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FRACTIONAL INTEGRAL FORMULAE INVOLVING THE PRODUCT OF A GENERAL CLASS OF POLYNOMIALS AND *I*-FUNCTION OF TWO VARIABLES-II

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ABSTRACT

In the present paper we establish some fractional integral formulae involving the product of a general class of polynomials and the *I*-function of two variables. Our results are quite general in character and a number of known and new formulae can be obtained as their particular cases. Several such intersting special and confluent cases of our main results are mentioned breifly.

1. Introduction. The object of the present paper is to obtain a few fractional integral formulae involving the product of a general class of polynomials and the *I*-function of two variables.

The fractional integral operator investigated by Erdélyi [1] and Kober [4] is defined as

$$I_x^{\eta,\upsilon}\{f(x)\}=\frac{x^{-\eta-\upsilon+1}}{\Gamma(\upsilon)}\int_0^x (x-t)^{\upsilon-1}t^{\eta-1}f(t)dt,\ Re(\upsilon)>0,\ \eta>0. \qquad ...(1.1)$$

Since in the *I*-function of two variables we use the notation I therefore in the operator $I_x^{\eta,\upsilon}$, instead of I we shall use U and in place of η we shall use ς and therefore the operator $I_x^{(\eta,\upsilon)}$ will be represented by $U_x^{\varsigma,\upsilon}$.

The *I*-function of two variables defined by Goyal and Agrawal [2] in the following manner:

$$I^{m_1,n_1:m_2,n_2;m_3,n_3}_{p,q:p_i^{(1)},q_i^{(1)},p_i^{(2)},q_i^{(2)};r}\left[\begin{matrix}Z_1\\ Z_2\end{matrix}\middle|\begin{matrix}[(e_p:E_p,E'_p)]:(a_j,\alpha_j)_{1,n_2}],(a_{ji},\alpha_{ji})_{n_2+1,p_i^{(1)}}];[(c_j,\gamma_j)_{1,n_3}],(c_{ji},\gamma_{ji})_{n_3+1,p_i^{(2)}}]\\ [(f_q:F_q,F'_q)]:(b_j,\beta_j)_{1,m_2}],(b_{ji},\beta_{ji})_{m_2+1,q_i^{(1)}}];(d_j,\delta_j)_{i,m_3}],(d_{ji},\delta_{ji})_{m_3+1,q_i^{(2)}}]\end{matrix}\right]=\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\Phi_1(\xi)\Phi_2(\eta)\Psi(\xi,\eta)Z_1^{\xi}Z_2^{\eta}d\xi d\eta \qquad ...(1.2)$$

where

$$\Phi_1(\xi)=\frac{\prod_{j=1}^{m_2}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n_2}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{j=1}^{r}\left[\prod_{j=m_2+1}^{q_i^{(1)}}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=n_2+1}^{p_i^{(1)}}\Gamma(a_{ji}-\alpha_{ji}\xi)\right]}. \qquad ...(1.2.1)$$

$$\Phi_2(\eta) = \frac{\prod_{j=1}^{m_3} \Gamma(d_j-\delta_j\eta) \prod_{j=1}^{n_3} \Gamma(1-c_j+\gamma_j\eta)}{\sum_{i=1}^{r} \left[\prod_{j=m_3+1}^{q_i^{(2)}} \Gamma(1-d_{ji}+\delta_{ji}\eta) \prod_{j=n_3+1}^{p_i^{(2)}} \Gamma(c_{ji}-\gamma_{ji}\eta) \right]}. \quad \dots (1.2.2)$$

$$\Psi(\xi,\eta) = \frac{\prod_{j=1}^{m_1} \Gamma(f_j-F_j\xi-F'_j\eta) \prod_{j=1}^{n_1} \Gamma(1-e_j+E_j\xi+E'_j\eta)}{\prod_{j=m_1+1}^{q} \Gamma(1-f_j+F_j\xi+F'_j\eta) \prod_{j=n_1+1}^{p} \Gamma(e_j-E_j\xi-E'_j\eta)}. \quad \dots (1.2.3)$$

Z_1, Z_2 are non zero complex variables; L_1, L_2 are two Mellin-Barne type contour integrals. Convergence conditions are

$$|arg\, z_1| < \frac{A_i\pi}{2}, \quad |arg\, z_2| < \frac{B_i\pi}{2}$$

$$A_i = \sum_{j=1}^{n_1} E_j - \sum_{j=n_1+1}^{p} E_j + \sum_{j=1}^{m_1} F_j - \sum_{j=m_1+1}^{q} F_j + \sum_{j=1}^{m_2} \beta_j - \sum_{j=m_2+1}^{q_i^{(1)}} \beta_{ji} + \sum_{j=1}^{n_2} \alpha_j - \sum_{j=n_2+1}^{p_i^{(1)}} \alpha_{ji} > 0$$

and

$$B_i = \sum_{j=1}^{n_1} E'_j - \sum_{j=n_1+1}^{p} E'_j + \sum_{j=1}^{m_1} F'_j - \sum_{j=m_1+1}^{q} F'_j + \sum_{j=1}^{m_3} \delta_j - \sum_{j=m_3+1}^{q_i^{(2)}} \delta_{ji} + \sum_{j=1}^{n_3} \gamma_j - \sum_{j=n_3+1}^{p_i^{(2)}} \gamma_{ji} > 0,$$

for $i = 1, \dots, r$.

A general class of polynomials $S_n^m[x]$ defined by Srivastava [8] as

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, x^k \qquad n = 0,1,2,\dots \quad \dots (1.3)$$

where m is an arbitrary positive integer and $A_{n,k}$ $(n,k \geq 0)$ are arbitrary constants, real or complex.

On specializing the coefficients of $A_{n,k}$; $S_n^m[x]$ yields a number of known polynomials as special cases.

Throughout the paper we shall use the following notations:

$P = m_2, n_2\,;\, m_3, n_3 \qquad , \qquad Q = p_i^{(1)}, q_i^{(1)}; p_i^{(2)}, q_i^{(2)} : r$

$V = (a_j,\alpha_j)_{1,n_2}],(a_{ji}, \alpha_{ji})_{n_2+1,p_i^{(1)}}]; (c_j,\gamma_j)_{1,n_3}], (c_{ji},\gamma_{ji})_{n_3+1,p_i^{(2)}}]$

$V' = (b_j,\beta_j)_{1,m_2}],(b_{ji}, \beta_{ji})_{m_2+1,q_i^{(1)}}];(d_j,\delta_j)_{1,m_3}], (d_{ji},\delta_{ji})_{m_3+1,q_i^{(2)}}]$

and the result for binomial expansion

$$(x+a)^k = a^k \sum_{i=0}^{\infty} \binom{k}{i} (x/a)i \,, \quad |x/a| < 1 \quad \dots (1.4)$$

The formula given by Ross [5] as:

$$U_x^{\zeta,\upsilon}\{x^{\lambda}\} = \frac{\Gamma(\lambda+\zeta)}{\Gamma(\lambda+\zeta+\upsilon)}\, x^{-1}, \quad Re\,(\lambda) > -\zeta\,. \qquad \ldots(1.5)$$

2. Main Results. In this section the following two fractional integral formulae for the I-function of two variables are established as :

(I)

$$U_x^{\zeta,\upsilon}\left[x^{\sigma}(x+\alpha)^{\rho}(x+\beta)^{\mu}\, S_n^m[x^a(x+\alpha)^b(x+\beta)^c] \times I\begin{bmatrix} x^{h_1}(x+\alpha)^{k_1}(x+\beta)^{l_1}\, z_1 \\ x^{h_2}(x+\alpha)^{k_2}(x+\beta)^{l_2}\, z_2 \end{bmatrix}\right]$$

$$= \alpha^{\rho}\beta^{\mu} \sum_{t_1,t_2=0}^{\infty} \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} \frac{\alpha^{bk-t_1}\,\beta^{ck-t_2}\, x^{\sigma+ak+t_1+t_2+ak}}{t_1!\, t_2!} \times$$

$$I_{p+3,q+3:Q}^{m_1,n_1+3:P}\left[\begin{matrix} x^{h_1}\alpha^{k_1}\beta^{l_1}\, z_1 \\ x^{h_2}\alpha^{k_2}\beta^{l_2}\, z_2 \end{matrix}\middle| \begin{matrix} X : V \\ X' : V' \end{matrix}\right] \qquad \ldots(2.1)$$

where

$X = (1-\sigma-ak-t_1-t_2-\zeta : h_1,h_2),(-\rho-bk : k_1,k_2), (-\mu-ck : l_1,l_2),(e_p : E_p, E'_p)$

and

$X' = (f_q : F_q, F'_q),(1-\sigma-ak-\zeta-\upsilon-t_1-t_2 : h_1,h_2),(-\rho-bk+t_1 : k_1,k_2),(-\mu-ck+t_2 : l_1,l_2)$

Provided that

(i) $Re\,(\upsilon) > 0$,

(ii) $Max\,\{\,|arg\,(x/\alpha)|,\ |arg\,(x/\beta)|\} < \pi$,

(iii) $Re\left[\sigma+ak+t_1+t_2+h_1 \underset{1\le j\le m_2}{Min}\left(\frac{b_j}{\beta_j}\right) + h_2 \underset{1\le j\le m_3}{Min}\left(\frac{d_j}{\delta_j}\right) + \zeta > 0.\right]$

(II)

$$U_x^{\zeta,\upsilon}\left[x^{\sigma}(x+\alpha)^{\rho}(x+\beta)^{\mu}\, S_n^m[x^a(x+\alpha)^b(x+\beta)^c] \times I\begin{bmatrix} x^{-h_1}(x+\alpha)^{-k_1}(x+\beta)^{-l_1}\, z_1 \\ x^{-h_2}(x+\alpha)^{-k_2}(x+\beta)^{-l_2}\, z_2 \end{bmatrix}\right]$$

$$= \alpha^{\rho}\beta^{\mu} \sum_{t_1,t_2=0}^{\infty} \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} \frac{\alpha^{bk-t_1}\,\beta^{ck-t_2}\, x^{\sigma+ak+t_1+t_2}}{t_1!\, t_2!} \times$$

$$I_{p+3,q+3:Q}^{m_1,n_1+3:P}\left[\begin{matrix} x^{-h_1}\alpha^{-k_1}\beta^{-l_1}\, z_1 \\ x^{-h_2}\alpha^{-k_2}\beta^{-l_2}\, z_2 \end{matrix}\middle| \begin{matrix} X : V \\ X' : V' \end{matrix}\right] \qquad \ldots(2.2)$$

where

$X = (e_p : E_p, E'_p), (\sigma+ak+t_1+t_2+\zeta+\upsilon : h_1,h_2),(1+\rho+bk-t_1 : k_1,k_2),(1+\mu+ck-t_2 : l_1,l_2)$,

and

$X' = (\sigma+ak+t_1+t_2+\zeta : h_1,h_2),(1+\rho+bk_1 : k_1,k_2),(1+\mu+ck : l_1,l_2),(f_q : F_q, F'_q)$,

provided that

(i) $Re\,(\upsilon) > 0$,

(ii) $Max\,\{\,|arg\,(x/\alpha)|,\ |arg\,(x/\beta)|\} < \pi$,

(iii) $Re\left[\sigma+ak+t_1+t_2-h_1 \min_{1\le j\le m_2}\left(\frac{b_j}{\beta_j}\right)-h_2 \min_{1\le j\le m_3}\left(\frac{d_j}{\delta_j}\right)\right]+\zeta>0.$

Proof. To establish the fractional integral formula (2.1), first we express the general class of polynomials in the series form given by (1.3) and I-function of two variables in terms of Mellin-Barne's type contour integrals on the left hand side of (2.1).

Let

$$A=U_x^{\zeta,\upsilon}\left[x^{\sigma}(x+\alpha)^{\rho}(x+\beta)^{\mu}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\,x^{ak}(x+\alpha)^{bk}(x+\beta)^{ck}\times\frac{1}{(2\pi\omega)^2}\right.$$
$$\left.\int_{L_1}\int_{L_2}\Phi_1(\xi)\Phi_2(\eta)\Psi(\xi,\eta)Z_1^{\xi}Z_2^{\eta}x^{h_1\xi+h_2\eta}(x+\alpha)^{k_1\xi+k_2\eta}(x+\beta)^{I_1\xi+I_2\eta}\,d\xi d\eta\right] \quad ..(2.3)$$

Interchanging the order of Barne's type contour integrals and the fractional integral involved in the expression (2.3). Collecting the powers of x, $(x+\alpha)$, $(x+\beta)$ in the expression thus obtained and expanding $(x+\alpha)^{\rho}$ and $(x+\beta)^{\mu}$ using the binomial expansion (1.4), changing the order of Barne's type contour integrals and series, which is permissible under conditions stated, we get

$$A=\sum_{t_1,t_2=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\Phi_1(\xi)\Phi_2(\eta)\Psi(\xi,\eta)Z_1^{\xi}Z_2^{\eta}\,\alpha^{\rho+bk+k_1\xi+k_2\eta}$$

$$\beta^{\mu+ck+I_1\xi+I_2\eta}\frac{\Gamma(\rho+bk+k_1\xi+k_2\eta+1)\,\Gamma(\mu+ck+I_1\xi+I_2\eta+1)}{t_1!\,t_2!\,(\rho+bk+k_1\xi+k_2\eta-t_1+1)\,\Gamma(\mu+ck+I_1\xi+I_2\eta-t_2+1)\,\alpha^{t_1}\alpha^{t_2}}\times$$

$$U_x^{\zeta,\upsilon}\{x^{\sigma+ck+t_1+t_2+h_1\xi+h_2\eta}\}\,d\xi d\eta \quad ...(2.4)$$

Now using the result (1.5), we arrive at the desired result (2.1). In the similar fashion we can easily establish the result (2.2).

3. Particular Cases

I. If we take $\alpha=\beta=1$ the result (2.1), we get

$$U_x^{\zeta,\upsilon}\left[x^{\sigma}(x+1)^{\rho+\mu}S_n^m[x^a(x+1)^{b+c}]\times I\begin{bmatrix}x^{h_1}(x+1)^{k_1+I_1}\,z_1\\ x^{h_2}(x+1)^{k_2+I_2}\,z_2\end{bmatrix}\right]$$

$$=\sum_{t_1,t_2=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\frac{x^{\sigma+ak+t_1}}{t_1!}I^{m_1,n_1+2:P}_{p+2,q+2:Q}\left[\begin{matrix}(x)^{h_1}z_1\\(x)^{h_2}z_2\end{matrix}\,\middle|\,\begin{matrix}Y:V\\Y':V'\end{matrix}\right], \quad ...(3.1)$$

where

$Y=(1-\sigma-ak-\zeta:h_1,h_2),(-\rho-\mu-bk-ck:k_1+I_1,k_2+I_2),(e_p:E_p,E'_p),$

$Y'=(f_q:F_q,F'_q),(1-\sigma-ak-\zeta-\upsilon-:h_1,h_2),(\rho-\mu-bk-ck+t_1:k_1+I_1,k_2+I_2),$

provided that

(i) $Re(\upsilon)>0$

(ii) $Max\ \{\ |arg\ (x/\alpha)|,\ |arg\ (x/\beta)|\} < \pi$

(iii) $Re\left[\sigma+ak+t_1+t_2+h_1 \min_{1\le j\le m_2}\left(\frac{b_j}{\beta_j}\right)+h_2 \min_{1\le j\le m_3}\left(\frac{d_j}{\delta_j}\right)+\zeta>0\right]$

(II) In (2.1) if we put $\mu = c = I_1 = I_2 = 0$, we get

$$U_x^{\zeta,\upsilon}\left[x^{\sigma}(x+\alpha)^{\rho}\, S_n^m[x^a\,(x+\alpha)^b]\times I\begin{bmatrix}x^{h_1}(x+\alpha)^{k_1}\,z_1\\ x^{h_2}(x+\alpha)^{k_2}\,z_2\end{bmatrix}\right]$$

$$=\sum_{t_1=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\frac{\alpha^{\rho+bk-t_1}\,x^{\sigma+ak+t_1}}{t_1!}I^{m_1,n_1+2:P}_{p+2,q+2:Q}\left[\begin{matrix}x^{h_1}\alpha^{k_1}z_1\\ x^{h_2}\alpha^{k_2}z_2\end{matrix}\,\middle|\,\begin{matrix}Y:V\\ Y':V'\end{matrix}\right] \quad \text{... (3.2)}$$

where

$Y=(1-\sigma-ak-\zeta-t_1:h_1,h_2),(-\rho-bk:k_1,k_2),(e_p:E_p,E'_p)$,

$Y'=(f_q:F_q,F'_q),(1-\sigma-ak-\zeta-\upsilon-t_1:h_1,h_2),(-\rho-bk+t_1:k_1,k_2)$,

provided that

(i) $Re\,(\upsilon)>0$,

(ii) $Re\left[\sigma+ak+t_1+h_1 \min_{1\le j\le m_2}\left(\frac{b_j}{\beta_j}\right)+h_2 \min_{1\le j\le m_3}\left(\frac{d_j}{\delta_j}\right)\right]+\zeta>0$

(iii) If we put $\rho = b = k_1 = k_2 = 0$ in (3.2), we get

$$U_x^{\zeta,\upsilon}\left[x^{\sigma}\,S_n^m[x^a]\,I\begin{bmatrix}x^{h_1}\ z_1\\ x^{h_2}\ z_2\end{bmatrix}\right]=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}x^{\sigma+ak}\,I^{m_1,n_1+1:P}_{p+1,q+1:Q}\left[\begin{matrix}x^{h_1}z_1\\ x^{h_2}z_2\end{matrix}\,\middle|\,\begin{matrix}Y:V\\ Y':V'\end{matrix}\right] \quad \text{... (3.3)}$$

where

$Y=(1-\sigma-ak-\zeta:h_1,h_2)(e_p:E_p,E'_p)$

$Y'=(f_q:F_q,F'_q),(1-\sigma-ak-\zeta-\upsilon:h_1,h_2)$

Provided that

(i) $Re\,(\upsilon)>0$,

(ii) $\left[Re\ \sigma+ak+h_1 \min_{1\le j\le m_2}\left(\frac{b_j}{\beta_j}\right)+h_2 \min_{1\le j\le m_3}\left(\frac{d_j}{\delta_j}\right)+\zeta>0\right]$

(iv) In (3.3), if we put $m_1=n_1=p=q=n_3=p_i^{(2)}=0$, $m_3=q_i^{(2)}=1$ and $z_2\to 0$ the result converts in the form of Saxena's [6] I-function of one variable

$$U_x^{\zeta,\upsilon}\left[x^{\sigma}\,S_n^m[x^a]\,I^{m_2,n_2}_{p_i^{(1)},q_i^{(1)}:r}[x^{h_1},z_1]\right]=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}x^{\sigma+ak}$$

$$I^{m_2,n_2+1}_{p_i^{(1)}+1,q_i^{(1)}+1:r}\left[x^{h_1}z_1\,\middle|\,\begin{matrix}(1-\sigma-ak-\zeta:h_1),V\\ V',(1-\sigma-ak-\zeta-\upsilon:h_1)\end{matrix}\right] \quad \text{...(3.4)}$$

Provided that

(i) $Re(\upsilon) > 0,$

(ii) $Re\left[\sigma+ak+h_1 \underset{1\le j\le m_2}{Min}\left(\frac{b_j}{\beta_j}\right)\right]+\zeta > 0.$

(V) In (3.4) if we take $m=1$, $A_{n,k}=\binom{n+\gamma}{n}\frac{(\gamma+\delta+n+1)_k}{(\gamma+1)_k}$ then the general class of polynomials $S_n^m[x^a]$ reduces to Jacobi polynomials,i.e., $S_n^1[x^a]\to P_n^{(\gamma,\delta)}[1-2.x^a]$

$$U_x^{\zeta,\upsilon}\left[x^\sigma P_x^{(\gamma,\delta)}[1-2x^a]\, I\,{}^{m_2,n_2}_{p_i^{(1)},q_i^{(1)}:r}\,[x^{h_1},z_1]\right] = \sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}\binom{n+\gamma}{n}$$

$$\frac{(\gamma+\delta+n+1)_k}{(\gamma+1)_k}x^{\sigma+ak}\; I\,{}^{m_2,n_2+1}_{p_i^{(1)}+1,q_i^{(1)}+1:r}\left[x^{h_1} z_1\left|\begin{matrix}(1-\sigma-ak-\zeta : h_1),V\\ V',(1-\sigma-ak-\zeta-\upsilon : h_1)\end{matrix}\right.\right] \quad (3.5)$$

Convergence conditions are same as in result (3.4).

In the similar manner we can establish the particular cases for the result (2.2).

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