Jñānābha, Vol. 30, 2000

APPROXIMATING COMMON FIXED POINTS OF TWO NONEXPANSIVE MAPPINGS IN BANACH SPACES

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(Received : December 20, 1999)

ABSTRACT

Let C be a nonempty subset closed convex subset of a real Banach space R and let U, T be nonexpansive mappings of C into itself. In this paper, we consider the following iteration procedure of Mann's type for approximating common fixed points of mappings U and T:

$$x_1 = x \in C,\ x_{n+1} = (1-\alpha_n)\, x_n + \frac{1}{n^2}\alpha_n \sum_{i,j=0}^{n-1} U^i\, T^j \text{ for every } n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. Using some ideas in the nonlinear ergodic theory, we prove that the iterates converge weakly to a common fixed point of the nonexpansive mappings T and U in a uniformly convex Banach space which satisfies Opial's condition or whose norm is Frechet differentiable.

1. Introduction. Let C be a nonempty closed convex subset of a Banach space R. Then, a mapping $T: C \to C$ is called nonexpansive if

$$\| Tx - Ty \| \leq \|x - y\| \qquad \text{for all } x,y \in C.$$

We denote by $F(T)$ the set of fixed points of T. Mann [5] introduced an iteration procedure for approximating fixed points of a mapping T in a Hilbert space as follows:

where $\{\alpha_n\}$ is a sequence in $[0,1]$. Later, Reich [7] discussed this iteration procedure in a uniformly convex Banach space whose norm is Frechet differentiable.

In this paper, we consider the following iteration procedure of Mann's type for approximating common fixed points of two nonexpansive mappings in a Banach space:

$$x_1 = x \in C,\ x_{n+1} = (1-\alpha_n)\, x_n + \frac{\alpha_n}{n^2} \sum_{i,j=0}^{n-1} U^i\, T^j x_n \text{ for every } n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ and U,T are nonexpansive mappings of C into itself. Using some ideas in the nonlinear ergodic theory and an inequality obtained by *Xu [9]*, we prove that the iterates converge weakly to a common fixed point of the two nonexpensive mappings in a uniformly convex Banach space which satisfies Opial's condition or whose norm is Frechet differentiable.

Through this paper, we assume that R is a real Banach space. We denote by R^* the dual space of R and also denote (y,x^*) the value of $x^* \in R^*$ at $y \in R$. We write $x_n \rightharpoonup x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{X_n\}$ of vectors converges weakly to x. Similarly $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) will symbolize strong convergence. We denote by N the set of all positive integers. For a subset A of R, coA and $\overline{coA}$ mean the convex hull of A and the closure of the convex hull of A, respectively. We say that R satisfies Opial's condition [6] if for any sequence $\{X_n\} \subset R$ with $x_n \rightharpoonup x \in R$, the inequality

$$\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\|$$

holds for every $y \in R$ with $y \neq x$. It is known that all Hilbert spaces and l^p with $1 < p < \infty$ satisfy Opial's condition. It is also known that every separable Banach space can be equivalently renormed so that it satisfies Opial's condition [3]. We also know that if the Banach space R has a duality mappings which is weakly sequentially continuous at 0, then R satisfies Opial's condition [4]. However, the space L^p with $1 < p < \infty$ and $p \neq 2$ do not satisfy Opial's condition [6]. Let R be a Banach space.

Then, the norm of R is said to be Gatcaux differentiable if

$$\lim_{t\to 0} (\| x + ty \| - \| x \|)/t$$

exists for each x and y in U_R, where $U_R = \{x \in R : \| x \| = 1\}$. It is said to be Frechet differentiable if for each x in U_R, this limit is attained uniformly for y in U_R. Let C be a closed convex subset of R and let T be a mapping of C into itself. Then, for each $\varepsilon > 0$, we define the set $F_\varepsilon(T)$ to be

$$F_\varepsilon(T) = \{x \in C : \| Tx - x \| \le \varepsilon\}$$

The following lemmas were proved in [2].

Lemma1. Let R be a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of R. Then, for each $\epsilon > 0$, there exits $\delta > 0$ such that

$$co\, F_\delta(T) \subset F_\varepsilon(T)$$

for every nonexpansive mappings T of C into itself.

Lemma.2 Let R be a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of R. Then,

$$\lim_{n\to\infty} \sup_{\substack{x \in C \\ T \in N(C)}} \|1/(n+1) \sum_{i=0}^{n} T^i x - T(1/(n+1) \sum_{i=0}^{n} T^i x)\| = 0,$$

where $N(C)$ denotes the set of all nonexpansive mappings of C into itself.

Lemma.3. Let C be a nonempty closed convex subset of a Banach space R. Let U and T be a nonexpansive mappings of C into itself such that $UT=TU$ and $F(S) \cap F(T) \neq \phi$. Now consider the following iteration scheme:

$$x_1 = x \in C,\ x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n\, 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n \qquad \ldots(1)$$

for every $n \in N$,

where $\{\alpha_n\}$ is a sequence in [0,1]. Then, for any $n \in N$ putting

$$T_n\, x = (1-\alpha_n)\, x + \alpha_n\, 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x \qquad \text{for every } n \in C,$$

the mapping T_n of C into itself is also nonexpansive. In fact, let $x,y \in C$. Then, we obtain

$$\| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j y \|$$

$$\leq 1/n^2 \sum_{i,j=0}^{n-1} \| U^i T^j x - U^i T^j y \|$$

$$\leq 1/n^2 \sum_{i,j=0}^{n-1} \| x-y \|$$

$$= \| x-y \|.$$

$$\|T_n x - T_n y\| = \|\{(1-\alpha_n)\, x + \alpha_n\, 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x\} -$$

$$\{(1-\alpha_n)\, y + \alpha_n\, 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j y\}] \|$$

$$\leq (1-\alpha_n)\ \|x-y\| + \alpha_n\ \|1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x -$$

$$1/n^2 \sum_{i,j=0}^{n-1} U^i T^j y \|$$

$$\leq (1-\alpha_n)\, \| x-y \| + \alpha_n \| x-y \|$$

$$= \| x-y \|.$$

Further, we have $F(T) \cap F(U) \subset F(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j) \subset F(T_n)$ for every $n \in N$ and hence $F(T) \cap F(U) \subset \bigcap_{n=1}^{\infty} F(T_n)$.

The iterates $\{x_n\}$ defined by (1) can be written as

$$x_{n+1} = T_n, T_{n-1} \ldots T_1 x_1, . \qquad \ldots (2)$$

Putting

$U_n = T_n\, T_{n-1} \ldots T_1 x_{n+1}$, we get

$$x_{n+1} = U_n\, x_1 \qquad \ldots (3)$$

Using Lemmas 1 and 2, we can prove the follwing lemma :

Lemma 3. Let C be a nonempty bounded closed convex Banach space R. Let U and T be nonexpansive mapping of C into itself with $UT = TU$. Then,

$$\lim_{n\to\infty} sup_{x\in C} \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - U(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j y) \| = 0,$$

and

$$\lim_{n\to\infty} sup_{x\in C} \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - T(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j y) \| = 0,$$

Proof. Let $\varepsilon > 0$. From Lemma 1, we know that there exits $\delta > 0$ such that

$$\overline{co}\, F_\delta (S) \subset F_\varepsilon (S) \qquad \dots (4)$$

for every nonexpansive mapping S of C into itself. From Lemma 2, we also have

$$\lim_{n\to\infty} sup_{y\in C} \|1/n \sum_{i=0}^{n-1} U^i y - U(1/n \sum_{i=0}^{n-1} U^i y) \| = 0.$$

Then, there exits $n_1 \in N$ such that

$$sup_{y\in C} \| 1/n \sum_{i=0}^{n-1} U^i y - U(1/n \sum_{i=0}^{n-1} U^i y) \| < \delta,$$

for every $n \geq n_1$. Then, we obtain that

$$1/n \sum_{i=0}^{n-1} U^i y \in F_\delta (U) \subset co\, F_\delta (U) \qquad \dots (5)$$

for every $y \in C$ and $n \geq n_1$. Let $1,p \in N$. Then, we have, for any $n \in N$ with $n>1,p$ and $x \in C$,

$$\| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - U(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x) \|$$

$$\leq \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - (1/n^2 \sum_{i,j=0}^{n-1} U^{i+l} T^{j+p} x) \|$$

$$+ \| 1/n^2 \sum_{i,j=0}^{n-1} U^{i+p} T^{j+p} x - U(1/n^2 \sum_{i,j=0}^{n-1} U^{i+l} T^{j+p} x) \|$$

$$+ \| U(1/n^2 \sum_{i,j=0}^{n-1} U^{i+p} T^{j+l} x - U(1/n^2 \sum_{i,j=0}^{n-1} U^{i+p} T^{j+p} x) \|$$

$$\leq 2 \| 1/n^2 \sum_{i=0}^{n-1} U^i T^j x - 1/n^2 \sum_{i=0}^{n-1} U^{i+l} T^{j+l} x) \|$$

$$+ \| 1/n^2 \sum_{i=0}^{n-1} U^{i+l} T^{j+p} x - U(1/n^2 \sum_{i=0}^{n-1} U^{i+p} T^{j+p} x) \|$$

and

$$I = \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - 1/n^2 \sum_{i,j=0}^{n-1} U^{i+l} T^{j+p} x) \|$$

$$= \| 1/n^2 (\sum_{i=0}^{n-1} \sum_{j=0}^{p-1} U^i T^j x + \sum_{i=0}^{l-1} \sum_{j=0}^{n-1} U^i T^j x - \sum_{i=n}^{n+l-1} \sum_{j=p}^{n-1} U^i T^j x - \sum_{i=1}^{n-1+l} \sum_{j=n}^{p+n-l} U^i T^j x) \|$$

$$\leq 1/n^2 (\sum_{i=0}^{n-1} \sum_{j=0}^{p-1} \| U^i T^j x \| + \sum_{i=0}^{l-1} \sum_{j=0}^{n-1} \| U^i T^j x \| - \sum_{i=n}^{n+l-1} \sum_{j=p}^{n-1}$$

$$\| U^i T^j x \| - \sum_{i=n}^{n+l-1} \sum_{j=p}^{p+n-1} \| U^i T^j x \|)$$

$\leq 1/n^2 \{ np + l(n-p) + l(n+p) + np \} M$

$\leq (2M (l+p)/n),$

where $M = sup_{z \in C} \| z \|$. Then, there exits $n_0 \in N$ such that $n_0 > max \{n1, l, p\}$ and $(2M(l+p)/n) < \varepsilon$ for every $n \geq n_0$.
This is implies that

$$I = \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - 1/n^2 \sum_{i,j=0}^{n-1} U^{i+l} T^{j+p} x) \| < \in$$

for every $x \in C$. Next, we prove that

$$1/n^2 \sum_{i,j=0}^{n-1} U^{i+m} T^{j+q} x \in \overline{co} F_\delta (U) .$$

for every $x \in C$, $m,q \in N$ and $n \geq n_1$. If not, we have that for some $m,q \in N$, $n \geq n_1$ and $x \in C$

$$1/n^2 \sum_{i,j=0}^{n-1} U^{i+m} T^{j+q} x \notin \overline{co} F_\delta (U) .$$

From the separation theorem, there exits $y^*_1 \in R^*$ such that

$$(1/n^2 \sum_{i,j=0}^{n-1} U^{i+m} T^{j+q} xy^*_1) < inf \{(z,y^*_1) : z \in \overline{co} F_\delta (U)\}. \quad ...(6)$$

Then from (5) we obtain

$$inf \{(z,y^*_1) : z \in \overline{co} F_\delta (U)\} \leq inf \{(1/n \sum_{i=0}^{n-1} U^i x y^*_1) : x \in C\}$$

$$\leq (1/n \sum_{i=0}^{n-1} U^i y, y^*_1)$$

for all $y \in C$ and $n \geq n_1$. Then we have that for any $j \in \{0,1,2,3, ...,n-1\}$

$$inf \{(z,y^*_1) : z \in \overline{co} F_\delta (U)\} \leq (1/n \sum_{i=0}^{n-1} U^i (U^m T^{j+q} x), y^*_1)$$

and hence

$$inf \{(z,y^*_1) : z \in co\, F_\delta (U)\} \leq 1/n \sum_{i=0}^{n-1} (1/n \sum_{i=0}^{n-1} U^i (U^m T^{j+q} x, y^*_1).$$

Therefore, From (6),

$$inf \{(z,y^*_1) : z \in \overline{co} F_\delta (U)\} \leq 1/n \sum_{j=0}^{n-1} (1/n \sum_{i=0}^{n-1} U^i (U^m T^{j+q} x, y^*_1).$$

$$< inf \{(z,y^*_1) : z \in \overline{co} F_\delta (U)\} .$$

This is contraction. Hence, from (4), we have

$$1/n^2 \sum_{i,j=0}^{n-1} U^{i+m} T^{j+q} x \in \overline{co} F_\delta (U) \subset F_\varepsilon (U) \quad ... (7)$$

for every $m,q \in N$, $x \in C$ and $n \geq n_1$ then, it follows from (7) that

$$sup_{x \in C} \| 1/n^2 \sum_{i,j=0}^{n-1} U^{i+m} T^{j+q} x - U \ (1/n^2 \sum_{i,j=0}^{n-1} U^{i+m} T^{j+q} x) \| < \varepsilon$$

$m,n \in N$

for every $n \geq n_1$. Hence, we obtain

$$\| \ 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - U \ (1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x) \| \ < 2\varepsilon + \varepsilon = 3\varepsilon$$

for every $x \in C$ and $n \geq n_0$. Therefore,

$$\lim_{n\to\infty} sup_{x \in C} \| \ 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - U \ (1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x) \| \ = 0.$$

Similary, we have

$$\lim_{n\to\infty} sup_{x \in C} \| \ 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - T (1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x) \| \ = 0.$$

Lemma 4. Let R be a Banach space and let C be a nonempty closed convex subset of R. Let U and T be nonexpansive mappings of C into itself such that $UT = TU$ and $F(U) \cap F(T) \neq \phi$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n (1/n^2) \sum_{i,j=0}^{n-1} U^i T^j x_n \qquad \text{for every } n \in N,$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. Let we a common fixed point of T and U. Then, $lim \ \| \ x_n - \omega \ \|$ exists.

Proof. Let ω be a common fixed point of T and U. Then, we have

$$\begin{aligned} \| \ x_{n+1} - \omega \ \| &= \| \ (1-\alpha_n)x_n + \alpha_n \, 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - \omega \ \| \\ &\leq (1-\alpha_n) \ \| \ x_n - \omega \ \| \ + \alpha_n \ \| \ 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - \omega \ \| \\ &\leq (1-\alpha_n) \ \| \ x_n - \omega \ \| \ + \alpha_n \ \| \ x_n - \omega \ \| \\ &= \| \ x_n - \omega \ \| \end{aligned}$$

and hence $\lim_{n\to\infty} \ \| \ x_n - \omega \ \|$ exists.

Lemma 5. Let C be a nonempty closed convex subset of a uniformly convex Banach space R. Let U and T be nonexpansive mappings of C into itself such that $UT = TU$ and $F(U) \cap F(T) \neq \phi$.

Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n \qquad \text{for every } n \in N,$$

where $0 \leq \alpha_n \leq a$ for some a with $0 < a < 1$. Then,

$$\lim_{n\to\infty} \ \| \ Tx_n - x_n \ \| \ = \lim_{n\to\infty} (Ux_n - x_n \ \| \ = 0.$$

In particular, $x_n \to y_0$ implies $y_0 \in F(T) \cap F(U)$.

Proof. For $x \in C$ and $f \in F(U) \cap F(T)$, put $r = \| \ x - f \ \|$ and set $X = \{u \in R : \ \| \ u - f \ \| \ \leq r\} \cap C$. then, X is a nonempty bounded closed convex subset of C which is T, U-invariant and contains $x_1 = x$. So, without loss of generality, we may assume that C is bounded. Let w be a common

fixed point of T and U. Then, there exits a continuous, strictly increasing and convex function $g : [0,\infty) [0,\infty)$
such that $g(0) = 0$ and

$$\| x_{n+1} - \omega \|^2 = \| (1-\alpha_n)(x_n - \omega) + \alpha_n (1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - \omega \|^2$$

$$\leq (1-\alpha_n) \| x_n - \omega \|^2 + \alpha_n \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - \omega \|^2$$

$$- \alpha_n (1-\alpha_n) g (\| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_n \|)$$

for all $n \in N$. Then, since $\alpha_n \leq a$, we have

$$\alpha_n (1-a) g (\| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_n \|$$

$$\leq \alpha_n (1-\alpha_n) g (\| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_n \|)$$

$$\leq (1-\alpha_n) \| x_n - \omega \|^2 + \alpha_n A | 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - \omega \|^2 - \| (x_{n+1} - \omega \|^2$$

$$\leq (1-\alpha_n) \| x_n - \omega \|^2 + \alpha_n \| x_n - \omega \|^2 - \| (x_{n+1} - \omega \|^2$$

$$\leq | |(x_n - \omega) \|^2 - \| (x_{n+1} - \omega \|^2.$$

So, from Lemma 4, we obtain

$$\lim_{n\to\infty} (1-\alpha_n) g (\| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_n \|) = 0.$$

Since g is continuous, strictly increasing and stisfies $g(0) = 0$, we have

$$\lim_{n\to\infty} (1-\alpha_n) \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_n \| = 0. \qquad \dots (8)$$

It follows from the definition of $\{x_n\}$ that

$$x_{n+1} - 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n = (1-\alpha_n)(x_n - 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n).$$

Since

$$\| Tx_{n+1} - x_{n+1} \| \leq \| Tx_{n+1} - T (1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n) \|$$

$$+ \| T (1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n \|$$

$$+ \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_{n+1} \|$$

$$\leq 2 \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_{n+1} \| + \| T (1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n)$$

$$- 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n \|$$

from (8) and Lemma 3, we have

$$\lim_{n\to\infty} \| Tx_n - x_n \| = 0.$$

Similary, we have

$$\lim_{n\to\infty} \| Ux_n - x_n \| = 0.$$

Assume $x_n \to y_n$. Then, since $I\text{-}T$ and $I\text{-}U$ are semiclosed by [1], we obtain that y_0 is a common fixed point of T and U.

3. Main Result

Theorem 1. Let R be a uniformly convex Banach space which satisfies Opial's condition or whose norm is Frechet differentiable. Let C be a nonempty closed convex subset of R. Let U and T be a nonexpansive mappings of C into itself such that $UT = TU$ and $F(U) \cap F(T) \neq \phi$.

Suppose $x_1 = x \in C$ *and* $\{x_n\}$ is given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(1/n^2) \sum_{i,j=0}^{n-1} U^i T^j x_n \qquad \text{for every } n \in N,$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,a]$ for some a with $0 < a < 1$, then $\{x_n\}$ converges weakly to a common fixed point z_0 of T and U.

Proof:. Let $x \in C$. We first assume that R satisfies Opial's condition. Let ω be a common fixed point of T and U. Then, from lemma 4, $\lim_{n\to\infty} \| x_n - \omega \|$ exits. As in the proof of Lemma 5, we may assume that C is bounded. Since R is reflexive, $\{x_n\}$ must contain a subsequence which converges weakly to a point in C. So, let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be a two subsequence of $\{x_n\}$ such that $x_{n_i} \to z_1$ and $x_{n_j} \to z_2$. Then, from Lemma 5, we have that z_1 and z_2 are common fixed points of T and U. Next, we show $z_1 = z_2$. If not, from Opial's condition,

$$\begin{aligned} \lim_{n\to\infty} \| x_n - z_1 \| &= \lim_{i\to\infty} \| x_{n_i} - z_1 \| \\ &< \lim_{i\to\infty} \| x_{n_j} - z_2 \| = \lim_{n\to\infty} \| x_n - z_2 \| \\ &\qquad\qquad = \lim_{j\to\infty} \| x_{n_j} - z_2 \| \\ &< \lim_{j\to\infty} \| x_{n_j} - z_1 \| - \lim_{n\to\infty} \| x_n - z_1 \| \end{aligned}$$

This is contradiction. Hence, we obtain $x_n \to y_0 \in F(T) \cap F(U)$.

Next, we assume that R has a Frechet differentiable norm. As in the proof of Lemma 5, we may assume that C is bounded. So, there exists a subsequence $\{x_n\}$ of $\{x_n\}$ such that $x_{n_i} \to y_0$. Then, from Lemma 5, we obtain $y_0 \in F(T) \cap F(U)$. Putting

$$T_n y = (1-\alpha_n)y + \alpha_n(1/n^2) \sum_{i,j=0}^{n-1} U^i T^j y$$

and

$U_n y = T_n T_{n-1} T_{n-2} \cdots T_1 y$ for all $n \in N$ and $y \in C$, form (3), we have

2. The Fractional Integral Operators. We recall here a few definitions and properties of operators used in solving the triple and quadruple integral equations. Lowndes [4] has defined the following operators:

(4) $$I_{\eta,\alpha}(a, b;\sigma) f(x) = \frac{\sigma x^{-\sigma(x+\eta)}}{\Gamma\alpha} \int_a^b (x^\sigma - t^\sigma)^{\alpha-1} t^{\sigma(\eta+1)-1} f(t)\, dt, \ \alpha > 0,$$

(5) $$= \frac{x^{1-\sigma(\alpha+\eta+1)}}{\Gamma(1+\alpha)} d/dx \int_a^b (x^\sigma - t^\sigma)^{\alpha} t^{\sigma(\eta+1)-1} f(t)\, dt, \ -1 < \alpha < 0,$$

(6) $$K_{\eta,\alpha}(c, d;\sigma) f(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma\alpha} \int_c^d (t^\sigma - x^\sigma)^{\alpha-1} t^{\sigma(1-\alpha-\eta)-1} f(t)\, dt, \ \alpha > 0,$$

(7) $$= \frac{x^{\sigma(\eta-1)+1}}{\Gamma(1+\alpha)} d/dx \int_c^d (t^\sigma - x^\sigma)^{\alpha} t^{\sigma(1-\alpha-\eta)-1} f(t)\, dt, \ -1 < \alpha < 0.$$

where $a < x < b, \quad \sigma > 0$.

From the theory of Abel integral equations it follows that the inverse operators are given by

(8) $$I^{-1}_{\eta,\alpha}(a, b;\sigma) f(x) = I_{\eta+\alpha,-\alpha}(a, b;\sigma) f(x)$$

(9) $$K^{-1}_{\eta,\alpha}(c, d;\sigma) f(x) = K^{-1}_{\eta+\alpha,-\alpha}(c, d;\sigma) f(x).$$

We require two lemmas given by Lowndes [4] which define the pairs of operators:

Lemma A . Let $I_{\eta,\alpha}(a, b;\sigma)$, $I^{-1}_{\eta,\alpha}(d, x;\sigma)$ be operators as defined earlier, then

(10) $$I^{-1}_{\eta,\alpha}(d, x;\sigma)\, I_{\eta,\alpha}(a, b;\sigma) f(x) = \frac{\sigma \sin(\alpha\pi)}{\pi} \frac{x^{-\sigma\eta}}{(x^\sigma - d^\sigma)} \int_a^b \frac{t^{\sigma(\eta+1)-1}(d^\sigma - t^\sigma)^\alpha}{(x^\sigma - t^\sigma)} f(t)\, dt,$$ provided $x > d \geq b > a$.

Lemma B . Let $K_{\eta,\alpha}(a, b;\sigma)$, $K^{-1}_{\eta,\alpha}(x,d;\sigma)$ be operators as defined earlier, then

(11) $$K^{-1}_{\eta,\alpha}(c, d;\sigma)\, K_{\eta,\alpha}(a, b;\sigma) f(x) = \frac{\sigma \sin(\alpha\pi)\, x^{\sigma(\alpha+\eta)}}{\pi\ (d^\sigma - x^\sigma)} x \int_a^b \frac{t^{\sigma(1-\alpha-\eta)-1}(t^\sigma - d^\sigma)^\alpha}{(t^\sigma - x^\sigma)} f(t)\, dt,$$ provided $x < d \leq a < b$.

Two well-known results [4] which play an important part in our solution are :

(12) $$M[I_{\eta,\alpha}(0, x;\sigma) f(x); s] = \frac{\Gamma(1+\eta-s/\sigma)}{\Gamma(1+\eta+\alpha-s/\sigma)} M[f(x); s],$$

(13) $$M[K_{\eta,\alpha}(x,\infty;\sigma) f(x); s] = \frac{\Gamma(\eta+s/\sigma)}{\Gamma(\eta+\alpha+s/\sigma)} M[f(x); s].$$

In what follows we are concerned with n-ranges of the variable x,

$y_0 \in \bigcap_{n=1}^{\infty} \overline{co}\,\{U_m x : m \geq n\}$ Hence, we have

$$y_0 \in \bigcap_{n=1}^{\infty} \overline{co}\,\{U_m x : m \geq n\} \cap F(T) \cap F(U) \subseteq \bigcap_{n=1}^{\infty} \overline{co}\,\{U_m x : m \geq n\} \cap \bigcap_{n=1}^{\infty} F(T_n).$$

So from Lemma 4, we have

$$\{y_0\} = \bigcap_{n=1}^{\infty} \overline{co}\,\{U_m x : m \geq n\} \cap \bigcap_{n=1}^{\infty} F(T_n).$$

Hence, we obtain $x_n \to y_0 \in F(T) \cap F(U)$.

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