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A UNIFIED PRESENTATION OF CERTAIN SUBCLASSES OF STARLIKE AND UNIFORMLY CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT

A unified class $T(\alpha,\beta)$ of starlike and uniformly convex functions in the open disk has been introduced. Number of theorems involving for example, sharp distortion inequalities and modified Hamdard Product (or convolution) of functions belonging to the class $T(\alpha,\beta)$ have been obtained. It is also shown how these theorems would apply to yield various results given in the literature.

1. Introduction. Let T denote the class of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \; a_k \geq 0 \tag{1}$$

Definition 1. The Class *UCT*. A function $f(z) \in T$ given by (1) is said to be uniformly convex, if it satisfies the inequality

$$1 + Re\,\{z f''(z)/f'(z)\} \geq |zf''(z)/f'(z)|, \; z \in U \tag{2}$$

We denote the class of such functions by *UCT*.

Definition 2. The Class *UCT* (α). A function $f(z)$ of the form (1) is said to be in $UCT(\alpha)$, $\alpha \geq 0$ if and only if

$$1+Re\,\{z f''(z)/f'(z)\} \geq \alpha\,|z f''(z)/f'(z)|, \; z \in U \tag{3}$$

Remark:

$UCT(1) = UCT$

$UCT(0) = C$, the class of convex functions with negaive coefficients

Inclusion Results.

$UCT(\alpha)$	$\subseteq$	UCT	For	$\alpha \geq 1$
UCT	$\subseteq$	$UCT(\alpha)$	For	$0 < \alpha \leq 1$
$UCT(\alpha)$	$\subseteq$	$UCT(\gamma)$	For	$\alpha \geq \gamma$

These results are straight forward from the definition.

Definition 3. The Class TS_p. A function $f(z)$ of the form (1) is said to be in TS_p if and only if

$$|zf'(z)/f(z) - 1| \le Re\{z f'(z) / f(z)\}\ z \in U \tag{4}$$

Definition 4.The Class TS_p (α) . A function $f(z)$ of the form (1) is said to be in $TS_p(\alpha)$ if and only if

$$\alpha\,|zf'(z)/f(z) - 1| \le Re\{z f'(z) / f(z)\}\ z \in U \tag{5}$$

Remark.

$TS_p(1) = TSp$

$TS_p(0) = T$

Inclusion Results.

$TS_p(\alpha)$	$\subseteq TS_p$	For	$\alpha \ge 1$
TS_p	$\subseteq TS_p(\alpha)$	For	$0 \le \alpha \le 1$
$TS_p(\alpha)$	$\subseteq TS_p(\gamma)$	For	$\alpha \ge \gamma$

These results follow easily from definition.

The classes $UCT(\alpha)$ and $TS_p(\alpha)$ were introduced by Murugusundaramurthy [1].

Murugusundaramurthy [1] gave the following lemmas :

Lemma 1. A function $f(z)$ defined by (1) is in $UCT(\alpha)$, if and only if

$$\sum_{k=2}^{\infty} k\,[k(\alpha+1) - \alpha]\,a_k \le 1 \tag{6}$$

The result (6) is *SHARP* for functions

$f_k(z) = z - z^k / k(\alpha+1) - \alpha$.

Lemma 2. A function $f(z)$ defined by (1) is in $TS_p(\alpha)$, if and only if,

$$\sum_{k=2}^{\infty} [k(\alpha+1) - \alpha]\,a_k \le 1. \tag{7}$$

The result (7) is *SHARP* for functions

$f_k(z) = z - z^k / [k(\alpha+1) - \alpha]$.

In view of Lemma 1 and Lemma 2, it would seem to be natural to introduce and study an interesting unification of the class $UCT(\alpha)$ and $TS_p(\alpha)$.

Thus we say that a function $f(z)$ defined by (1) belongs to the class $T(\alpha, \beta)$ if and only if

$$\sum_{k=2}^{\infty} [k(\alpha+1) - \alpha]\,[1 - \beta + \beta k]\,a_k \le 1 \tag{8}$$

for some $\alpha \ge 0$ and some β $(0 \le \beta \le 1)$.

Since,

$$1 - \beta + \beta k \ge 1 \qquad (0 \le \beta \le 1)\quad (k = 2, 3, \ldots)$$

Therefore,

$$T(\alpha, 0) = TS_p(\alpha),$$

and

$$T(\alpha, 1) = UCT(\alpha).$$

The object of this paper is to present a unified study of the $UCT(\alpha)$ and $TS_p(\alpha)$ by proving various interesting properties and characteristics of the general class $T(\alpha, \beta)$.

2. Distortion Inequalities.

Theorem 1. If a function $f(z)$ defined by (1) is in the class $T(\alpha, \beta)$, then

$$r - \frac{r^2}{(\alpha+2)(1+\beta)} \le |f(z)| \le r + \frac{r^2}{(\alpha+2)(1+\beta)}. \tag{9}$$

and

$$1 - \frac{2r}{(\alpha+2)(1+\beta)} \le |f(z)| \le 1 + \frac{2r}{(\alpha+2)(1+\beta)}, \quad |z| = r. \tag{10}$$

The estimates are sharp.

Proof. Since,

$$(\alpha+2)(1+\beta) \sum_{k=2}^{\infty} a_k \le \sum_{k=2}^{\infty} [k(\alpha+1)-\alpha][1-\beta+\beta k]\, a_k \le 1,$$

we have,

$$\begin{aligned} |f(z)| &\le |z| + |z|^2 \sum_{k=2}^{\infty} a_k \\ &\le |z| + \frac{1}{(\alpha+2)(1+\beta)} |z|^2 \\ &\le r + \frac{r^2}{(\alpha+2)(1+\beta)}. \end{aligned}$$

Likewise $|f(z)| \ge r - \dfrac{r^2}{(\alpha+2)(1+\beta)}$.

This proves the estimate (9) of the theorem 2.

We note that,

$$\begin{aligned} \frac{(\alpha+2)(1+\beta)}{2} \sum_{k=2}^{\infty} a_k &\le \sum_{k=2}^{\infty} [1+\alpha/k][1-\beta+\beta k]k\, a_k \\ &\le \sum_{k=2}^{\infty} \frac{[k(\alpha+1)-\alpha]}{k} [1-\beta+\beta k]k\, a_k \\ &\le 1. \end{aligned}$$

Now we have

$$\begin{aligned} |f'(z)| &\le 1 + |z| \sum_{k=2}^{\infty} k\, a_k \\ &\le 1 + \frac{2}{(\alpha+2)(1+\beta)} |z| \\ &\le 1 + \frac{2r}{(\alpha+2)(1+\beta)}. \end{aligned}$$

Likewise $|f'(z)| \ge 1 - \dfrac{2r}{(\alpha+2)(1+\beta)}$.

$$\Rightarrow \qquad \gamma\,(k+1)-k \;\le\; [\,k\,(\alpha+1)-\alpha\,]^2\,[1-\beta+\beta k]$$

$$\Rightarrow \qquad \gamma \;\le\; \frac{[\,k\,(\alpha+1)-\alpha\,]^2\,[1-\beta+\beta k]-k}{\gamma+1}\,.$$

If we set

$$\phi\,(k)=\frac{[\,k\,(\alpha+1)-\alpha\,]^2\,[1-\beta+\beta k]-k}{\gamma+1}\,.$$

We easily see that $\phi\,(k)$ is an increasing functionj of k.
on setting $k=2$, we get

$$\gamma \;\le\; \phi\,(2)=(\alpha+2)^2\,(1-\beta)-2.$$

This completes the proof.

Finally taking the function given by

$$f_j(z)=z-\frac{1}{(\alpha+2)\,(1+\beta)}\,z^2.$$

The result of the theorem 2 is *SHARP*.

Remark. Putting $\beta=0$ and $\beta=1$ in the theorem 2, we obtain the corresponding results for the classes $TS_p(\alpha)$ and $UCT\,(\alpha)$ given earlier by Murugusundaramurthy [1].

Theorem 3. Let the function $f_j(z)$ $(j=1,2)$ be in the class $T(\alpha,\beta)$, then the function $h(z)$ defined by

$$h(z)=z-\sum_{k=2}^{\infty}\,(a_{1,k}^2+a_{2,k}^2)\,z^k$$

belongs to the class $T\,(\gamma,\beta)$, where

$$\gamma=\frac{(\alpha+2)^2\,(1+\beta)-4}{2}\,.$$

Proof. Since,

$$\sum_{k=2}^{\infty}\,[\,k\,(\alpha+1)-\alpha\,]^2\,[1-\beta+\beta k]^2\,a_{j,k}^2$$

$$\le\Big[\sum_{k=2}^{\infty}\,[\,k\,(\alpha+1)-\alpha\,]\,[1-\beta+\beta k]\,a_{j,k}\Big]^2\le 1\;(j=1,2)$$

we have,

$$\sum_{k=2}^{\infty}\,[\,k\,(\alpha+1)-\alpha\,]^2\,[1-\beta+\beta k]^2\,[a_{1,k}^2+a_{2,k}^2\,]\le 2.$$

Thus, it is sufficient to find a largest γ such that

$$[k\,(\gamma+1)-\gamma]\,[1-\beta+\beta k]\le[k\,(\alpha+1)-\alpha\,]^2\,[1-\beta+\beta k]^2\,/2\,,$$

that is,

$$\gamma\,(k-1)-k\le[k\,(\alpha+1)-\alpha]^2\,[1-\beta+\beta k]\,/2,$$

$$\gamma\,(k-1)\;\le\;\frac{[\,k\,(\alpha+1)-\alpha\,]^2\,[1-\beta+\beta k]-2k}{2}$$

This proves the estimate (10) of the theorem 2.

The bounds are *SHARP* since the equalities are attained for the function

$$f(z) = z - \frac{1}{(\alpha+2)(1+\beta)} z^2 .$$

Remark. Putting $\beta = 0$ and $\beta = 1$ in theorem 2, we obtain the corresponding results for the classes $TS_p(\alpha)$ and $UCT(\alpha)$ given earlier by Murugusundaramurthy [1].

3. Modified Hadamard Product (Convolution Product)

Definition. Let the function $f_j(z)$ $(j = 1,2,)$ ε T is of the form

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{j,k} z^k, \ (a_{j,k} \geq 0, j = 1,2) \tag{11}$$

The modified Hadamard Product (or convolution) of $f_1(z)$ and $f_2(z)$ is denoted by $f_1 * f_2(z)$ and defined by

$$f_1 * f_2(z) = z - \sum_{k=2}^{\infty} a_{1,k}\, a_{2,k}\, z^k \tag{12}$$

Theorem 2. Let the $f_j(z)$ $(j = 1,2)$ be in the class $T(\alpha,\beta)$. Then $f_1 * f_2(z)$ belongs to the class $T(\gamma,\beta)$, where $\gamma = (\alpha+2)^2(1+\beta) - 2$.

Proof. Applying the technique used earlier by Schild and Silverman [2], we need to find the largest γ such that

$$\sum_{k=2}^{\infty} [k(\gamma+1) - \gamma][1-\beta+\beta k]\, a_{1,k}\, a_{2,k} \leq 1.$$

Since $f_j(z)$ $(j= 1,2)$ ε $T(\alpha,\beta)$, therefore, we have

$$\sum_{k=2}^{\infty} [k(\alpha+1) - \alpha][1-\beta+\beta k]\, a_{j,k} \leq 1 \ (j = 1,2).$$

Therefore, by Cauchy Schwarz inequality, we have

$$\sum_{k=2}^{\infty} [k(\alpha+1) - \alpha][1-\beta+\beta k]\sqrt{a_{1,k}\, a_{2,k}} \leq 1 \tag{13}$$

Thus, it is sufficient to show that

$$[k(\gamma+1) - \gamma][1-\beta+\beta k]\, a_{1,k}\, a_{2,k} \leq [k(\alpha+1) - \alpha][1-\beta+\beta k]\sqrt{(a_{1,k}\, a_{2,k})}$$

that is,

$$\sqrt{(a_{1,k}\, a_{2,k})} \leq \frac{k(\alpha+1) - \alpha}{k(\gamma+1) - \gamma}$$

that is, if

$$\frac{1}{[k(\alpha+1) - \alpha][1-\beta+\beta k]} \leq \frac{k(\alpha+1) - \alpha}{k(\gamma+1) - \gamma} \quad \text{(from (13))}$$

that is, if $\quad [k(\gamma+1) - \gamma] \leq [k(\alpha+1) - \alpha]^2 [1-\beta+\beta k]$

that is,

$$\gamma \leq \frac{[k(\alpha+1)-\alpha]^2 [1-\beta+\beta k]-2k}{2(k-1)} .$$

If we set,

$$\Psi(k) = \frac{[k(\alpha+1)-\alpha]^2 [1-\beta+\beta k]-2k}{2(k-1)} .$$

We easily see that $\Psi(k)$ is an increasing function of k. On setting $k = 2$, we get,

$$\gamma \leq \Psi(z) = \frac{(\alpha+2)^2 (1+\beta)-4}{2} .$$

This completes the proof.

Remark. Taking $\beta = 0$ and $\beta = 1$ we get the corresponding results for the classes $TS_p(\alpha)$ and $UCT(\alpha)$ given earlier by Murugusundaramurthy [1].

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