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SOME n-TUPLE INTEGRAL EQUATIONS INVOLVING INVERSE MELLIN TRANSFORMS

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ABSTRACT

The solution of n-tuple integral equations when n is even involving inverse Mellin transform has been obtained by reducing them to a system of Fredholm integral equations.

1. Introduction. In this paper we solve the n-tuple integral equations when n is even

$$(1)\qquad M^{-1}\left[\frac{\Gamma(1+\eta-s/\sigma)}{\Gamma(1+\eta+\alpha-s/\sigma)}\,\phi(s)\,;\,x\right]=f_i(x),\ a_{i-1}<x<a_i,\ (i=1,3,\ldots,n-1)\ \text{and}\ a_0=0,$$

$$(2)\qquad M^{-1}\left[\frac{\Gamma(\xi+s/\delta)}{\Gamma(\xi+\beta+s/\delta)}\,\phi(s)\,;\,x\right]=g_i(x),\ a_{i-1}<x<a_i,\ (i=2,4,\ldots,n)\ \text{and}\ a_n=\infty,$$

where $\alpha, \beta, \xi, \eta, \delta, > 0$, $\sigma > 0$ are real parameters, g_i $(i = 2, 4 \ldots, n)$ are known functions, $\phi(s)$ is to be determined and

$$(3)\qquad M\,[h(x)\,;\,s] = H(s),$$
$$M^{-1}\,[H(s)\,;\,x] = h(x),$$

denote the Mellin transform of $h(x)$ and its inversion formula respectively.

The above equations are an extension of the triple integral equations solved by Lowndes [4], quardruple integral equations solved by Dwivedi, Kushwaha and Trivedi [8], and six integral equations solved by Dwivedi, Shukla and Shukla [9] by means of a systematic applications of some slightly extended forms of Erdèlyi-Köber operators of fractional integration [2].

Using the properties of some slightly extended forms of the Erdèlyi-Köber operators given in [4]. We show in purely formal manner that the solution of the n-tuple integral equations can be expressed in terms of solution of Fredholm intergral equation of the second kind. The mothod of solution employed here will be seen to follow closely that used by Ahmad [7] to obtain the solution of som quadruple integral equation involving Bessel functions.

namely

(14) $I_i = \{x : a_{i-1} \le x < a_i\}$, $(i = 1,2,3,..,n)$ *and* $(a_0 = 0,\ a_n \to \infty)$

and we shall write any function $f(x)$, $x \ge 0$, in the form

(15) $f(x) = \sum_{i=1}^{n} f_i(x)$,

where

(16) $f_i(x) = \begin{cases} f(x), & x \in Ii \\ 0, & \text{otherwise} \end{cases} \quad i = 1,2,3,\ldots,n.$

and similarly for g.

3. Solution of the Integral Equations. Using the notation of equations (15) and (16) we can write the n-tuple integral equations (when n is even) (1) and (2) as

(17) $M^{-1}\left[\dfrac{\Gamma(1+\eta-s/\sigma)}{\Gamma(1+\eta+\alpha-s/\sigma)}\ \phi(s)\ ;\ x\right] = f(x),$

(18) $M^{-1}\left[\dfrac{\Gamma(\xi+s/\delta)}{\Gamma(\xi+\beta+s/\delta)}\ \phi(s)\ ;\ x\right] = g(x),$

where $f_1, f_3, \ldots, f_{n-1}$ and $g_2, g_4, \ldots, g_n$ are prescribed functions while f_2, $f_4, \ldots, f_n$ and $g_1, g_3, \ldots, g_{n-1}$ are unknown functions to be determined. If we write

(19) $\phi(s) = M[\phi(x); s]$,

and use the formulae (12) and (13) we find that equations (17) and (18) assume the operational form

(20) $I_{\eta,\alpha}(0, x; \sigma)\, \phi(x) = f(x)$

(21) $K_{\xi,\beta}(x,\infty; \delta)\, \phi(x) = g(x).$

Using the formulae (8) and (9) and solving the above equations for $\phi(x)$ be obtain

(22) $\phi(x) = I_{\eta,\alpha}(0, x; \sigma)\ f(x)$

(23) $\phi(x) = K_{\xi+\beta,-\beta}(x,\infty; \delta)\ g(x).$

We proceed to determine ϕ. The subscripts on all the operators I's will be supposed to have the subscripts $(\eta,\alpha;\sigma)$ understand and all K's to have subsript $(\xi, \beta; \delta)$. Evaluating (22) on $I_1, I_2, I_3, \ldots, I_{n-1}$ we get

(24) $\phi_1 = \binom{x}{a_0} I^{-1} f_1,$

(25) $\phi_2 = \binom{a_1}{a_0} I^{-1} f_1 + \binom{x}{a_1} I^{-1} f_2,$

(26) $\phi_3 = \binom{a_1}{a_0} I^{-1} f_1 + \binom{a_2}{a_1} I^{-1} f_2 + \binom{x}{a_2} I^{-1} f_3,$

... ...

... ...

(27) $\phi_{n-1} = \binom{a_1}{a_0} I^{-1} f_1 + \binom{a_2}{a_1} I^{-1} f_2 + \ldots + \binom{x}{a_{n-2}} I^{-1} f_{n-1}$,

where $a_0 = 0$

Evaluating (23) on $I_2, I_3, \ldots, I_n$ we get

(28) $\phi_2 = \binom{a_2}{x} K^{-1} g_2 + \binom{a_3}{a_2} K^{-1} g_3 + \ldots + \binom{a_n}{a_{n-1}} K^{-1} g_n$.

(29) $\phi_3 = \binom{a_3}{x} K^{-1} g_3 + \binom{a_4}{a_3} K^{-1} g_4 + \ldots + \binom{a_n}{a_{n-1}} K^{-1} g_n$.

... ...

... ...

(30) $\phi_{n-2} = \binom{a_{n-2}}{x} K^{-1} g_{n-2} + \binom{a_{n-1}}{a_{n-2}} K^{-1} g_{n-1} + \binom{a_n}{a_{n-1}} K^{-1} g_n$.

(31) $\phi_{n-1} = \binom{a_{n-1}}{x} K^{-1} g_{n-1} + \binom{a_n}{a_{n-1}} K^{-1} g_n$.

(32) $\phi_n = \binom{a_n}{x} K^{-1} g_n$.

where $a_n \to \infty$.

Since f_1 and g_n are known, ϕ_1 and ϕ_n can be determined by equations (24) and (32).

We now solve (25) for f_2 and substitute its value in (26) to get

(33) $\phi_3 = \binom{a_1}{a_0} I^{-1} f_1 + \binom{a_2}{a_1} I^{-1} \binom{x}{a_1} I \,[\, \phi_2 - \binom{a_1}{a_0} I^{-1} f_1 \,] + \binom{a^3}{a_2} I^{-1} f_3$.

Similarly

(34) $\phi_{n-1} = \binom{a_1}{a_0} I^{-1} f_1 + \binom{a_2}{a_1} I^{-1} \binom{x}{a_1} I \,[\, \phi_2 - \binom{a_1}{a_0} I^{-1} f_1 \,] + \binom{a_3}{a_2} I^{-1} f_3 + \ldots$

$+ \binom{a_{n-1}}{a_{n-2}} I^{-1} f_{n-1}$,

We solve equation (31) for g_{n-1} and substitute its value in (30) to get

(35) $\phi_{n-2} = \binom{a_{n-2}}{x} K^{-1} g_{n-2} + \binom{a_{n-1}}{a_{n-2}} K^{-1} \binom{a_{n-1}}{x} K \,[\, \phi_{n-1} - \binom{a_n}{a_{n-1}} K^{-1} g_n \,]$

$+ \binom{a_n}{a_{n-1}} K^{-1} g_n$.

Similarly from equation (28) we find out the value of ϕ_2

(36) $\phi_2 = \binom{a_2}{x} K^{-1} g_2 + \binom{a_3}{a_2} K^{-1} \binom{a_3}{x} K \,[\, \phi_3 - \binom{a_4}{a_3} K^{-1} g_4 - \ldots - \binom{a_n}{a_{n-1}}$

$K^{-1} g_n] + \ldots + \binom{a_n}{a_{n-1}} K^{-1} g_n$.

We have thus arrived at n–tuple simultaneous equations (24), (36), (33), ... , (35), (34) and (32) associated with n-tuple unknown functions $\phi_1, \phi_2, \phi_3, \ldots, \phi_{n-2}, \phi_{n-1}$ and ϕ_n respectively.

From these equations we can obtain the values of these unknowns and hence the solution of the problem will be determined by equation (19).

4. An Application

We shall consider now the n-tuple integral equations, when n is even which are extension of triple integral equations solved by Lowndes [4] :

(37) $\int_0^\infty u^{-2n}\Psi(u)J_{2q}(ux)du = F_i(x),\ a_{i-1} < x < a_i,\ (i = 1,3,\ldots n-1)$ *and* $a_0=0$

(37) $\int_0^\infty \Psi(u)J_{2q}(ux)du = 0,\ a_{i-1} < x < a_i,\ (i = 2,4,\ldots n)$ *and* $a_n \to \infty$

where $J_{2p}(ux)$ is the Bessel function of the first kind of order $2p$, $F_1(x)$, $F_3(x),\ldots, F_{n-1}(x)$ are prescribed functions and $\Psi(u)$ is to be determined. When $p = q$ and $a_{i>3,\ldots n} \to \infty$ these are equations investigated by Ahmad [7]. We now show, in a fairly straight forward manner, that the above equations can be transformed into equations of the type (1) to (2) with $g_i = 0,\ (i = 2,4,\ldots n,)$

Denoting the Mellin transforms of $\Psi(u)$ by

(39) $M[\Psi(u); s] = \Psi(s)$

and using the result

(40) $M[\xi^{-2n} j_{2q}(\xi); s] = 2^{s-1-2n}\dfrac{\Gamma(q-n+s/2)}{\Gamma(1+n+q-s/2)},$

we have, on applying the Faltung theorem for Mellin transforms that the integral equations (37) and (38) can be written in the form

(41) $M^{-1}\left[\dfrac{\Gamma(1+p-s/2)}{\Gamma(1+n+q-s/2)}\phi(s); x\right] = 2^{1+2n}x^{-2n}F_i(x),\ a_{i-1} < x < a_i,$
$(i = 1,3\ldots n-1)$ *and* $a_0 = 0,$

(41) $M^{-1}\left[\dfrac{\Gamma(p+s/2)}{\Gamma(q-n+s/2)}\phi(s); x\right] = 0,\ a_{i-1}<x<a_i,\ (i = 2,4,\ldots,n)\&\ a_0 \to\infty$

where

(43) $\phi(s) = 2^s\left[\dfrac{\Gamma(q-n+s/2)}{\Gamma(1+p-s/2)}\right]\Psi(1-s).$

These are the same as equations (1) and (2) with

$\sigma = \delta = 2,\ \xi = \eta = p,\ \alpha = q-p+n,\ \beta = q-p-n,$

and

(44) $f_i(x) = 2^{1+2n}x^{-2n}F_i(x),\ i = 1,3,\ldots, n-1$
$g_i = 0,\ i = 2,4,\ldots, n.$

Using the results of the previous section we have therefore, that the solution of equations (41) and (42) can be found in terms of a function $\phi(x)$ by

(45) $\phi(s) = M[\phi(x); s],$

where the functions $\phi_1, \phi_2, \phi_3, \ldots, \phi_{n-2}, \phi_{n-1}$ and ϕ_n are obtained from equations (24), (36), (33), ..., (35), (34) and (32) with parameters ξ, η etc. given by equations (44).

Finally, in order to find the solution of the integral equations (37) and (38) in terms of $\phi(x)$, we proceed in the following way.

From equation (39) we have that the solution is

$$(46) \quad \psi(u) = M^{-1}[\psi(s); u],$$

$$= M^{-1}\left[2^{s-1}\frac{\Gamma(1/2+p+s/2)}{\Gamma(1/2+q-n-s/2)} M\{\phi(x); 1-s\}; u\right],$$

on using equations (43) and (44). Inverting the order of integration in the last equation we get

$$(47) \quad \psi(u) = \int_0^\infty \phi(x)\, M^{-1}\left[2^{s-1}\frac{\Gamma(1/2+p+s/2)}{\Gamma(1/2+q-n-s/2)}; ux\right] dx,$$

$$= \int_0^\infty \left(\frac{ux}{2}\right)^{1+n+p-q} \phi(x)\, J_{p+q-n}(ux)\, dx.$$

after applying the results (40). When $p = q$, this solution is exactly the same as that found by Ahmad [7].

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