be obtained.

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