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EXISTENCE OF MAXIMAL ELEMENTS AND APPLICATIONS

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Recall that a binary relation F on a set C is a subset of $C \times C$ or a mapping of C into itself. It is written yFx or $y \in Fx$ to mean that y stands in relation F to x.

A maximal element of F is a point x such that no point y satisfies $y \in Fx$, i.e., $Fx = \phi$. Thus the set of maximal elements is

$$\{x \in C : Fx = \phi\} = \bigcap_{x \in C} (C \setminus F^{-1}x),$$

where

$$F^{-1}x = \{y \in C : x \in Fy\}.$$

The following well known theorem is due to Sonnenschein [13].

Theorem 1 *Let C be a compact convex subset of R^n and F a binary relation on C satisfying*

(i) $x \notin coFx$ for all $x \in C$ (co stands for convex hull),

(ii) if $y \in F^{-1}x$, then there exists some $x_1 \in C$ (possibly $x_1 = x$) such that $y \in intF^{-1}x_1$.

Then F has a maximal element.

Bergstrom proved the following [1].

Theorem 2 *Let C be a nonempty, compact convex subset of R^n and $F: C \to 2^C$ a preference map satisfying*

(i) $x \notin coFx$ for each $x \in C$,

(ii) F is lower semicontinuous on C.

Then F has a maximal element.

We note that the results on maximal elements are useful in fixed point theory, variational inequalities, complementarity problems and best approximation (see [2] and [12]).

Ky Fan's Lemma [6]. *Let C be a nonempty compact convex subset of R^n and $F : C \to 2^C$ a multifunction such that*

(i) Fx is convex for each $x \in C$,

(ii) F has open graph,

(iii) $x \notin Fx$ for each $x \in C$.

Then F has a maximal element.

We prove the following for maximal elements :

Theorem 3. *Let C be a nonempty closed convex subset of a Hausdorff topological vector space X and $F:C \to 2^C$ satisfy*

(i) $x \notin Fx$ for each $x \in C$,

(ii) Fx is closed for each $x \in C$,

(iii) $F^{-1}(y) = \{x \in C : y \in Fx\}$ is convex for each $y \in C$,

(iv) C can be covered by some finite number of closed sets $Fx_1, Fx_2, \ldots, Fx_n$.

Then F has a maximal element.

Proof.

Let $Fx \neq \phi$ for each $x \in C$. Define $G(x) = C \setminus Fx$. Then $G(x)$ is open for each $x \in C$ since Fx is closed. By (iv) $C = \cup_{i=1}^{n} Fx_i$, so $\cap_{i=1}^{n} Gx_i = \cap_{i=1}^{n} (C/Fx_i) = (\cup_{i=1}^{n} Fx_i)^c = \phi$. Therefore G is not a KKM-map.

Hence, there exists a finite set $\{x_1, x_2, \ldots, x_k\}$ of C such that

$$z = \sum_{i=0}^{k} \lambda_i x_i \not\subset \bigcup_{1}^{k} Gx_i,$$

where $\lambda_i = 0$ and $\sum \lambda_i = 1$.

so $z \in \cap_{i=1}^{k} Fx_i$ and $x_i \in F^{-1}z$, $i = 1, 2, \ldots, k$. Since $F^{-1}z$ is convex so

$$z = \sum_{i=0}^{k} \lambda_i x_i \in F^{-1}z$$

implying that $z \in Fz$, a contradiction to hypothesis (i). So $Fx = \phi$.

The following result is a corollary :

Corollary 1. *Let C be a nonempty closed convex subset of a topological vector space X and $F:C \to 2^C$ an upper semicontinuous convex-valued map. Further, assume that there is some finite subset B of C such that $Fx \cap B \neq \phi$ for every $x \in C$, and $x \notin Fx$ for each $x \in C$. Then F has a maximal element.*

Proof.

Let $G : C \to 2^C$ be defined by

$$Gx = F^{-1}x = \{y \in C : x \in Fy\}.$$

Since F is upper semicontinuous, each $G(x)$ is closed. Now, $G^{-1}(y) = (F^{-1}y)^{-1} = Fy$ is convex.

Also C can be covered by $\{G(x) : x \in B\}$, finitely many closed sets.

By hypothesis $x \notin Fx$ for $x \in C$. So by Theorem 3, F has a maximal element.

In the following, the KKM-map principle is applied :

Theorem 4. *Let C be a nonempty convex subset of a topological vector space X, and $F: C \to 2^C$ satisfy*

(i) $x \notin coFx$ *for each* $x \in C$,

(ii) *if* $y \in F^{-1}x$, *then there exists some* $x_1 \in C$ *(possibly* $x_1 = x$*) such that* $y \in intF^{-1}x_1$,

(iii) *C has a nonempty compact convex subset D such that the set*

$$B = \{x \in C : y \notin Fx \text{ for all } y \in D\}$$

is compact.

Then F has a maximal element.

Proof. Define $G(x) = C \setminus int\, F^{-1}(x)$ for each $x \in C$. Then $G(x)$ is closed for each $x \in C$. We claim that G is a KKM-map. Let $z \in co(x_1, x_2, \dots, x_n)$. *If* $z \notin \cup_{i=1}^{n} Gx_i$ then $z \notin Gx_i$, $i = 1,2,\dots n$, that is, $z \in F^{-1}x_i$, $i = 1,2,\dots,n$. Thus $x_i \in Fz$ and $z \in coFz$, contradiction to (i). Hence G is a KKM-map.

Now,

$$\begin{aligned} B &= \{x \in C : y \notin Fx \text{ for all } y \in D\} \\ &= \{x \in C : x \notin F^{-1}y \text{ for all } y \in D\} \\ &= \{x \in C : x \in Gy \text{ for all } y \in D\} \\ &= \cap_{y \in D} Gy \neq \phi , \end{aligned}$$

that is , $x_0 \in \cap_{y \in D} Gy$.

Thus $x_0 \notin F^{-1}y$ for all $y \in C$, that is, $y \notin Fx_0$ for all $y \in C$ and $Fx_0 = \phi$.

The following results are derived as corollaries :

Corollary 2. *Let C be a nonempty convex subset of a topological vector space E and $F : C \to 2^C$ satisfy:*

(i) $x \notin Fx$,

(ii) *Fx is convex for each* $x \in C$,

(iii) $F^{-1}(y) = \{x \in C : y \in Fx\}$ *is open in C for each* $y \in C$,

(iv) *C has a nonempty compact convex subset D such that the set*

$$B = \{x \in C: y \notin Fx \text{ for all } y \in D\}$$

is compact.

Then F has a maximal element.

The following is due to Lin [9,10]:

Corollary 3. *Let C be a nonempty subset of a topological vector space E and $F : C \to 2^C$ satisfy*

(i) $x \notin co\, F(x)$ *is convex for each* $x \in C$,

(ii) $F^{-1}(y) = \{x \in C : y \in Fx\}$ *is open for each* $y \in C$,

(iii) *C has a nonempty compact convex subset D such that the set*

$$B = \{x \in C: y \notin Fx \text{ for all } y \in D\}$$

is compact.

Then F has a maximal element.

In case C is a nonempty compact convex subset of a topological vector space X and $F : C \to 2^C$ satisfy

(i) $x \notin coF(x)$ for each $x \in C$,

(ii) $F^{-1}(y) = \{x \in C : y \in Fx\}$ is open for each $y \in C$,

then F has a maximal element.

Corollary 4. *Let C be a nonempty compact convex subset of a normed linear space X and $f : C \to X$ a continuous map. Then there is a $y_0 \in C$ such that*

$$||y_0 - fy_0|| = d(fy_0, C),$$

where $d(x,C) = \inf \{||x - y|| : y \in C\}$.

The following variant of Hartman and Stampacchia [7] theorem is very useful in variational problems.

Theorem 5. *Let C be a closed bounded convex subset of a Hilbert space H. If $f : C \to H$ is a continuous and monotone map, then there exists a $y_0 \in C$ such that $\langle fy_0, y_0 - x\rangle \leq 0$ for all $x \in C$.*

Recall that f is monotone on C if $\langle fx - fy, x - y \rangle \geq 0$ for all $x, y \in C$.

The fixed point theorem due to Browder [3] follows as a corollary.

Corollary 5. *If C is a closed bounded convex subset of a Hilbert space H and $f : C \to C$ a nonexpansive map then f has a fixed point.*

Put $F = 1 - f$. Then $F : C \to H$ is continuous and monotone; hence there is a $y_0 \in C$ such that $(Fy_0, y_0 - x) \leq 0$ for all $x \in C$, i.e., $\langle (1-f)\, y_0,$

$$||fx_0-y|| = ||fx_0-\lambda x_0-(1-\lambda)fx_0|| \\ = ||\lambda||\,||x_0-fx_0||<||x_0-fx_0||,$$

a contradiction, so $x_0 = fx_0$.

Using Fan's best approximation theorem, one could prove the following variational inequality due to Hartman and Stampacchia [7].

Theorem 10. Let *C be a compact convex subset of* R^n *and* $f: C \to R^n$ *a continuous function.*

Then there exists an $x_0 \in C$ *such that*

$$< fx_0,\, y-x_0 > \geq 0 \text{ for all } y \in C.$$

Proof. Let $g = I - f: C \to R^n$. Then g is a continuous function. By Theorem 9, there exists a $y_0 \in C$ such that

$$||gy_0 - y_0|| \leq ||gy_0-x|| \text{ for all } x \in C,$$

that is,

$$< gy_0 - y_0,\, y_0-x> \geq 0 \text{ for all } x \in C.$$

Thus

$$< y_0 - fy_0 - y_0,\, y_0-x> \geq 0 \text{ for all } x \in C,$$

that is

$$<-fy_0,\ y_0 - x > \geq 0 \text{ for all } x \in C,$$

Hence

$$<fy_0,\ x - y_0 > \geq 0 \text{ for all } x \in C.$$

Note. If $f = I-g$, then

$<y_0- gy_0,\ x - y_0 > \geq 0$ for all $x \in C$,

gives that g has a fixed point by taking $x = gy_0$.

Theorem 9 is useful in proving the existence of zeros of a given function.

Theorem 11. *Let C be a compact convex subset of* R^n *and* $f: C \to R^n$ *a continuous function. Let* $x - \lambda fx \in C$ *for all* $x \in C$ *and for some* $\lambda > 0$.

Then there is an $x_0 \in C$ *such that* $fx_0 = 0$ *[14].*

Proof. Let $gx = x - \lambda fx$. Then by Theorem 9 there is an $x_0 \in C$ such that

$$||gx_0-x_0|| = d(gx_0,C).$$

In the case $x - \lambda fx \in C$ for all $x \in C$, then $gx_0 \in C$ and $gx_0 = x_0$, that is, $x_0 - \lambda fx_0 = x_0$, so $fx_0 = 0$.

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$z \in Pz$. This contradicts assumption (i) and hence P has a maximal element.

The following result due to Ky Fan [5] is proved by using the existence of maximal elements.

Theorem 8. *Let C a compact convex subset of R^n and let $A \subset C \times C$ be a closed set such that*

(i) for each $x \in C$, $(x,x) \in A$,

(ii) for all $y \in C$, $\{x \in C : (x,y) \notin A\}$ is convex or ϕ.

Then there is a $\bar{y} \in C$ such that $C \times \{\bar{y}\} \subset A$.

Proof. Define F on C by

$$Fy = \{x \in C : (x,y) \notin A\}$$

Then $y \notin Fy$, and Fy is convex for each y.

F has open graph since $(x,y) \notin A$ so $(x,y) \in A^c$ is an open set. So by Ky Fan's Lemma F has a maximal element.

Hence, there exists a $\bar{y} \in C$ such that $F\bar{y} = \phi$, that is, $C \times \{\bar{y}\} \subset A$.

We prove the best approximation theorem by making use of maximal elements and then derive a few fixed point theorems.

The following is Fan's best approximation theorem [6]:

Theorem 9. *Let C be a compact convex subset of R^n and $f : C \to R^n$ a continuous function. Then there exists an $x_0 \in C$ such that*

$$||x_0 - fx_0|| = d(fx_0, C) \leq ||fx_0 - y|| \text{ for all } y \in C.$$

Proof. Define F on C by

$$y \in Fx \text{ if and only if } ||y - fx|| < ||x - fx||.$$

Therefore, by Ky Fan's Lemma we get that F has a maximal element. Since Fx is convex, $x \notin Fx$ and F has open graph; that is, $F\bar{x} = \phi$ for $\bar{x} \in C$. Hence

$$||\bar{x} - f\bar{x}|| \leq ||f\bar{x} - y|| \text{ for all } y \in C.$$

Note. In case $f : C \to C$ in Theorem 9, then f has a fixed point.

In Theorem 9, if $\bar{x} \neq f\bar{x}$, then the following condition will guarantee that f has a fixed point.

If $\bar{x} \neq f\bar{x}$, then the line segment $[\bar{x}, f\bar{x}]$ has at least two points of C.

With this additional hypothesis f has a fixed point.

For instance, by Theorem 9 there is an $x_0 \in C$ such that $||x_0 - fx_0|| = d(fx_0, C)$.

Let $x_0 \neq fx_0$. Set $y = \lambda x_0 + (1-\lambda) fx_0$, $0 < \lambda < 1$. Then

$y_0 - x> \leq 0$ for all $x \in C$. If $x = fy_0$ then we get that $y_0 = fy_0$.

If C is not a weakly compact set, then the following statement gives a result similar to [7].

Theorem 6. *Let C be a nonempty convex subset of a Hilbert space H and $f : C \to H$ continuous monotone map. If C has a weakly compact, convex subset C_0 such that the set*

$$B = \{y \in C : <fy, y - x> \leq 0 \text{ for all } x \in C_0\}$$

is weaklky compact, then there is a $y \in C$ such that

$$<fy_0, y_0 - x> \leq 0 \text{ for all } x \in C.$$

For the following terminologies, see Mehta [11].

As regards the preference correspondence P one assigns for each bundle x in the consumption set, $P(x)$ is regarded as the set of all bundles that are strictly preferred to x, P is irreflexive binary relation. P has a maximal element x_0 if Px_0 is empty, i.e. if there is no bundle strictly preferred to x_0.

In the end, a result for maximal elements in Mathematical Economic is given. Mehta [11] has discussed results of this nature and has shown the existence of maximal elements for an economics agent.

Theorem 7. *Let $P : C \to 2^C$ be a preference correspondence where C is a nonempty closed convex subset of a Hausdorff topological vector space E. Suppose that*

(i) $x \notin Px$ for each $x \in C$,

(ii) Px is closed for each $x \in C$,

(iii) $P^{-1}y = \{x \in C : y \in Px\}$ is convex for each $y \in C$,

(iv) assume that C is covered by finitely many closed sets.

Then there is a maximal element for P.

Proof. Suppose there is no maximal element so

$$P(x) \neq \phi \text{ for all } x \in C.$$

Define $G : C \to 2^C$ by $Gx = C \setminus Px$. Then G is an open valued map for each $x \in C$.

Since $C = \bigcup_1^n Px_i$, so $\bigcap_1^n Gx_i = \bigcap_1^n (C/Px_i) = \cup_1^n Px_i^c = \phi$.

So G is not a *KKM*-map. Recall that if G is an open valued *KKM*-map then the family $\{Gx : x \in C\}$ has the finite intersection property.

Therefore $z = \sum_1^n \alpha_i x_i \notin \bigcup_1^n Gx_i$, and $z \in \bigcap_1^n Px_i$.

Thus $x_i P^{-1}z$. But $P^{-1}z$ is convex so $z = \sum_1^n \alpha_i x_i \in P^{-1}z$, *i.e.*,

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