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ON RELATIVELY SINGULAR MAPS

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ABSTRACT

In this paper we introduce the concept of relatively singular maps between locally compact spaces and obtain various results concerned to singular maps in this new setting.

1. Introduction. Throughout the paper a space will mean a locally compact Hausdorff space and a map will mean a continuous map.

The notion of the singular set of a mapping was defined and investigated by G.T. Whyburn [14] and G.L. Cain [2,3]. Later the idea became quite useful to obtain a compactification of a locally compact space X with a given compact space K as a remainder. The compactification obtained by this method is called a *singular compactification.* This laid a new step in the theory of compactifications, one of the most active areas of researches in general topology. A few celebrated names in this area are Banaschewski, Cain, Chandler, Conway, Faulkner, Magill Jr., Walker and Whyburn [2,3,4,5,6,7,8,12, 13,14].

A point y in Y is called a *singular point* of $f : X \to Y$ if the closure $cl\, f^{-1}(V)$ of $f^{-1}(V)$ is not compact for any neighbourhood V of y. The set of singular points of f is denoted by $S(f)$. Clearly it is a closed set of Y. If for a map $f{:}X \to Y$, $S(f) = Y$, f is called a *singular map.*

Let $f : X \to Y$ and $\alpha{:}Y \to Y$ be a self map. A point $y \in Y$ is called a *relatively singular point* of the map $f(w.r.t.\alpha)$ if the closure $cl\, (\alpha\, o\, f)^{-1}(U)$ of $(\alpha o f)^{-1}(U)$ is not compact for any neighbourhood U of y. If every point of Y is relatively singular, we call f to be *relatively singular map* ($w.r.t.\alpha$ $:Y \to Y$).

The singular map and relatively singular map may differ but when α is equal to identity map on Y, both concepts coincide. A singular map is always relatively singular but the converse is not true.

1.1 Example. The indentity map $I{:}R^2 \to R^2$ is not singular but I is relatively singular with respect to the first projection $\pi_1 : R^2 \to R$.

In section *2* we show that the product of a relatively singular map with any map is relatively singular. Also the induced map of a relatively singular map $f:X \to Y$ on respective cones remains relatively singular. Moreover an analogue of Pasting lemma is obtained for relatively singular maps.

A G-space is a Hausdorff space X on which a topological group G acts continuously. The orbit space of a G-space is denoted by X/G and the map which assigns to x in X its orbit is the orbit map on X. An equivariant map f from a G-space X to a G-space Y induces a continuous map f_G on orbit spaces which sends the orbit of x in X to the orbit of $f(x)$. Let H be a compact subgroup of G and let X be an H-space. Then for $h \varepsilon H$ and $(g,x) \varepsilon G\times X$, $h(g,x) = (gh^{-1}, h.x)$ defines an action of H on $G\times X$; the orbit space $G\times_H X$ of the H-space $G\times X$ is called the *twisted product* of G and X. For H-spaces X and Y, an equivariant map $f : X \to Y$ determines a continuous map $f_H : G\times_H X \to G\times_H Y$, which sends $[g,x]$ in $G\times_H X$ to $[g,f(x)]$ in $G\times_H X$ [1].

A map $f : X \to Y$ will be called *compact* if the inverse image of a compact set of Y is a compact set of X.

We consider G-spaces and equivariant maps in section 3 and obtain that an equivariant relatively singular map from a G-space X to a G-space Y induces a relatively singular map on orbit spaces.

2. Relatively Singular maps and Various Structures

In this section we obtain some results for relatively singular maps which are already established for singular maps [10].

2.1 Proposition. Let $f : X \to Y$, $h : X' \to Y'$ and $\alpha: Y \to Y$ be maps, If f is relatively singular (*w.r.t.* α), then $f \times h : X \times X' \to Y\times Y'$ defined by

$$(f \times h)(x, x') = \{f(x), h(x')\}$$

is relatively singular (*w.r.t.*$\alpha\times I$) where $I : Y' \to Y'$ is the idnetity map.

Proof. Suppose that $f \times h : X \times X' \to Y \times Y'$ is not relatively singular (*w.r.t.*$\alpha x I$). Then there exists a basic open set $U\times U'$ of $Y\times Y'$ such that $cl[(\alpha\times I) \, o \, (f\times h)]^{-1} (U \times U')$ is compact.

Since

$$cl[(\alpha\times I) \, o \, (f\times h)]^{-1} (U \times U')$$

$$= cl\,(\alpha o f)^{-1} (U) \times cl\,(I \times h)^{-1} (U')$$

$$\Rightarrow cl\,(\alpha \, o \, f)^{-1} (U) \text{ is compact.}$$

A contradiction to the hypothesis.

The following proposition gives an analogue of Pasting Lemma for relatively singular maps.

2.2 Proposition. Let X, Y be locally compact spaces and A,B be closed set of X such that $A \cup B = X$. Let $f{:}A \to Y$ and $g{:}B \to Y$ be two relatively singular maps $(w.r.t. \alpha{:}Y \to Y)$ such that $f(x) = g(x)$ for $x \varepsilon A \cap B$. Then the map $h{:}X \to Y$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \varepsilon A \\ g(x) & \text{if } x \varepsilon B \end{cases}$$

is relatively singular $(w.r.t. \alpha{:}Y \to Y)$.

Proof. Let U be an open set of Y. Then $(\alpha o h)^{-1}(U) = (\alpha\, o\, f)^{-1}(U) \cup (\alpha o g)^{-1}(U) \Rightarrow cl\,(\alpha o h)^{-1}(U) = cl(\alpha\, o\, f)^{-1}(U) \cup cl\,(\alpha o g)^{-1}(U)$.

Since f and g are relatively singular maps, $cl\,(\alpha\, o\, f)^{-1}(U)$ and $cl\,(\alpha o g)^{-1}(U)$ both are noncompact.

$\Rightarrow (\alpha\, o\, h)^{-1}(U)$ is non-compact.

2.3 Definition. Let X be a topological space, the *Cone* T_X over X is the quotient space $(X \times I)/A$, where A is the subset of $X \times I$ given by $X \times \{1\}$.

2.4 Proposition. Let $f: X \to Y$ be a reltively singular map $(w.r.t. \alpha{:}Y \to Y)$. If the quotient maps p,q are compact maps, then the induced map $Tf{:}\ Tx \to Ty$ is relatively singular $(w.r.t. T\alpha{:}Ty \to Ty)$.

Proof Consider the following commutative diagram.

$$\begin{array}{ccc} X \times I & \xrightarrow{(\alpha \times I)\, o\, (f \times I)} & Y \times I \\ p \downarrow & & \downarrow q \\ Tx & \xrightarrow[T\alpha\, o\, Tf]{} & Ty \end{array}$$

If $f: X \to Y$ be relatively singular $w.r.t. \alpha{:}Y \to Y$, then $f \times I : X \times I\ Y \times I$ be relatively singular $w.r.t. \alpha \times I{:}Y \times I \to Y \times I$.

Suppose $Tf{:}\ Tx \to Ty$ is not relatively singular, then there is an open set U of Ty satisfying that $cl\,(T\alpha o Tf)^{-1}(U)$ is compact.

$\Rightarrow p^{-1}(cl(T\alpha o\ Tf)^{-1}(U))$ is compact. From continuity of q and commutativity of the above diagram, it follows that

$$\begin{aligned} p^{-1}(cl(T\alpha o\ Tf)^{-1}(U)) &\supseteq cl\,(p^{-1}(T\alpha o\ Tf)^{-1}(U)) \\ &= cl\,(p^{-1}(Tf^{-1}(T\alpha^{-1}(U)))) \\ &= cl\,[(T\alpha\, o\, Tf)\, o\, (p)]^{-1}(U) \\ &= cl\,[q o \{(\alpha \times I)\, o\, (f \times I)\}]^{-1}(U) \\ &= cl\,[(\alpha \times I)\, o\, (f \times I)]^{-1} q^{-1}(U) \end{aligned}$$

Thus $cl\,[(\alpha \times I)\, o\, (f \times I)]^{-1} q^{-1}(U)$ is compact, a contradiction to the hypothesis that $f \times I$ is relatively singular.

3. Relatively Singular Maps and G-spaces. In this section X and Y will denote G-spaces, where G is a compact topological group and the spaces X and Y are respectively locally compact and compact.

3.1 Proposition. Let $f{:}X\to Y$ be an equivariant and relatively singular map ($w.r.t.\alpha{:}Y\to Y$). Then the induced map $f_G{:}X/G\to Y/G$ is relatively singular ($w.r.t.\alpha_G{:}Y/G\to Y/G$).

Proof. Consider the following commutative diagram.

$$\begin{array}{ccc} X & \xrightarrow{\alpha\, o\, f} & Y \\ q_x\downarrow & & \downarrow q_y \\ X/G & \xrightarrow[\alpha_G\, o\, f_G]{} & Y/G \end{array}$$

where q_x and q_y are orbit maps.

Suppose that f_G is not relatively singular. Then there is a point $G(y)$ in Y and an open set U of Y/G containing $G(y)$ such that $cl(\alpha_G o f_G)^{-1}(U)$ is compact. Since q_x is a compact map,

$q_x^{-1}(cl(\alpha_G o f_G)^{-1}(U))$ is compact. From continuity of q_x and commutativity of the above diagram, it follows that

$$q_x^{-1}(cl(\alpha_G o f_G)^{-1}(U)) \supseteq cl\,(q_x^{-1}\,(\alpha_G o f_G)^{-1}(U))$$
$$= cl\,(q_x^{-1}(f_G^{-1}(\alpha_G^{-1}(U))))$$
$$= cl\,((\alpha_G o f_G)\, o q_x)^{-1}(U)$$
$$= cl\,(q_y\, o\, (\alpha o f))^{-1}(U)$$
$$= cl\,(f^{-1}\,(\alpha^{-1}(q_y^{-1}\,(U)))).$$

Thus $cl(f^{-1}(\alpha^{-1}(q_y^{-1}(U))))$ is compact, a contradication to the hypothesis that f is relatively singular.

3.2. Proposition. Let $f{:}X\to Y$ be an equivariant and relatively singular map ($w.r.t.\alpha{:}Y\to Y$). Then the induced map $f_H{:}G\times_H X\to G\times_H Y$ is relatively singular $w.r.t.\alpha_H{:}G\times_H Y\to G\times_H Y$.

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} G\times X & \xrightarrow{(I_G\times\alpha)\, o\, (I_G\times f)} & G\times Y \\ q_x\downarrow & & \downarrow q_y \\ G\times_H X & \xrightarrow[\alpha_H\, o\, f_H]{} & G\times_H Y \end{array}$$

Suppose that f_H is not relatively singular. Then there is an open set U of $G\times_H Y$ such that $cl\,(\alpha_H o f_H)^{-1}(U))$ is compact, then $q_x^{-1}\,(cl\,(\alpha_H\, o\, f_H)^{-1}(U))$ is also compact. From continuity of q_x and commutativity of the above diagram it follows that $q_x^{-1}(cl(\alpha_H o f_H)^{-1}(U)) \supseteq cl\,(q_x^{-1}\,(\alpha_H o f_H)^{-1}(U))$

$$= cl\,(q_x^{-1}(f_H^{-1}(\alpha_H^{-1}(U))))$$
$$= cl\,((\alpha_H o f_H)\, o q_x)^{-1}(U)$$
$$= cl\,(q_y\, o\, (((I_G \times \alpha)\, o\, (I_G \times f)))^{-1}(U)$$
$$= cl\,[((I_G \times \alpha)\, o\, (I_G \times f))^{-1}\, q_y^{-1}\,(U))]$$

is compact, a contradiction to the hypothesis that $I_G \times f$ is relatively singular.

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