JÑĀNABHA ज्ञानाभ

Section A Volume 2

1972

Published by :

The Vijnana Parishad

D. V. Postgraduate College

(Kanpur University)

Orai. U.P., India

J n ā n a b h a, Sect. A, Vol. 2, July 1972.

ON A CLASS OF GENERALIZED HYPERGEOMETRIC DISTRIBUTIONS*

By

H. M. SRIVASTAVA

Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada

and

J P. SINGHAL

Department of Mathematics, University of Jodhpur, Jodhpur, India

(*Received on 24th July, 1972*)

SUMMARY

In the course of an attempt to present a unified theory of the classical probability distributions, the authors introduce and study here a general family of statistical probability distributions involving Fox's H-function. In particular, the distribution function, the characteristic function, distributions of the largest order statistic and of the ratio of two independent stochastic variables having the probability functions in this family of statistical probability distributions, etc., are investigated.

---

*This work was supported in part by the National Research Council of Canada under Grant A-7353.

See Abstract 70 T-F21 in Notices Amer. Math. Soc. **17** (1970), p. 969.

Several interesting properties of this class of generalized hypergeometric distributions are also pointed out.

1. INTRODUCTION.

In the literature on probability theory a large number of statistical distributions have been studied, in varying details, from time to time because of their enormous practical applications. Indeed there are frequent instances of studies of some general classes of statistical distributions, such as the general hypergeometric distribution, the generalized beta and gamma distributions, and so on. Recently, MATHAI and SAXENA [4] have introduced what they call a generalized hypergeometric distribution whose probability density function involves the Gaussian ordinary hypergeometric function ${}_2F_1$ (see, e. g., [1], p. 56). A limiting form of this probability distribution would lead to what is well-known in the literature as the general hypergeometric distribution the density function of which involves Kummer's confluent hypergeometric function ${}_1F_1$ [loc. cit., p. 248]. Thus it is readily seen that almost all classical statistical distributions, such as the generalized beta and gamma distributions, the generalized F-distribution, Student's t-distribution, the normal distribution, the exponential distribution, the waiting time distribution, the logistic distribution, and the distributions that go with the names of Cauchy, Raleigh and Weibull, can be derived as specialized or limiting cases of the so-called generalized hypergeometric distribution.

The motivation of the present paper is manifold. A critical analysis of the aforementioned work of Mathai and Saxena would reveal, among other things, the fact that it is only the generalized F-distribution which follows as a particular case of the distribution they have studied. In order to deduce the other classical statistical distributions from theirs, they obviously have had to recourse to a certain limiting process by means of which their generalized hypergeometric distribution would reduce, first of all, to the known hypergeometric distribution involving Kummer's ${}_1F_1$ function. Also in the study of the characteristic function and of the distribution of the ratio of two independent stochastic variables with probability functions expressed in terms of Gauss' ${}_2F_1$, they have to bring in a wider class of functions, viz. Fox's H-function defined by [2, p. 408]

$$(1) \qquad H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1, A_1), \ldots, (a_p, A_p) \\ \\ (b_1, B_1), \ldots, (b_q, B_q) \end{matrix} \right]$$

$$= \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^{m} \Gamma(b_j - B_j \zeta) \prod_{j=1}^{n} \Gamma(1 - a_j + A_j \zeta)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + B_j \zeta) \prod_{j=n+1}^{p} \Gamma(a_j - A_j \zeta)} z^{\zeta}\, d\zeta,$$

where an empty product is interpreted as unity, $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$, the A_j and B_j are all positive, the poles of the integrand of (1) are simple, C is a suitable contour of Mellin-Barnes' type which runs from $\tau - i\infty$ to $\tau + i\infty$ with indentations, if necessary, in such a manner that all the poles of $\Gamma(b_j - B_j\zeta)$, $j = 1, \ldots, m$, are to the right, and those of $\Gamma(1 - a_j + A_j\zeta)$, $j = 1, \ldots, n$, to the left, of C, and the integral in (1) converges if

$$(2) \qquad |\arg(z)| < \tfrac{1}{2}\pi\Delta,$$

with

$$(3) \qquad \Delta \equiv \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0.$$

These conditions will be assumed throughout the present paper, and for convenience, we shall abbreviate the first member of (1) by

$$H_{p,q}^{m,n}[z].$$

Evidently

$$(4) \quad H_{p,q}^{m,n}[z] = H_{q,p}^{n,m}\left[\frac{1}{z}\,\middle|\, \begin{matrix} (1-b_1, B_1), \ldots, (1-b_q, B_q) \\ (1-a_1, A_1), \ldots, (1-a_p, A_p) \end{matrix}\right],$$

$$(5) \quad z^{\delta} H_{p,q}^{m,n}[z] = H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1+\delta A_1, A_1), \ldots, (a_p+\delta A_p, A_p) \\ (b_1+\delta B_1, B_1), \ldots, (b_q+\delta B_q, B_q) \end{matrix}\right]$$

and

$$(6) \quad H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1, 1), \ldots, (a_p, 1) \\ (b_1, 1), \ldots, (b_q, 1) \end{matrix}\right] = G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right),$$

where $G^{m,n}_{p,q}(z)$ denotes the G-function of MEIJER [5].

It may be of interest to note that since [1, p. 215]

$$(7) \quad {}_pF_q \begin{bmatrix} a_1,\ldots, a_p; \\ b_1,\ldots, b_q; \end{bmatrix} z = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} \cdot G^{1,p}_{p,q+1}\left(-z \,\middle|\, \begin{matrix} 1-a_1,\ldots, 1-a_p \\ 0,\ 1-b_1,\ldots, 1-b_q \end{matrix}\right),$$

both Gauss' ${}_2F_1$ and Kummer's ${}_1F_1$ can be recovered from the G-function, and hence also from the H-function, by merely specializing the parameters appropriately, there being no limiting processes involved. Moreover, Meijer's G-functions given by (6) do find several interesting applications in statistical distribution problems (cf., e. g., [3]). Thus it would seem quite natural to consider, in this paper, a general family of statistical probability distributions having the probability density function

$$(8) \quad p(x) = U\left[\alpha, \beta, \gamma : (a_p, A_p)_n ; (b_q, B_q)_m\right] x^{\beta-1} H^{m,n}_{p,q}\left[\alpha x^{\gamma}\right],$$

for $x > 0$, $\gamma > 0$,

$$(9) \quad -\min_{1 \leqslant j \leqslant m}\left(\frac{b_j}{B_j}\right) < \frac{\beta}{\gamma} < -\max_{1 \leqslant j \leqslant n}\left(\frac{a_j-1}{A_j}\right),$$

and $p(x) = 0$ elsewhere ; wherein, for convenience,

$$(10) \quad U\left[\alpha, \beta, \gamma : (a_p, A_p)_n ; (b_q, B_q)_m\right]$$

$$= \frac{\gamma\, \alpha^{\beta/\gamma} \prod_{j=m+1}^{q} \Gamma\left(1-b_j-\frac{\beta}{\gamma}B_j\right) \prod_{j=n+1}^{p} \Gamma\left(a_j+\frac{\beta}{\gamma}A_j\right)}{\prod_{j=1}^{m} \Gamma\left(b_j+\frac{\beta}{\gamma}B_j\right) \prod_{j=1}^{n} \Gamma\left(1-a_j-\frac{\beta}{\gamma}A_j\right)},$$

it being understood that the parameters involved are so restricted that $p(x)$ remains positive.

For several interesting properties of the H-function, in addition to relationships (4) and (5) above, one may refer, for instance, to FOX [2, pp. 408-429] and SRIVASTAVA and DAOUST [7, pp. 451-453]. Indeed its special case, the G-function, has been treated extensively, for instance, by MEIJER [5] and ERDÉLYI et al. [1, pp. 206-222].

It seems worthwhile to remark in passing that in our systematic study of the generalized statistical probability distribution associated with (8) we do not encounter any higher transcendental functions other than the H-functions themselves.

2. THE CHARACTERISTIC FUNCTION.

The characteristic function of $p(x)$ is given by

$$(11) \qquad \phi(t) = E[e^{itX}] = \int_0^\infty e^{itx}\, p(x)\, dx,$$

where $i = \sqrt{-1}$, and E stands for 'mathematical expectation'.

On substituting for $p(x)$ from (8), if we replace the H-function by its contour integral (1), invert the order of integration, and then evaluate the inner Eulerian integral, we shall obtain the characteristic function as

$$(12) \qquad \begin{aligned} \phi(t) &= U[\alpha, \beta, \gamma : (a_p, A_p)_n ; (b_q, B_q)_m] \\ &\cdot (-it)^{-\beta}\, H^{m,\, n+1}_{p+1,\, q}\left[\alpha(-it)^{-\gamma} \;\middle|\; \begin{matrix} (1-\beta, \gamma), (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{matrix}\right], \end{aligned}$$

where $U[\ldots]$ is given by (10).

The derivatives of the H-function, occurring on the right-hand side of (12), can be expressed fairly easily in terms of H-functions themselves. Thus it would be quite straightforward to evaluate the moments and related measures for the general family of probability distributions defined by (8).

3. THE DISTRIBUTION FUNCTION.

The distribution function $F(x)$ or the cumulative probability function for the probability density function $p(x)$ is given by

$$(13) \qquad F(x) = \int_{-\infty}^{x} p(t)\, dt$$

$$= U\,[\alpha,\ \beta,\ \gamma : (a_p,\ A_p)_n\ ;\ (b_q,\ B_q)_m] \int_0^x t^{\beta-1}\ H_{p,\,q}^{m,\,n} \left[a\,t^{\gamma} \right] dt.$$

Now substitute the contour integral for the H-function and change the order of integration. Then evaluate the inner integral and interpret the resulting expression in terms of the H-function.
We thus find that

$$F\,(x) = U\left[\alpha,\ \beta,\ \gamma : (a_p,\ A_p)_n\ ;\ \ (b_q,\ B_q)_m\right]$$

(14)

$$.\, x^{\beta}\, H_{p+1,\ q+1}^{m,\ n+1} \left\{ a x^{\gamma} \,\middle|\, \begin{matrix} (1-\beta,\gamma),\ (a_1,\ A_1),\ ..(a_p,\ A_p) \\ (b_1,\ B_1),\ldots,\ (b_q,\ B_q),\ (-\beta,\gamma) \end{matrix} \right\}$$

4. DISTRIBUTION OF THE LARGEST ORDER STATISTIC.

The density function for the largest order statistic for a sample of size N from a population $f\,(x)$ is given by

$$(15)\qquad g\,(x) = N \left[\int_{-\infty}^{x} f\,(t)\ dt \right]^{N-1} .\, f(x).$$

Thus if we let $f\,(x)$ be the same as $p\,(x)$ defined by (8), then the density function will be given by

$$(16)\qquad g\,(x) = N\,U\,[\alpha,\ \beta,\ \gamma : (a_p,\ A_p)_n\ ;\ (b_q,\ B_q)_m]$$

$$.\, x^{\beta-1}\ \{F\,(x)\}^{N-1}\ H_{p,\,q}^{m,\,n} \left[a x^{\gamma} \right],$$

where $F\,(x)$ is defined by (14) above.

The special case of (16) when $N = 2$ is worthy of note. Indeed, for a sample of size 2, the density function $g\,(x)$ is expressible as the product of two H-functions having the same argument.

5. DISTRIBUTION OF THE RATIO.

Consider the distribution of the ratio of two independent stochastic variables X and Y whose probability density functions $p\,(x)$ and $q\,(y)$ belong to the same family as in (8) above. Thus if we suppose $p\,(x)$ to be given by (8) itself, we may write analogously

(17) $$q(y) = U\left[\lambda, \rho, \sigma : (e_r, E_r)_\nu; (g_s, G_s)_\mu\right]$$

$$\cdot y^{\sigma-1} H^{\mu,\nu}_{r,s}\left[\lambda y^\sigma \left| \begin{matrix} (e_1, E_1), \ldots, (e_r, E_r) \\ (g_1, G_1), \ldots, (g_s, G_s) \end{matrix} \right.\right],$$

for $y > 0$, $\sigma > 0$,

(18) $$-\min_{1 \leqslant j \leqslant \mu} \left(\frac{g_j}{G_j}\right) < \frac{\rho}{\sigma} < -\max_{1 \leqslant j \leqslant \nu} \left(\frac{e_j - 1}{E_j}\right),$$

and $q(y) = 0$ elsewhere ; where

(19) $$\Omega \equiv \sum_{j=1}^{\nu} E_j - \sum_{j=\nu+1}^{r} E_j + \sum_{j=1}^{\mu} G_j - \sum_{j=\mu+1}^{s} G_j > 0,$$

it being understood, as before, that the parameters involved are so constrained that $q(y)$ remains positive.

If we put $W = X/Y$, then

(20) $$V = \log W = \log X - \log Y,$$

and the characteristic function for V is given by

$$\phi_V(t) = E\left[e^{itV}\right] = \int_0^\infty \int_0^\infty e^{it(\log x - \log y)}\, p(x)\, q(y)\, dx\, dy$$

(21) $$= \int_0^\infty x^{it} p(x)\, dx. \int_0^\infty y^{-it} q(y)\, dy.$$

On substituting for $p(x)$ and $q(y)$ from (8) and (17) respectively, if we evaluate the resulting integrals in (21) by making an appeal to the Mellin transform of the H-function [cf., e. g., [7], p. 452, formula (2.5)], we shall at once arrive at the elegant result

(22) $$\phi V(t) = \frac{U\left[\alpha, \beta, \gamma : (a_p, A_p)_n ; (b_q, B_q)_m\right]}{U\left[\alpha, \beta + it, \gamma : (a_p, A_p)_n ; (b_q, B_q)_m\right]}$$

$$\cdot \frac{U\left[\lambda, \rho, \sigma : (e_r, E_r)_\nu \; ; (g_s, G_s)_\mu\right]}{U\left[\lambda, \rho\text{-}it, \sigma : (e_r, E_r)_\nu \; ; (g_s, G_s)_\mu\right]} \cdot$$

Now the Fourier transform of $\phi_V(t)$ would give us the density function $p_V(x)$ of V. Thus

$$(23) \qquad p_V(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_V(t)\,dt.$$

If we substitute for $\phi_V(t)$ from (22) and put $\rho - i\,t = \zeta$, then the second member of (23) will transform itself into a contour integral of Mellin-Barnes' type which can readily be interpreted as an H-function by means of (1).

In order to obtain the density function $p_W(x)$ of W we apply the transformation $W = e^V$, and we finally have

$$(24) \qquad p_W(x) = U[\alpha, \beta, \gamma : (a_p, A_p)_n \; ; (b_q, B_q)_m]$$

$$\cdot\, U\left[\lambda, \rho, \sigma : (e_r, E_r)_\nu \; ; (g_s, G_s)_\mu\right] \frac{x^{-\rho}\, \alpha^{-(\beta+\rho)/\sigma}}{\gamma\sigma}$$

$$\cdot H^{m+\nu,\, n+\mu}_{p+s,\; q+r}\left[\frac{x\, \alpha^{1/\gamma}}{\lambda^{1/\sigma}} \,\middle|\, \begin{matrix} \left(\delta_1, \frac{A_1}{\gamma}\right), \ldots, \left(\delta_n, \frac{A_n}{\gamma}\right)\left(1\text{-}g_1, \frac{G_1}{\sigma}\right), \ldots, \left(1-g_s, \frac{G_s}{\sigma}\right), \left(\delta_{n+1}, \frac{A_{n+1}}{\gamma}\right), \ldots, \left(\delta_p, \frac{A_p}{\gamma}\right) \\ \left(\omega_1, \frac{B_1}{\gamma}\right), \ldots, \left(\omega_m, \frac{B_m}{\gamma}\right)\left(1 - e_1, \frac{E_1}{\sigma}\right), \ldots, \left(1-e_r, \frac{E_r}{\sigma}\right), \left(\omega_{m+1}, \frac{B_{m+1}}{\gamma}\right), \ldots, \left(\omega_q, \frac{B_q}{\gamma}\right) \end{matrix} \right],$$

for $x > 0$ and $p_W(x) = 0$ elsewhere; where, for the sake of brevity,

$$(25) \qquad \begin{cases} \delta_j = a_j + \left(\frac{\beta+\rho}{\gamma}\right) A_j,\; j = 1, \ldots, p, \\ \omega_j = b_j + \left(\frac{\beta+\rho}{\gamma}\right) B_j,\; j = 1, \ldots, q. \end{cases}$$

REFERENCES

[1] ERDÉLYI, A., et al. : Higher Transcendental Functions, Vol. I, McGraw-Hill, New York-Toronto-London, 1953.

[2] FOX, C. : The G and H functions as symmetrical Fourier kernels, Trans. Amer. Math. Soc. **98** (1961), pp. 395-429.

[3] KABE, D. G. : Some applications of Meijer-G functions to distribution problems in statistics, Biometrika. **45** (1958), pp. 578-580.

[4] MATHAI, A. M., and SAXENA, R. K. : On a generalized hypergeometric distribution, Metrika. **11** (1966), pp. 127-132.

[5] MEIJER, C. S. : On the G-function I-VIII, Nederl. Akad.Wetensch. Proc. Ser. A **49** (1946), pp. 227-237, 344-356, 457-469, 632-641, 765-772,936-943, 1063-1072, 1165-1175.

[6] PATIL, G. P. : A characterization of the exponential type distribution, Biometrika. **50** (1961), pp. 205-207.

[7] SRIVASTAVA, H. M., and DAOUST, M. C. : Certain generalized Neumann expansions associated with the Kampé de Fériet function, Nederl. Akad. Wetensch. Proc. Ser. A **72** = Indag. Math. **31** (1969), pp. 449-457.

[8] STACY, E. W. : A generalization of the gamma distribution, Ann. Math. Statist. **33** (1962), pp. 1187-1192.

Jñānabha, Sect. A, Vol. 2, July 1972.

P. HUMBERT'S CONFLUENT HYPERGEOMETRIC FUNTION ϕ_1 (α, β; γ; x, y)

By

R. TREMBLAY and M. L. LAVERTU

Department de Mathematiques

Universite Laval

Quebec, P. Q., Canada.

(*Received on 24th April, 1972*)

ABSTRACT :

Some known results involving Laguerre polynomials and Humbert's function are generalized.

1— INTRODUCTION.

A few years ago, Abdul-Halim and Al-Salam [1] obtained the expansion

$$(1.1) \qquad \phi_1 (\alpha, \beta; \gamma; -u, -uv) = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\gamma)_r} u^r L_r^{(-\beta-r)} (v)$$

where, as usual, the Laguerre polynomials are defined by

([6], p. 200)

$$(1.2) \qquad L_n^{(\alpha)} (x) = \frac{(1+\alpha)_n}{n!} \, {}_1F_1 \left[\begin{array}{c|c} -n & x \\ 1+\alpha & \end{array} \right].$$

${}_1F_1$ is the confluent hypergeometric function and ϕ_1 $(\alpha, \beta; \gamma; x, y)$ is the confluent form of Appell's function F_1 ([4], Vol II, p. 444)

$$(1.3) \qquad \phi_1(\alpha, \beta; \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}$$

We prove here the more general result

$$1.4) \quad \phi_1\left(\alpha, \beta; \gamma; -u, -u \sum_{i=1}^{n} w_i\right) = \sum_{r_1,\ldots,r_n=0}^{\infty} \frac{(\alpha)_{r_1+\ldots+r_n}}{(\gamma)_{r_1+\ldots+r_n}} u^{r_1+\ldots+r_n}$$

$$\times L_{r_1}^{(\beta_1-\beta-r_1)}(w_1) \ldots L_{r_{n-1}}^{(\beta_{n-1}-\beta_{n-2}-r_{n-1})}(w_{n-1})\, L_{r_n}^{(-\beta_{n-1}-r_n)}(w_n)$$

with $(\beta_r - \beta_{r-1}) \neq 0, -1, -2, \ldots$ for $r=1, 2, \ldots, n$ where $\beta_0 = \beta$, $\beta_n = 0$ and $|x| < 1$, $(\gamma) \neq 0, -1, -2, \ldots$.

We also obtain generalizations of known formulas which may be new, e. g. (2.1), (2.5), (3.2), (3.4), (3.7), (3.8), (3.9) and (3.12). Note that for the sake of brevity in most of these formulas, we do not mention explicitly the restrictions on the parameters involved. They can easily be obtained from the restrictions mentioned in connection with formulas (1.4) or (2.4). We refer the reader to standard texts, e. g. [5], for definitions and notation used in connection with hypergeometric functions.

2. PROOF OF THE MAIN RESULT.

Since it is no more difficult, we shall obtain slightly more general result than (1.4). Consider the identity

$$(2.1) \quad \phi_1(\alpha, \beta; \gamma; x, y) = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\gamma)_r} (-x)^r \, L_r^{(\beta_1-\beta-r)}\left\{(1-\sigma)\frac{y}{x}\right\}$$
$$\phi_1(\alpha+r, \beta_1; \gamma+r; x, \sigma y),$$

which is easily established as follow. From (1.2) and the fact that

$$(-r)_k = \frac{(-1)_k \, r!}{(r-k)!} \; (0 \leq k \leq r) \text{ and}$$

$$\frac{(1+\beta_1-\beta-r)_r}{(1+\beta_1-\beta-r)_k} = (-1)^{r+k} (\beta-\beta_1)_{r-k},$$

the Laguerre polynomial of (2.1) can be written

(2.2)

$$L_r^{(\beta_1-\beta-r)}\left\{(1-\sigma)\frac{y}{x}\right\} = (-1)^r \sum_{k=0}^{r} \frac{(\beta-\beta_1)_{r-k}}{(r-k)!\,k!} \left((1-\sigma)\frac{y}{x}\right)^k.$$

With (2.2) and the definition (1.3), the right-side assumes the form

(2·3)

$$\sum_{r,\,m,\,n=0}^{\infty} \sum_{k=0}^{r} \frac{(\alpha)_{r+m+n}}{(\gamma)_{r+m+n}} \frac{(\beta-\beta_1)_{r-k}\,(\beta_1)_m}{(r-k)!\,k!\,m!\,n!} x^{m+r-k}\, y^{n+k}\, \sigma^n\, (1-\sigma)^k$$

$$= \sum_{r,\,k,\,m,\,n=0}^{\infty} \frac{(\alpha)_{r+m+n+k}\,(\beta-\beta_1)_r\,(\beta_1)_m}{(\gamma)_{r+m+n+k}\; r!\,k!\,m!\,n!} x^{m+r}\, y^{n+k}\, \sigma^n\, (1-\sigma)^k$$

$$= \sum_{r,\,k=0}^{\infty} \sum_{m=0}^{r} \frac{(\alpha)_{r+k}}{(\gamma)_{r+k}} \frac{(\beta-\beta_1)_{r-m}\,(\beta_1)_m}{(r-m)!\,(k-n)!\,m\,n!} x^r\, y^k\, \sigma^n\, (1-\sigma)^{k-n}$$

$$= \sum_{r,\,k=0}^{\infty} \frac{(\alpha)_{r+k}\,(\beta-\beta_1)_r}{(\gamma)_{r+k}} \frac{x^r}{r!} \frac{((1-\sigma)\,y)^k}{k!}\, {}_1F_0\left(\begin{matrix}-k\\ -\end{matrix}\,\middle|\, \frac{-\sigma}{1-\sigma}\right)$$

$$ {}_2F_1\left(\begin{matrix}-r,\ \beta_1\\ 1+\beta_1-\beta-r\end{matrix}\,\middle|\, 1\right)$$

Gauss's summation theorem ([6], p. 49) implies (2·1) immediately.

Thus applying (2·1) n times we obtain our main result

(2.4)

$$\phi_1(\alpha, \beta; \gamma; x, y) = \sum_{r_1,\ldots,r_n=0}^{\infty} \frac{(\alpha)_{r_1+\ldots+r_n}}{(\gamma)_{r_1+\ldots+r_n}} (-x)^{r_1+\ldots+r_n}$$

$$\times L_{r_1}^{(\beta_1-\beta-r_1)}\left((1-\sigma_1)\frac{y}{x}\right)\ldots L_{r_n}^{(\beta_n-\beta_{n-1}-r_n)}\left((1-\sigma_n)\,\sigma_1\ldots\sigma_{n-1}\frac{y}{x}\right)$$

$$\times\, \phi_1(\alpha+r_1+\ldots+r_n, \beta_n; \gamma+r_1+\ldots r_n; x, \sigma_1\ldots\sigma_n y),$$

which holds for $(\beta_i-\beta_{i-1}) \neq 0, -1, -2, \ldots$ for $i = 1, 2, \ldots, n$

where $\beta_0 = \beta$ and $|x| < 1$, $(\gamma) \neq 0, -1, -2, \ldots\ldots$.

If $x = -u, \quad y = -u \sum_{j=1}^{n+1} w_j$ and

$$\sigma_i = 1 - \frac{W_i}{W_1 + \cdots + W_{n+1}} \quad \text{for } i = 1, 2, \ldots, n.$$

Eq (2.4) becomes

$$\phi_1\left(\alpha, \beta; \gamma; -u, -u \sum_{j=1}^{n+1} w_j \right) = \sum_{r_1, \ldots, r_n = 0}^{\infty} \frac{(\alpha)_{r_1 + \ldots + r_n}}{(\gamma)_{r_1 + \ldots + r_n}} u^{r_1 + \ldots + r_n}$$

(2·5) $$\times L_{r_1}^{(\beta_1 - \beta - r_1)}(w_1) \ldots L_{r_n}^{(\beta_n - \beta_{n-1} - r_n)}(w_n)$$

$$\times \phi_1 (\alpha + r_1 + \ldots\ldots + r_n, \ \beta_n; \ \gamma + r_1 + \ldots + r_n; \ -u, \ -u w_{n+1}).$$

Our relation (2·5) contains very many interesting special cases, listed in Sec. 3 of this paper. In particular we obtain (1·4) for $\beta_n = w_{n+1} = 0$.

3— SOME SPECIAL CASES.

If we recall that

(3.1) $$\phi_1 \ (\alpha, \beta; \gamma; x, 0) = {}_2F_1 \left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix} \middle| x \right),$$

then with $w_{n+1} = 0$ in (2·5), we obtain

(3·2)

$$\phi_1 \ (\alpha, \beta \ ; \gamma; -u, -u \sum_{j=1}^{n} w_j) = \sum_{r_1, \ldots, r_n = 0}^{\infty} \frac{(\alpha)_{r_1 + \ldots + r_n}}{(\gamma)_{r_1 + \ldots + r_n}} u^{r_1 + \ldots + r_n}$$

$$L_{r_1}^{(\beta_1 - \beta - r_1)}(w_1) \ldots L_{r_n}^{(\beta_n - \beta_{n-1} - r_n)}(w_n) \quad {}_2F_1 \left(\begin{matrix} \alpha + r_1 + \ldots + r_n, \beta_n \\ \gamma + r_1 \ldots + r_n \end{matrix} \middle| -u \right);$$

as before we have two cases in which the ${}_2F_1$ can be summed. If $u = 1$ and $\gamma = 1 + \alpha - \beta_n$, then Kummer's summation theorem [(6), p. 68]

(3.3) $${}_2F_1 \left(\begin{matrix} \alpha, \beta \\ 1 + \alpha - \beta \end{matrix} \middle| -1 \right) = \frac{\Gamma(1 + \alpha - \beta) \ \Gamma(1 + \frac{1}{2} \alpha)}{\Gamma(1 + \frac{1}{2} \alpha - \beta) \ \Gamma(1 + \alpha)} \quad \text{Re}(\beta) < 1$$

and $1 + \alpha - \beta \neq 0, -1, -2, \ldots\ldots$

reduces (3.2) to

$$(3.4)\quad \phi_1\left(a, \beta; 1+a-\beta_n; -1, -\sum_{j=1}^{n} w_j\right)=\frac{\Gamma(1+a-\beta_n)}{\Gamma(a)}$$

$$\sum_{r_1,\ldots,r_n=0}^{\infty} \frac{\Gamma(1+\frac{1}{2}(a+r_1+\ldots+r_n))}{\Gamma(1-\beta_n+\frac{1}{2}(a+r_1+\ldots+r_n))\,(a+r_1+\ldots+r_n)}$$

$$\times\quad L_{r_1}^{(\beta_1-\beta-r_1)}(w_1)\ldots L_{r_n}^{(\beta_n-\beta_{n-1}-r_n)}(w_n)$$

with the supplementary conditions Re $(\beta_n) < 1$ and Re $(\beta) < 1$ for convergence.

In the second case, if $u=-1$ and Re $(\gamma-a-\beta_n) > 0$, we apply Gauss's summation theorem [(6), p. 49] to the right side of (3.2) and recalling that [(5), Vol I, p. 239]

$$(3.5)\quad \phi_1(a, \beta; \gamma; 1, y)=\frac{\Gamma(\gamma)\,\Gamma(\gamma-a-\beta)}{\Gamma(\gamma-a)\,\Gamma(\gamma-\beta)}\,{}_1F_1\left[\begin{matrix} a \\ \gamma-\beta \end{matrix}\,\middle|\, y\right] \text{ with}$$

$$\text{Re }(\gamma-a-\beta) > 0,$$

we obtain

$$(3{\cdot}6)\quad {}_1F_1\left(\begin{matrix} a \\ \gamma-\beta \end{matrix}\,\middle|\, \sum_{j=1}^{n} w_j\right) = \frac{\Gamma(\gamma-\beta)\,\Gamma(\gamma-a-\beta_n)}{\Gamma(\gamma-\beta_n)\,\Gamma(\gamma-a-\beta)}$$

$$\sum_{r_1,\ldots,r_n=0}^{\infty} \frac{(a)_{r_1+\ldots+r_n}\,(1-)^{r_1+\ldots+r_n}}{(\gamma-\beta_n)_{r_1+\ldots+r_n}}$$

$$\times L_{r_1}^{(\beta_1-\beta-r_1)}(w_1)\ldots L_{r_n}^{(\beta_n-\beta_{n-1}-r_n)}(w_n).$$

However in the above, if $a=-s$ and if $\gamma-\beta-1=a_1+\ldots+a_{n+1}$, $\gamma-\beta_n-1=a_{n+1}$ and $\beta_i-\beta_{i-1}=a_i$ for $i=1, 2,\ldots, n$ with $\beta_0=\beta$, then *E*q. (3·6) may be written

$$(3.7)\quad L_s^{(a_1+\ldots+a_{n+1})}\left(\sum_{j=1}^{n} w_j\right) = \sum_{r_1+\ldots+r_n=0}^{s}\binom{a_{n+1}+s}{a_{n+1}+r_1+\ldots+r_n}$$

$$\times L_{r_1}^{(a_1-r_1)}(w_1)\ldots L_{r_n}^{(a_n-r_n)}(w_n)$$

where as usual

$$\binom{\alpha}{\beta} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)\,\Gamma(1+\alpha-\beta)}$$

Moreover, if $w_i = 0$ for $i = 1, 2, \ldots, n$, all polynomials reduce immediately to binomial coefficients and we obtain the interesting result

(3·8)

$$\binom{a_1+\ldots+a_{n+1}+s}{s} = \sum_{r_1+\ldots+r_n=0}^{s} \binom{a_{n+1}+s}{a_{n+1}+r_1+\quad+r_n} \binom{a_1}{r_1} \binom{a_n}{r_n}.$$

Let us now return to the main relation (2.5) and, as before, put $u=-1$, $\alpha=-s$, $\gamma-\beta-1=a_1+\ldots+a_{n+1}$ and $a_i=\beta_i-\beta_{i-1}$ for $i=1, 2,\ldots, n$ with $\beta 0=\beta$ but with $w_{n+1} \neq 0$. Then $\gamma-\beta_n-1=a_{n+1}$ and the equation becomes

(3.9)

$$L_s^{(a_1+\ldots+a_{n+1})}\left[\sum_{j=1}^{n+1} w_j\right] =$$

$$\sum_{r_1+\ldots+r_n=0}^{s} L_{s-r_1-\ldots-r_n}^{(a_{n+1}+r_1+\ldots+r_n)}(w_{n+1})\, L_{r_1}^{(a_1-r_1)}(w_1)\ldots L_{r_n}^{(a_n-r_n)}(w_n)$$

Equation (3.9) is more general than (3.7), the special case is recovered from (3.9) with $w_{n+1}=0$. In (3.7), (3.8) and (3.9) the sum is extended over all sets of non-negative integers $r_1, r_2, \ldots, r_{n-1}$, and r_n having sums eqal to 0, 1,..., $s-1$ and s.

Another special case can be obtained. If we let $\beta \to \beta_1$ in (2.1) and apply the transformation

(3.10) $$\phi_1(\alpha, \beta; \gamma; x, y) = (1-x)^{-\beta}\; e^{y}\; \phi_1\left(\gamma-\alpha, \beta; \gamma; \frac{x}{x-1}, -y\right),$$

which is a confluent case of one of the known transformations of the function F_1 (Appell's first type) ([5], Vol. *I*, *p*. 239), we obtain

$$(3.11)\quad \phi_1\left(\gamma-a,\beta;\gamma;\frac{x}{x-1},-y\right)=e^{y(\sigma-1)}$$

$$\sum_{r=0}^{\infty}\frac{(a)_r}{(\gamma)_r}\frac{[(1-\sigma)y]^r}{r!}\phi_1\left(\gamma-a,\beta;\gamma+r;\frac{x}{x-1},-\sigma y\right).$$

In the above, letting $a=\gamma-\delta$, $x=\frac{-v}{1-v}$, and $y=-w$,

we have

$$(3\cdot12)\quad \phi_1(\delta,\beta;\gamma;v,w)=e^{-v(\sigma-1)}\sum_{r=0}^{\infty}\frac{(\gamma-\delta)_r}{(\gamma)_r}\frac{((\sigma-1)w)^r}{r!}$$

$$\phi_1(\delta,\beta;\gamma+r;v,\sigma w).$$

This inverse relation can be obtained similarly. Moreover if $\beta=0$ and $\sigma=v$, we obtain a known result for the confluent hypergeometric function ${}_1F_1$ ([7], eq. 2·3·13, p. 23).

Many other special cases for two and three variables can be obtained from the relation (3·11), which are already special cases of other general known results. For example the special cases for three variables can be obtained from the bilinear relation (3·1) and (3.2) in ([8], p. 70) in terms of Kampé de Fériet's double hypergeometric functions. The special cases for two variables can be obtained from the well known formulas of Brafman ([2], eq. 27, p. 947) and Chaundy ([3], eq. (25), p. 62) which are also special cases of the more general formulas (3.1) and (3.2) in ([8], p. 70).

ACKNOWLEDGMENT

We finally thank Dr. H. M. Srivastava for his guidance in the preparation of this paper.

REFERENCES

[1] N. Abdul-Halim and W. A. Al-Salam (1963), "Double Euler Transformation of certain hypergeometric functions", Duke Math. Jour. **30**, p. 51-62.

[2] F. Brafman (1951), "Generating functions of Jacobi and related polynomials", Proc. Amer. Math. Soc.. **2** p. 942-949.

[3] T. W. Chaundy (1943), "An extension of hypergeometric functions", Quart. J. Math. Oxford Ser. **14**, p. 55-78.

[4] A. Erdelyi (1954), "Tables of integral Transforms", McGraw-Hill Book Co., N. Y. —Toronto—London.

[5] A. Erdelyi (1953), "Higher Transcendental Functions", Vol. I, II, Mc Graw-Hill Book Co., N. Y.—Toronto—London.

[6] E. D. Rainville (1965), "Special functions", Macmillan.

[7] L. J. Slater (1960), "Confluent hypergeometric functions", Cambridge.

[8] H. M. Srivastava (1969), "Infinite series of certain products involving Appell's double hypergeometric functions", Glasnik Mat. Ser. III, 4 (24), p. 67-73.

Jñānabha, Sect. A, Vol. 2, July 1972.

GENERALIZED HERMITE POLYNOMIALS

By

R. C. SINGH CHANDEL

Department of Mathematics, D. V. (Post-graduate) College, Orai, U. P., India

(*Received on 6th July, 1971*)

1. INTRODUCTION.

Recently, Lahiri [3, 4, 5] studied the polynomials which are defined by the generating function*

$$e^{vxt-t^m} = \sum_{n=0}^{\infty} \frac{H_{n,m,v}(x)t^n}{n!}, \tag{1.1}$$

where m is a positive integer.

In this paper; we shall extend some of the results due to Lahiri [4] with the help of the operator $x^2 \frac{d}{dx}$ or by the modified operator $x^k \frac{d}{dx}$, studied in [1] and [2] respectively.

2. APPLICATION OF THE OPERATOR $\Omega = x^2 \frac{d}{dx}$.

For the operator $\Omega = x^2 \frac{d}{dx}$, it is easy to show that, formally,

$$e^{t\Omega}\{f(x)\} = f\left(\frac{x}{1-xt}\right), \tag{2.1}$$

where $f(x)$ admits power series expansion.

---

*More general sequences of polynomials have already been studied in the literature. See, for instance, Srivastava [6, 7].

Now consider the result [4, (3.1)]

$$\left(\frac{d}{dx}\right)^s \quad [H_{n, m, v} (x)] = \frac{v^s \, n \, ! \, H_{n-s, m, v} (x)}{(n-s) \, !} .$$

Replacing x by $1/x$, we get

$$(-\Omega)^s \quad H_{n, m, v}(1/x) = \frac{v^s \, n \, ! \, H_{n-s, m, v}(1/x)}{(n-s) \, !} ,$$

from which we can further write

$$e^{t\Omega} \, \{H_{n, m, v}(1/x)\} = \sum_{s=0}^{\infty} \frac{v^s \, n \, ! \, (-t)^s}{(n-s) \, ! \, s \, !} H_{n-s, m, v}(1/x).$$

Applying (2.1) and replacing x by $\frac{1}{x}$, we obtain the summation formula

$$(2.2) \quad H_{n, m, v} (x+t) = \sum_{s=0}^{n} \binom{n}{s} v^s \, t^s \, H_{n-s, m, v} (x) \quad .$$

Now replacing t by x and x by 0, we obtain another result

$$(2.3) \quad H_{n, m, v} (x) = \sum_{s=0}^{n} \binom{n}{s} v^s \, x^s \, H_{n-s, m, v}(0).$$

In particular, on replacing t by $\frac{t}{v}$ in (2.2), we get

$$(2.4) \quad H_{n, m, v} \left(x + \frac{t}{v} \right) = \sum_{s=0}^{n} \binom{n}{s} t^s \, H_{n-s, m, v}(x) \quad .$$

Similarly, on replacing x by $\frac{x}{v}$ in (2.3), we derive

$$(2.5) \quad H_{n, m, v} \left(\frac{x}{v} \right) = \sum_{s=0}^{n} \binom{n}{s} x^s \, H_{n-s, m, v}(0)$$

3. APPLICATION OF THE MODIFIED OPERATOR $x^k \frac{d}{dx}$.

Recently, using the operator $x^k \frac{d}{dx}$ in our previous paper [2], we studied the polynomials $T_n^{\alpha, k}(x, r, p)$ defined by the Rodrigues' formula*

$$T_n^{\alpha, k}(x, r, p) = x^{-\alpha} e^{px^r} \left(x^k \frac{d}{dx}\right)^n \left\{ x^{\alpha} e^{-px^r} \right\} .$$

Here k is not necessarily a positive integer; it may be any real number.

For the operator $\Omega_x = x^{\alpha} \frac{d}{dx}$, we can easily prove

$$(3.1) \quad e^{t\Omega_x} \{ f(x) \} = f \left[\frac{x}{\left(1-(\alpha-1)\, x^{\alpha-1} t \right)^{1/\alpha-1}} \right],$$

where α is any real number except 1, and $f(x)$ admits power series expansion.

Now consider the corrected version of formula (9.1 of Lahiri [4] :

$$(3.2) \quad \left(x^{\alpha m+1} \frac{d}{dx} \right)^r \left[x^{-\alpha n} H_{n, m, v}(x^{\alpha}) \right] = \frac{n!(\alpha m)^r}{(n-mr)!} x^{-\alpha(n-mr)} H_{n-mr, m, v}(x^{\alpha}) .$$

From the above result, we write

$$\sum_{r=0}^{\infty} \frac{n!(\alpha m)^r}{(n-mr)!} \frac{t^r}{r!} x^{-\alpha(n-mr)} H_{n-mr, m, v}(x^{\alpha}) .$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \left(x^{\alpha m+1} \frac{d}{dx} \right)^r \left\{ x^{-\alpha n} H_{n, m, v}(x^{\alpha}) \right\}$$

$$= e^{t\left(x^{\alpha m+1} \frac{d}{dx} \right)} \left\{ x^{-\alpha n} H_{n, m, v}(x^{\alpha}) \right\}$$

---

* A similar polynomial system was studied earlier by Srivastava and Singhal [8].

$$= \left[\frac{x}{(1-am\, x^{am}t)^{1/am}}\right]^{-an} H_{n,m,v} \left[\frac{x}{(1-am\, x^{am}t)^{1/am}}\right]^{a}.$$

Now replacing t by $\dfrac{t}{amx^{am}}$, and finally x^a by x, we get

$$(3.3) \qquad (1-t)^{n/m} H_{n,m,v}\left(\frac{x}{(1-t)^{1/m}}\right) = \sum_{r=0}^{[n/m]} \frac{t^r\, n\,!}{r\,!\,(n-mr)\,!} H_{n-mr,m,v}(x),$$

which is believed to be new.

Applying the same techniques, the above result can also be obtained from [4, (8.2)]

$$(3.4) \qquad \left(\frac{x^{m+1}}{m}\frac{d}{dx}\right)^r \left[\frac{x^{-n} H_{n,m,v}(x)}{n\,!}\right] = \frac{x^{-n+rm}}{(n-rm)\,!} H_{n-rm,m,v}(x).$$

Now we recall our earlier result [2]:

$$D^n = \sum_{j=0}^{n} \binom{n}{j} T_{n-j}^{a,k}(x,r,p)\, \Omega_x^j,$$

where $D = ax^{k-1} - prx^{k+r-1} + \Omega_x$ and $\Omega_x = x^k \dfrac{d}{dx}$.

Replace k by $m+1$ in the above result, and denote

$$ax^m - prx^{m+r} + x^{m+1}\frac{d}{dx} \text{ by } \phi \text{ and } x^{m+1}\frac{d}{dx} \text{ by } \theta.$$

We thus obtain

$$\phi^s = \sum_{j=0}^{s} \binom{s}{j} T_{s-j}^{a,m+1}(x,r,p)\, \theta^j,$$

whence we have

$$\phi^{s}\left\{\frac{x^{-n}\,H_{n,\,m,\,v}(x)}{n\,!}\right\}=\sum_{j=0}^{s}\binom{s}{j}\;T^{\alpha,\,m+1}_{s-j}\,(x,r,p)$$

$$\theta^{j}\left\{\frac{x^{-n}\,H_{n,\,m,\,v}(x)}{n\,!}\right\}.$$

Now applying (3.4) on the right-hand side of the above result,

we get

$$(3.5)\qquad \phi^{s}\left\{\frac{x^{-n}\,H_{n,\,m,\,v}(x)}{n\,!}\right\}$$

$$\sum_{j=0}^{min\,(s,[n/m])}\binom{s}{j}\frac{x^{-n+jm}\,m^{j}}{(n-jm)\,!}\;H_{n-jm,\,m,\,v}(x)$$

$$T^{\alpha,\,m+1}_{s-j}\,(x,r,p),$$

which appears to be new.

In our previous paper [2], we also have a result

$$\Omega_x^{\,n}=\sum_{j=0}^{n}\binom{n}{j}\,T_{n-j}^{\,-\alpha,\,k}\,(x,r,-p)\;D^{j}\;.$$

Replying k by $m+1$, we write

$$\theta^{\,n}\left\{\frac{x^{-s}\,H_{s,\,m,\,v}(x)}{s!}\right\}=\sum_{j=0}^{n}\binom{n}{j}\,T_{n-j}^{\,-\alpha,\,m+1}\,(x,r,-p)\;\phi^{j}\left\{\frac{x^{-s}H_{,\,m,\,v}(x)}{s!}\right\}.$$

Now using (3.4), we establish the result

$$(3.6)\qquad \frac{m^{n}\,x^{-s+nm}}{(s-nm)!}\;H_{s-nm,\,m,\,v}(x)$$

$$=\sum_{j=0}^{n}\binom{n}{j}\;T^{-\alpha,\,m+1}_{n-j}\,(x,r,-p)\;\phi^{j}\left\{\frac{x^{-s}\,H_{s,\,m,\,v}(x)}{s!}\right\}$$

We can easily prove

$$(3.7) \quad \theta^{s}(u.v) = \sum_{j=0}^{s} \binom{s}{j} \theta^{s-j}(u) \quad \theta^{j}(v).$$

Thus we have

$$\theta^{s}\left\{\frac{x^{-n} H_{n, m, v}(x)}{n!} \cdot \frac{x^{-p} H_{p, m, v}(x)}{p!}\right\}$$

$$= \sum_{j=0}^{s} \binom{s}{j} \theta^{s-j} \left(\frac{x^{-n} H_{n, m, v}(x)}{n!}\right) \theta^{j} \left(\frac{x^{-p} H_{p, m, v}(x)}{p!}\right)$$

$$= \sum_{j=0}^{s} \binom{s}{j} \frac{m^{s-j} x^{-n+(s-j)m}}{(n-(s-j)m)!} H_{n-(s-j)m, m, v}(x)$$

$$\times \frac{m^{j} x^{-p+jm}}{(p-jm)!} H_{p-jm, m, v}(x) .$$

Hence we obtain

$$(3.8) \quad \theta^{s}\left(\frac{x^{-n-p}}{n!\, p!} H_{n, m, v}(x) \quad H_{p, m, v}(x)\right)$$

$$= \sum_{j=0}^{s} \binom{s}{j} \frac{m^{s} x^{-n-p+sm}}{(n-(s-j)m)!\ (p-jm)!} H_{n-(s-j)m, m, v}(x) \quad H_{p-jm, m, v}(x)$$

Remark:—Particularly for $m=v=2$, all the results of this section will be reduced to similar results for Hermite polynomials.

4. EXTENSION OF THE GENERATING FUNCTION.

From the generating relation (1.1), we write

$$\sum_{n=0}^{\infty} H_{n, m, v}(x+y) \frac{t^{n}}{n!} = e^{v(x+y)-t^{m}}$$

$$= e^{vty} \sum_{k=0}^{\infty} H_{k,m,v}(x) \frac{t^k}{k!}$$

$$= \sum_{n=0}^{\infty} \frac{(vyt)^n}{n!} \sum_{k=0}^{\infty} H_{k,m,v}(x) \frac{t^k}{k!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(vy)^{n-k}}{(n-k)!\,k!} H_{k,m,v}(x) t^n \quad .$$

Hence

$$(4.1) \quad H_{n,m,v}(x+y) = \sum_{k=0}^{n} \binom{n}{k} (vy)^{n-k} H_{k,m,v}(x) \quad .$$

Also

$$(4.2) \quad H_{n,m,v}\left(x+\frac{y}{v}\right) = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} H_{k,m,v}(x) \quad .$$

Put $x=0$ in the above result, so that

$$(4.3) \quad H_{n,m,v}\left(\frac{y}{v}\right) = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} H_{k,m,v}(0) \quad .$$

Again from (1.1), we have

$$\sum_{n=0}^{\infty} H_{n,m,(u+v)}(x) \frac{t^n}{n!} = e^{(u+v)xt - t^m}$$

$$= e^{uxt} \sum_{k=0}^{\infty} H_{k,m,v}(x) \frac{t^k}{k!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(ux)^{n-k}}{(n-k)!\,k!} H_{k,m,v}(x) t^n \quad .$$

Therefore,

$$(4.4) \quad H_{n, m, u+v}(x) = \sum_{k=0}^{n} \binom{n}{k} (ux)^{n-k} H_{k, m, v}(x) .$$

Similarly

$$(4.5) \quad H_{n, m, u+v}(x) = \sum_{k=0}^{n} \binom{n}{k} (vx)^{n-k} H_{k, m, u}(x) .$$

Replacing u by u/x in (4.4) and v by v/x in (4.5), we derive

$$(4.6) \quad H_{n, m, \frac{u}{x}+v}(x) = \sum_{k=0}^{n} \binom{n}{k} u^{n-k} H_{k, m, v}(x) ,$$

and

$$(4.7) \quad H_{n, m, u+\frac{v}{x}}(x) = \sum_{=0}^{n} \binom{n}{k} v^{n-k} H_{k, m, u}(x) .$$

Putting $u=0$ in (4.7), we further derive

$$(4.8) \quad H_{n, m, \frac{v}{x}}(x) = \sum_{k=0}^{n} \binom{n}{k} v^{n-k} H_{k, m, 0}(x) .$$

Now comparing (4.2) and (4.6), we get

$$(4.9) \quad H_{n, m, \frac{u}{x}+v}(x) = H_{n, m, v}\left[x+\frac{u}{v} \right] .$$

ACKNOWLEDGMENTS

I wish to record my deepest and sincerest feelings of gratitude to Dr. B. B. Lal, Principal, D. V. College, Orai, for the facilities that he has provided me. My sincere thanks are also due to Professor H. M. Srivastava of the University of Victoria for his suggestions regarding the paper.

REFERENCES

[1] Chandel, R. C. S., "Operational representations and hypergeometric functions of three variables" : Proc. Nat. Acad. Sci., India, **39** Sect. A (1969), 217-222.

[2] Chandel, R. C. S., "A new class of polynomials" : Indian J. Math. (To appear).

[3] Lahiri, M., "On a generalisation of Hermite Polynomials" : Proc. Amer. Math. Soc., **27** (1971), 117-121.

[4] Lahiri, M., "Some recurrence relations and the differential for the generalised Hermite polynomial" : J. Sci. Res. Banaras Hindu Univ., **22** (1969-70), 78—83.

[5] Lahiri, M., "Integrals involving generalised Hermite polynomial and its integral representations" : Ganita, **21** (1970), 69-74.

[6] Srivastava, H. M., "Finite summation formulas associated with a class of generalized hypergeometric polynomials" : J. Math. Anal. Appl., **23** (1968), 453-458.

[7] Srivastava, H. M., "On q-generating functions and certain formulas of David Zeitlin" : Illinois J. Math., **15** (1971), 64-72.

[8] Srivastava, H. M. and Singhal, J. P., "A class of polynomials defined by generalized Rodrigues' formula" : Ann. Mat. Pura Appl. Ser. IV, **90** (1971), 75-85.

J ñ ā n a b h a, Sect. A, Vol. 2, July 1972.

GRAVITATIONAL INSTABILITY OF AN INFINITELY EXTENDING VISCOUS LAYER SURROUNDED BY NON-CONDUCTING MATTER

By

K. M. SRIVASTAVA and JAIPAL

Department of Mathematics, University of Jodhpur, Jodhpur.

(*Received on 27th August, 1972*)

ABSTRACT

We have studied the magneto-gravitational instability of a rotating infinite fluid layer of finite thickness. The layer is surrounded by a rotating, infinitely extending non-conducting matter. The dispersion relation for equal kinematic viscosities has been derived and studied in various situations. The cases of short and long wave lengths have also been discussed. It is observed that the rotation has no effect on the equilibrium if the viscosity is predominant. In case of short waves also rotation plays no role.

1. INTRODUCTION

The importance of the study of gravitational and magneto-gravitational stability of fluid layers of finite thickness in the astronomical context was pointed out by Shafranow[1]. Ognesyan[2,3] studied the same in the presence of a uniform magnetic field and Chakraborty[4] extended his investigations to study the effeet of uniform rotation. Uberoi[5] considered the same problem in the presence of infinitely extending surrounding non-conducting fluid on both sides of the layers. Srivastava and Sharma[7] extended the problem of Uberoi to consider the effect of finite conductivity. The aim of the present paper is to extend our later investigations to include the effect of unform rotation of both the layer and the surrounding matter.

2. LINEARIZED EQUATIONS AND THEIR SOLUTIONS.

Consider an infinitely extending plane layer of viscous, gravitating and ideally conducting fluid mass of uniform density. The thickness of the layer is taken to be *2h*. It is supposed that the xoy plane coincides with the equilibrium middle level of the layer. The z-axis is directed upwards and is normal to the unperturbed fluid surfaces. This layer is surrounded by a non-conducting fluid of constant density ρ_0. The entire matter is incompressible.

In the static state the conducting fluid is supposed to be immersed in an external uniform magnetic field of strength $\bar{H}_0$ along the x-axis. The layer and the surrounding liquid both are assumed to be rotating about z-axis with uniform angular velocity.

The linearised hydro-magnetic equations for the conducting layer governing the small departures from equilibrium are Equations of momentum.

(2.1) $$\frac{\partial \bar{u}}{\partial t} + 2\bar{\Omega} \times \bar{v} = -\nabla\pi_1 + \frac{1}{4\pi\rho} (\bar{H}_0 \cdot \nabla) \bar{h} + \nu\nabla^2 \bar{u},$$

(2.2) $$\pi_1 = \frac{\delta p}{\rho} + \delta v + \frac{\bar{H}_0 . \bar{h}}{4\pi\rho},$$

Equation of continuity :

(2.3) $$\operatorname{div} \bar{v} = 0,$$

Maxwell's equations :

(2.4) $$\operatorname{div} \bar{h}_1 = 0,$$

(2.5) $$\frac{\delta h}{\delta t} = \operatorname{curl} (\bar{v} \times \bar{H}_0), \quad \bar{H}_0 = [H_0, 0, 0],$$

Poisson's Equation :

(2.6) $$\nabla^2 \delta V_1 = 0,$$

where δp, $\bar{v}$, $\bar{h}_1$, δV_1 denote the perturbations in velocity field (initially assumed to be zero), magnetic field and gravitational potential.

With the same notations as above the corresponding linearized equations for the non-conducting medium are,

(2.7) $$\frac{\delta v}{\delta t} + 2\bar{\Omega} \times \bar{u} = -\nabla\pi_e + \nu_0 \nabla^2 \bar{u},$$

(2.8) $\pi_e = \frac{\delta p_e}{\rho} + \delta v$,

(2.9, 10) $\operatorname{div} \bar{u} = 0$, $\operatorname{div} \bar{h}_e = 0$,

(2.11, 12) $\operatorname{curl} \bar{h}_e = 0$, $\nabla^2 \delta v_e = 0$,

where, now δp_e , $\bar{u}$, $\bar{h}_e$ and δv_e denote corresponding perturbations in the matter surrounding the layer.

To study the stability of the system, it is assumed that all the physical quantities vary as

(2.13) $F(x, y, z, t) = f(z) \exp. [(\sigma t + \overline{ik_1 x + k_2 y})]$.

Equation (2.5) can be written as

(2.14) $\bar{h} = \frac{ik_1 H_0}{\sigma} \bar{v}$.

Substituting for $\bar{h}$ from (2.14) in (2.1) and taking its divergence, we have

(2.15) $\nabla^2 \pi_i = 2 \bar{\Omega} \nabla \times \bar{v} = 2 \Omega \zeta_i$,

where ζ_i is the z component of the vorticity, given by

(2.16) $\zeta_i = ik_1 v_y - ik_2 v_x$.

The equation (2.1) can be written as

(2.17) $(A - \nu\nabla^2) \bar{v} + 2 \bar{\Omega} \times \bar{v} = -\nabla \pi_i$.

From equation (2.13), we have

(2.18) $\frac{\delta^2}{\delta x^2} = - k_1^2 , \frac{\delta^2}{\delta y^2} = - k_2^2 , \frac{\delta}{\delta t} = \sigma$.

Equation (2.17) breaks up into three component equations

(2.19) $(B - \nu D^2) v_x - 2 \Omega v_y = - ik_1 \pi_i$

(2.20) $(B - \nu D^2) v_y - 2 \Omega v_x = - ik_2 \pi_i$

(2.21) $(B-\nu D^2)\, v_z \qquad = -D\pi_1$

and the equation of continuity gives

(2.22) $ik_1 v_x + ik_2 v_y + D v_z = 0$

where $B = A + \nu\, (k_1^2 + k_2^2) = A + \nu k^2$

(2.23) $A = \left(\sigma + \dfrac{\Omega^2 A}{\sigma} \right), \; \Omega^2 A = \dfrac{uk_1^2\, H_o^2}{4\pi\rho}$

From equations (2.15, 2.16 and 2.19 — 2.22) we have

(2.24) $(D^2 - m^2)\, v_z = 1/\nu\, D\, \pi_1$

(2.25) $(D^2 - k^2)\, \pi_1 = 2\, \Omega\, \zeta_1$

(2.26) $(D^2 - m^2)\, Dv_z - \dfrac{2\Omega}{\nu}\, \zeta_1 = \dfrac{k^2}{\nu}\, \pi_1$

(2.27) $(D^2 - m^2)\, \zeta_1 - \dfrac{2\,\Omega}{\nu}\, Dv_z = 0$

(2.28) where $m^2 = B/\nu$.

From equations (2.24 — 2.27) we obtain

(2.29) $\left[(D^2 - k^2)\, (D^2 - m^2)^2 - \dfrac{4\Omega^2}{\nu^2}\, D^2 \right] \pi_1 = 0\,.$

The solution of (2.29), considering the symmetry of u_z, is written as

(2.30) $\pi_1 = C_1 \sinh pz.$

(2.30a) where p is a positive root of $(p^2 - k^2)\, (p^2 - m^2)^2 - \dfrac{4\Omega^2}{\nu^2}\, p^2 = 0.$

From equations (2.24) and (2.30) the solution for v_z is written as

(2. 1) $v_z = C_2 \cosh mz + \dfrac{p\, C_1 \cosh pz}{\nu\, (p^2 - m^2)}$,

(2.32) $\zeta_1 = \dfrac{p^2 - k^2}{2\Omega}\, \pi_1$

The solution for δv_i is written from equation (2.6)

$$\delta v_i = C_3 \sinh kz \,, \tag{2.33}$$

where the constant C_3 is determined from the conditions of continuity.

The corresponding solutions for π_e , u_z and ζ_e for the non-conducting fluid are

$$\pi_e = E_1 e^{-p'z} \;, \; z>h$$

$$\pi_e = {}^-E_1 e^{p'z} \;, \; z<h \tag{2.34}$$

$$u_z = E_2 e^{m'z} - \frac{p' E_1 e^{-p'z}}{(p'^2 - m'^2)} \;, \; z>h$$

$$u_z = E_2 e^{m'z} - \frac{p' E_1 e^{p'z}}{(p'^2 - m'^2)} \;, \; z<h \tag{2.35}$$

and

$$\zeta_e = \frac{p'^2 - k^2}{2\Omega} \pi_e \,, \tag{2.36}$$

where

$$m'^2 = \frac{\sigma}{\nu_0} + k^2 \,, \tag{2.37}$$

$$(p'^2 - k^2)(p'^2 - m'^2)^2 - \frac{4\Omega^2}{\nu_0{}^2} p'^2 = 0 \,. \tag{2.37a}$$

The solution for δv_e is

$$\delta v_e = E_3 e^{-kz} \,, \; z > h \,,$$

$$\delta v_e = - E_3 e^{+kz} \,, \; z < h \tag{2.38}$$

The solution for $\bar{h}_e$ is written from (2.10), (2.11)

$$\bar{h}_e = \nabla (Le^{-kz}) \,, \; z > h, \tag{2.39}$$

$$\bar{h}_e = \nabla (-Le^{-kz}) \,, \; z < h.$$

3. THE BOUNDARY CONDITIONS.

The perturbed interfaces between the conducting and non-conducting fluids considering the perturbations symmetrical about the mid-level of the layer are given by

(3.1) $z_1 = h + (\delta z) \exp. [\, (\sigma\, t + \overline{ik_1 x + k_2 y}\,)\,]$,

and

$z_2 = -h + (\delta z) \exp. [\, (\sigma\, t + \overline{ik_1 x + k_2 y}\,)\,]$,

where (Δz) is the amplitude of the displacement at the interfaces.

The perturbation $\delta \hat{n}_0$ in the unit normal n_0 is given by

(3.2) $\delta \hat{n}_0 = (-ik_1 \Delta z, -ik_2 \Delta z, 0) \exp. (\sigma\, t + \overline{ik_1 x + k_2 y})$.

The following conditions should be satisfied at the perturbed interfaces.

(i) gravitational potential is continuous i.e.

(3.3) $$[\, V\,] = 0$$

(ii) the normal component of the gradient of the gravitational potential is continuous, i.e.

(3.4) $\hat{n}.[\, \nabla V\,] = 0$,

(iii) the velocity should be compatible with the assumed form of the deformed interfaces

(3.5) $\hat{n}.[\, \bar{u}\,] = 0$,

(iv) the component of the magnetic field normal to the deformed interfaces must be continuous ,

(3.6) $\hat{n}.[\, \bar{B}\,] = 0$,

(v) all the perturbed quantities for the non-conducting medium must be bounded.

(vi) the tangential viscous stresses must be continuous

(3.7) $\left[\left\{ \nu\,(D^2+k^2) + \frac{\Omega^2 A}{\sigma} \right\} v_z \right] = 0$,

(vii) the normal component of the total stress tensor must also be continuous on the deformed interfaces

$$[P \{ \pi - \delta V - 2\, \nu D u_z \}] = 0 \,, \tag{3.8}$$

(viii) integrating equation (2.24) over an infinitesimal element of z at the interface, we get

$$[\nu D v_z - \pi] = 0 \tag{3.9}$$

where $\hat{n}$ denotes the unit normal vector directed in the conducting fluid, the square brackets denote the jump in the enclosed quantity upon crossing the interface from the non-conducting to the conducting fluid.

4. DISPERSION RELATION AND DISCUSSION.

The conditions (3.8) and (3.9) applied on equation (2.35) and (3.43) yield.

$$\delta V_1 = \frac{-4\pi G}{k\sigma} (\rho - \rho_0)\, e^{-kh} \,.\, (u_z)_{z=h} \sinh kz \tag{4.1}$$

and

$$\delta V_e = \frac{-4\pi G}{k\sigma} (\rho - \rho_0)\, e^{-kh} \, (u_z)_{z=h} \, e^{-kz} \,,\ z > h$$

$$\delta V_e = \frac{+4\pi G}{k\sigma} (\rho - \rho_0) e^{-kh} \, (u_z)_{z=h} \, e^{kz} \,,\ z < h \tag{4.2}$$

The boundary condition (3.45) gives the perturbation in the magnetic field in the non-conducting fluid to be zero.

The boundary conditions (3.6), (3.7), (3.8) and (3.9) yield four homogeneous equations in four unknowns. The condition for the existence of non-trivial solutions yields the dispersion relation

$$m \tanh X \Bigg[\left\{ \nu(p^2+k^2) + \frac{\Omega^2 A}{\sigma} - \nu_0(p'^2+k^2) \right\} \left\{ \frac{\rho_0}{\rho p'} (m'-p') - \frac{2m'}{p'} \right\} \tag{4.3}$$

$$- (m'+p') \left\{ \frac{\nu}{p} (p^2 - m^2) = \frac{F'}{\sigma k} - \frac{2m'^2 \nu_0}{p'} + \frac{\nu_0 \rho_0}{\rho p'} (p'^2 + m'^2) \right\} \Bigg]$$

$$+ \frac{m^2}{p} \tanh y \Bigg[\left\{ \nu(m^2+k^2) + \frac{\Omega^2 A}{\sigma} - \nu_0(p'^2+k^2) \right\} \left\{ \frac{m'}{p'} \left(\frac{p^2 - m^2}{m^2} \right) \right.$$

$$-\frac{\rho_0}{\rho p'}(m'-p)\Big\}+(m'+p')\Big\{\frac{-F'}{k\sigma}+\frac{\nu_0 p_0}{\rho p'}\overline{p'^2+m'^2}-\frac{\nu_0 m'^2}{p'\,m^2}\overline{-p^2+m^2}\Big\}\Big]$$

$$+\frac{m'}{p'}(m^2-p^2)\left[\frac{\nu_0\rho_0}{\rho}(m'+p')-\frac{F}{\sigma k}\right]=0$$

where

$$X=mh,\ y=ph,\ \alpha=kh,$$

(4.4)

$$F'=4\pi G\rho\left(1-\frac{\rho_0}{\rho}\right)\alpha\left[1-\frac{1-\rho_0/\rho}{\alpha\,(1+\coth\alpha)}\right].$$

Making $\nu=\nu_0$ equation (4.3) reduces to

$$(4.5)\quad X\tanh x\left[\left\{(y^2-z^2)+\frac{L^2}{x^2-\alpha^2}\right\}\right]\alpha\left\{\frac{\rho_0}{\rho}\ \frac{x-z}{z}-\frac{2x}{z}\right\}-(x+z)$$

$$\left\{\frac{\alpha}{y}(y^2-X^2)-\frac{F}{x^2-\alpha^2}-\frac{2x^2\alpha}{z}+\frac{\rho_0\alpha}{\rho\,z}(z^2+x^2)\right\}\Big]+\frac{X^2}{y}\tanh y\left[\left\{(X^2-z^2)\right.\right.$$

$$+\left.\frac{L^2}{x^2-\alpha^2}\right\}\alpha\left\{\frac{x}{z}\ \frac{y^2+X^2}{X^2}-\frac{\rho_0}{\rho}\ \frac{x-y}{z}\right\}+(x+z)\left\{\frac{-F}{x^2-\alpha^2}+\frac{\rho_0}{\rho}\ \frac{\alpha}{z}(x^2+z^2)\right.$$

$$\left.\left.-\frac{\alpha}{z}\ \frac{x^2}{X^2}(y^2+X^2)\right\}\right]+\frac{x}{z}(X^2-y^2)\left[\frac{\rho_0}{\rho}\alpha\,(x+z)-\frac{F}{x^2-\alpha^2}\right]=0,$$

with $x=m'h,\ z=p'h,$

$$(4.6)\quad L^2=\frac{\Omega^2 Ah^4}{\nu^2},\ F=4\pi G\rho\left(1-\frac{\rho_0}{\rho}\right)\left(\frac{h^2}{\nu}\right)^2.\alpha\left[1-\frac{1-\rho_0/\rho}{(1+\coth\alpha)}\right],$$

and

$$(4.7)\quad X^2=x^2+\frac{L^2}{x^2-\alpha^2},\ \frac{\sigma h^2}{\nu}=x^2-\alpha^2\ .$$

The dispersion eq. (4.5) is discussed for certain limiting cases

(i) k and m small (ii) k large (iii) $\nu\to\infty$.

Case (i) *k* and *m* small.

The roots of the eq. (2.30*a*) neglecting $k^2\,m^4$ are given by

(4.8) $p=\frac{1}{2}\left[(k^2+2m^2)\pm\left\{(k^2+2m^2)^2-4\left((2k^2m^2+m^4-\frac{4\Omega^2}{\nu^2}\right)\right\}^{1/2}\right]$

the dispersion eq. (4.5) reduces to

(4.9) $X^2=y^2$ i.e. $m^2=p^2$.

Eq. (4.9) with the help of eq. (4.8) gives

(4.10) $\sigma^2-\frac{4\ \Omega^2}{\nu^2k^2\ \sigma}+\Omega^2A=0$

of which both the roots are positive. Thus we find that long wave perturbations make the system unstable.

Case (ii) k **large.**

From the eq. (2.30a) we find that for short wave perturbation, the effect of rotation is negligible. Making rotation evanescent, this case reduces to the problem of our paper[8] which has fully been discussed there. Thus for short waves, the unstable modes when the density ρ_0, of the surrounding medium is less than the density ρ, of the layer are shown in figures 1, 2, 3 and 4 for $\rho_0/\rho=0$ and 0.5. It is also seen that the system is physiclly unstable for $\rho_0/\rho>1$. In the limiting case of high viscosity the system exhibits marginal stabil'ty. It must be mentioned that the dispersion relation in this case is obtained independently taking $\Omega=0$.

Case (iii) $\nu=\nu_0=\infty$ **(The case of high viscosity).**

When viscosity is paramount, we put

(4.11) $X=\alpha+\delta$, $X=\alpha+\delta'$

where δ and δ' are small quantities and y and z tend to α. Also equations (2.57) and (2.58) give

(4.12) $2\alpha\delta=\frac{\sigma h^2}{\nu}$, $2\alpha\delta'=\frac{\sigma h^2}{\nu}+\frac{L^2}{\sigma k^2/\nu}$

(4.13) $\frac{\sigma h^2}{\nu}=-\frac{2}{3}\frac{F}{\alpha^2}$

which implies that σ is negative and first order. The perturbations are purely damped in this case. We also observe that rotation has no effect in this case.

REFERENCES

[1] V. S. Shafranov, Ann. D. Astrophys., (1960), **23**, 979.

[2] R. S. Oganesyan, Soviet Astronomy, (1961) a, **4**, 434.

[3] R. S. Oganesyan, Soviet Astronomy, (1961) b, **4**, 634.

[4] B. B. Chakraborty, Symp. Magnetohyd., (1963).

[5] C. Uberoi, J. I. I. Sc. (India), (1964), **46**, 1, 11.

[6] M. Kruskal and M. Schwarzschild, Proc. Roy. Soc., (1954) A, 223, 348.

[7] K. M. Srivastava and R. C. Sharma, Acta Astronomica (1969) **19**, 291.

J ñ ā n a b h a, Sect. A, Vol. 2, July 1972.

CONTIGUOUS RELATIONS AND RELATED FORMULAS FOR THE H-FUNCTION OF FOX

By

R. G. Buschman†

(*Received on 27th October, 1972*)

Abstract. The set of contiguous relations for the H-function of Fox are obtained by a unified treatment in a direct and simple manner, the connections among these formulas are discussed, and the connections with some scattered results in the literature are given. Several finite series involving H-functions can be generated from each of these contiguous relations; some examples are included for illustration.

1. Introduction. The H-function was introduced by C. Fox [5] and it is usually defined in terms of the contour integral

$$H^{m,n}_{p,q}\left[z \;\middle|\; \begin{matrix}\{(a_i, \alpha_i)\}\\ \{(b_i, \beta_i)\}\end{matrix}\right]=$$

$$=\frac{1}{2\pi i}\int_C \frac{\prod_{i=1}^{n}\Gamma(1-a_i+\alpha_i s)\prod_{i=1}^{m}\Gamma(b_i-\beta_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i-\alpha_i s)\prod_{i=m+1}^{q}\Gamma(1-b_i+\beta_i s)}\, z^s\, ds,$$

in which we use the notations $\{(a_i, \alpha_i)\}$ and $\{(b_i, \beta_i)\}$ to denote, respectively, the sets of parameters $(a_1, \alpha_1), \ldots, (a_p, \alpha_p)$ and $(b_1, \beta_1), \ldots, (b_q, \beta_q)$. The description of the path of integration and the conditions which the parameters must satisfy are described in [5]. An extension of the definition is given in [14]. If the second parameter of every pair is equal to 1, then the H-function reduces to the G-function of Meijer, a summary of the properties of which appear in the books [4] and [11].

†*Department of Mathematics, University of Victoria, Victoria, B. C., Canada; on sabbatical leave from the University of Wyoming, Laramie, WY 82070, U. S. A.*

A number of recurrence formulas for the H-function have appeared, scattered in recent literature. Some of these can be given the name of contiguous relations; where they involve a unit shift in any of the first parameters of the pairs, the terminology thus corresponds to that used for the Gauss hypergeometric function. Various methods have been used in order to derive such formulas; for example, see [1], [2], [3], [6], [7], [8], [9], [10], [12], and [15]. Often the method used involves the evaluation of a complicated integral of a product of the H-function and some other function and the use of an identity for the second function. Many of the results contain restrictive conditions on the second parameters of the pairs. In section 2 we collect the 30 contiguous relations. In a unified treatment they are obtained by simple direct derivations and without the severe restrictions on the second parameters. Further, the connections with certain other formulas which have appeared in the literature are discussed.

Finite series which involve the H-function also have appeared in a number of places, for example, in [3], [13], and [15]. Similar methods were used and often with severe restrictions placed upon the second parameters of the pairs. In section 3 we discuss the manner in which various simple forms of finite series can be obtained directly from our contiguous relations and how the complicated generalizations of some known series can also be obtained, again with fewer restrictions on the parameters.

Inasmuch as certain determinants appear throughout this work, we shall introduce simplifying notations as, for example

$$d(b_1, a_p-k)=\det\begin{bmatrix} b_1 & a_p-k \\ \beta_1 & \alpha_p \end{bmatrix},$$

in which we display the first row of the determinant by our notation. The second row of the determinant is always to be filled in with the appropriate α,s and β,s in order to correspond to those a,s and b,s of the first row. We assume that none of these determinants equals zero, which is consistent with the restrictions in the definition of the H-function. To further simplify the notational problems we merely write H for the function as given in the defining relation; then, for example, we use $H[b_1+1]$ to denote the contiguous function in which b_1 is replaced by b_1+1, but with all other parameters left unchanged, where further it is to be understood that if both H and $H[b_1+1]$ appear, then $m \geqq 1$.

2. Contiguous relations. We first note from the definition that the replacement of b_1 by b_1+1 and the application of the recurrence formula $\Gamma(z+1)=z\Gamma(z)$ is equivalent to the introduction of the multipiler $b_1-\beta_1 s$ into the contour integral format for H. Similarlv, the replacement of b_q by b_q+1 introduces $-b_q+\beta_q s$, of a_1 by a_1-1 introduces $1-a_1+\alpha_1 s$, and of a_p by a_p-1 introduces $a_p-1-\alpha_p s$. Consequently, we can simply form a 3-term recurrence involving undetermined coefficients.

$$AH[b_1+1]+BH[a_p-1]=CH$$

and then require that

$$A(b_1-\beta_1 s)+B(a_p-1-\alpha_p s)=C$$

be an identity in s. Hence A, B and C can be evaluated in order to obtain the contiguous relation numbered (1) in our list. In a similar manner all of the other listed contiguous relations can be obtained.

(1) $\alpha_p H[b_1+1]-\beta_1 H[a_p-1]=d(b_1, a_p-1)H$

(2) $\alpha_p H[a_1-1]+\alpha_1 H[a_p-1]=-d(a_1-1, a_p-1)H$

(3) $\beta_q H[a_1-1]-\alpha_1 H[b_q+1]=-d(a_1-1, b_q)H$

(4) $\beta_q H[b_1+1]+\beta_1 H[b_q+1]=d(b_1, b_q)H$

(5) $\alpha_1 H[b_1+1]+\beta_1 H[a_1-1]=d(b_1, a_1-1)H$

(6) $\beta_q H[a_p-1]+\alpha_p H[b_q+1]=d(a_p-1, b_q)H$

(7) $\beta_2 H[b_1+1]-\beta_1 H[b_2+1]=d(b_1, b_2)H.$

(8) $\alpha_2 H[a_1-1]-\alpha_1 H[a_2-1]=-d(a_1-1, a_2-1)H$

(9) $\alpha_{p-1} H[a_p-1]-\alpha_p H[a_{p-1}-1]=d(a_p-1, a_{p-1}-1)H$

(10) $\beta_{q-1} H[b_q+1]-\beta_q H[b_{q-1}+1]=-d(b_q, b_{q-1})H$

(11) $d(a_p-1, b_q) H[a_1-1]-d(b_q, a_1-1) H[a_p-1]=-d(a_1-1, a_p-1) H[b_q+1]$

(12) $d(a_p-1, b_q) H[b_1+1]+d(b_q, b_1) H[a_p-1]=d(b_1, a_p-1) H[b_q+1]$

(13) $d(a_1-1, b_q) H[b_1+1]-d(b_q, b_1) H[a_1-1]=d(b_1, a_1-1) H[b_q+1]$

(14) $d(a_1-1, a_p-1)H[b_1+1]-d(a_p-1, b_1)H[a_1-1] = -d(b_1, a_1-1)$
$H[a_p-1]$

(15) $d(b_2, b_3)H[b_1+1]+d(b_3, b_1)H[b_2+1] = -d(b_1, b_2)H[b_3+1]$

(16) $d(a_2-1, a_3-1)H[a_1-1]+d(a_3-1, a_2-1)H[a_2-1]=$
$-d(a_1-1, a_2-1)H[a_3-1]$

(17) $d(a_{p-1}-1, a_{p-2}-1)H[a_p-1]+d(a_{p-2}-1, a_p-1)H[a_{p-1}-1]$
$=-d(a_p-1, a_{p-1}-1)H[a_{p-2}-1]$

(18) $d(b_{q-1}, b_{q-2})H[b_q+1]+d(b_{q-2}, b_q)H[b_{q-1}+1]=-d(b_q, b_{q-1})$
$H[b_{q-2}+1]$

(19) $d(a_p-1, b_1)H[a_{p-1}-1]+d(b_1, a_{p-1}-1)H[a_p-1]$
$=-d(a_{p-1}-1, a_p-1)H[b_1+1]$

(20) $d(b_q, a_1-1)H[b_{q-1}+1]+d(a_1-1, b_{q-1})H[b_q+1]=-d(b_{q-1}, b_q)H[a_1-1]$

(21) $d(a_2-1, a_p-1)H[a_1-1]+d(a_p-1, a_1-1)H[a_2-1]=d(a_1-1, a_2-1)$
$H[a_p-1]$

(22) $d(b_2, b_q)H[b_1+1]+d(b_q, b_1)H[b_2+1]=d(b_1, b_2)H[b_q+1]$

(23) $d(a_{p-1}-1, a_1-1)H[a_p-1] + d(a_1-1, a_p-1)H[a_{p-1}-1] =$
$d(a_p-1, a_{p-1}-1)H[a_1-1]$

(24) $d(b_{q-1}, b_1)H[b_q+1]+d(b_1, b_q)H[b_{q-1}+1]=d(b_q, b_{q-1})H[b_1+1]$

(25) $d(a_2-1, b_q)H[a_1-1]+d(b_q, a_1-1)H[a_2-1]=-d(a_1-1, a_2-1)H[b_q+1]$

(26) $d(b_2, a_p-1)H[b_1+1]+d(a_p-1, b_1)H[b_2+1]=-d(b_1, b_2)H[a_p-1]$

(27) $d(a_2-1, b_1)H[a_1-1]+d(b_1, a_1-1)H[a_2-1]=d(a_1-1, a_2-1)H[b_1+1]$

(28) $d(b_2, a_1-1)H[b_1+1]+d(a_1-1, b_1)H[b_2+1]=d(b_1, b_2)H[a_1-1]$

(29) $d(a_{p-1}-1, b_q)H[a_p-1]+d(b_q, a_p-1)H[a_{p-1}-1]=d(a_p-1, a_{p-1}-1)$
$H[b_q+1]$

(30) $d(b_{q-1}, a_p-1)H[b_q+1]+d(a_p-1, b_q)H[b_{q-1}+1]=d(b_q, b_{q-1})H[a_p-1]$

Alternatively, it should be noted that many of these relations are closely connected. The H-function satisfies the identity

$$H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix}\{(a_i, \alpha_i)\} \\ \{(b_i, \beta_i)\}\end{matrix}\right] = H^{n,m}_{q,p}\left[z^{-1} \,\middle|\, \begin{matrix}\{(1-b_i, \beta_i)\} \\ \{(1-a_i, \alpha_i)\}\end{matrix}\right]$$

so that, for example, (3) and (4) can be obtained, respectively, from (1) and (2) and beyond formula (6) each odd numbered formula leads in the same manner to the even numbered successor. Actually we could consider formulas (1) and (2) as basic and then derive all of the others from the two of them along with this transformation formula. For example, (5) can be obtained by combining (1) and (2) and similarly (6) from (2) and (3). A double application of (1) leads to (7) and to (9), (11) comes from (2) and (3), (13) from (3) and (4), (15) from a triple application of (7), and similarly (17) from (9). Formula (19) can be obtained by a double application of (1) and similarly (21), (23), (25), (27), and (29) from (2), (2), (3), (5), and (6), respectively.

If we set all of the α's and β's equal to 1 we obtain the special cases for the G-function of Meijer, where now, for example, $d\ (b_1, a_p-k)$ is simply the difference, $b_1-(a_p-k)$. Since MacRoberts' E-function and the generalized hypergeometric series ${}_pF_q$ are special cases of the G-function, some contiguous relations for these functions now also follow.

Some of the formulas have appeared in various format and are scattered in the literature; some examples which are known to us follow. Formulas (1)-(6) were earlier derived by P. Anandani [2] from differentiation formulas and without unnecessary restrictions on the α's and β's; cases of (1)—(4) and (6), but with the restriction of equality of those α's and β's related to the a's and b's of the contiguity of the particular formula, were also given by P. Anandani [3] as special cases of certain finite sums. Formulas related to (5) and (6) were obtained from differentiation formulas by B. M. Agrawal [1]. Formula (2) has been derived using integrals involving the identities connecting generalized Bessel functions by Aruna Srivastava and K. C. Gupta [15]; cases which follow directly from formulas (3), (8) and (25) were given earlier by K. C. Gupta [7]. U. C. Jain [10] gives a special case of (28) with $\alpha_1 = \beta_1 = \beta_2$; P.C. Golas [6] obtains the special case $\alpha_1=\alpha_p=\beta_1=\beta_2$ of formula (1) from integrals and identities involving the Gauss hypergeometric function. A result which is a simple combination of (5) and (14), but with the restriction $\alpha_1=\alpha_p=\beta_1$ was obtained from integrals and identities involving Laguerre polynomials by S. L Mathur [12]. Other recurrences of four and more terms which have appeared can be obtained from combinations of our contiguous relations, such as those in [6] and [12].

Relations analogous to those for the Gauss hypergeometric function in which the coefficients are polynomials of degree one in z are not available for the general H-function, since from

$$zH^{m,\,n}_{p,\,q}\left[z \;\middle|\; \begin{matrix}\{(a_i,\,\alpha_i)\}\\ \{(b_i,\,\beta_i)\}\end{matrix}\right] = H^{m,\,n}_{p,\,q}\left[z \;\middle|\; \begin{matrix}\{(a_i+\alpha_i,\,\alpha_i)\}\\ \{(b_i+\beta_i,\,\beta_i)\}\end{matrix}\right]$$

it is seen that contiguous type functions appear only for the special case of the G-function.

3. **Finite Series.** Certain finite series can be obtained from the contiguous relations by the formation of collapsing series. We illustrate for the case of formula (1), the others can be obtained, although beyond formula (10) the coefficients become quite messy in comparison with their first formulas. If we pair the terms which involve a_p and first write

$$\beta_1\, H\,[b_1+k-1,\, a_p-1] = a_p\, H\,[b_1+k,\, a_p] - d\,(b_1+k-1,\, a_p-1)\, H\,[b_1+k-1,\, a_p]$$

and then note that

$$d\,(b_1+k-1,\, a_p-1) = a_p\, \Gamma\,(b_1+k-(a_p-1)\;\beta_1/a_p)/\Gamma\,(b_1+k-1-(a_p-1)\;\beta_1/a_p)$$

we can sum on k and collapse the resulting series on the right. Consequently, we obtain

$$\text{(1a)}\quad \frac{\beta_1}{a_p}\,\Sigma^n_{k=1}\frac{H\,[b_1+k-1,\, a_p-1]}{\Gamma\,(b_1+k-(a_p-1)\;\beta_1/a_p)} = \frac{H\,[b_1+n,\, a_p]}{\Gamma\,(b_1+n-(a_p-1)\;\beta_1/a_p)} - \frac{H\,[b_1,\, a_p]}{\Gamma\,(b_1-(a_p-1)\;\beta_1/a_p)}.$$

If we similarly begin with the form

$$a_p\, H\,[b_1+1,\, a_p-k+1] = \beta_1\, H\,[b_1,\, a_p-k] - d\,(b_1,\, a_p-k)\, H\,[b_1,\, a_p-k+1]$$

(here it seems more convenient to decrease the indices, since one of the a's is involved), then a similar formula can be derived,

$$\text{(1b)}\quad \frac{a_p}{\beta_1}\,\Sigma^n_{k=1}\frac{(-1)^k\, H\,[b_1+1,\, a_p-k+1]}{\Gamma\,(b_1\;a_p/\beta_1-(a_p-1)+k)} = \frac{(-1)^n\, H\,[b_1,\, a_p-n]}{\Gamma\,(b_1\;a_p/\beta_1-(a_p-1)+n)} - \frac{H\,[b_1,\, a_p]}{\Gamma(b_1\;a_p/\beta_1-(a_p-1))}.$$

The third pairing of terms in the form

$$d\,(b_1+k,\, a_p+k-1)\;H[b_1+k,\, a_p+k] = \beta_1\;H[b_1+k,\, a_p+k-1] - a_p\, H\,[b_1+k-1,\, a_p+k]$$

leads to a series in which both parameters are involved in the summation,

$$(1c)\quad \Sigma_{k=1}^{n} \alpha_p^{k-1} \beta_1^{n-k} d(b_1+k, a_p+k-1)\ H[b_1+k, a_p+k]$$
$$= \alpha_p^n H[b_1+n+1, a_p+n] - \beta_1^n H[b_1+1, a_p].$$

P. N. Rathie [13] has obtained a formula of this third type for the G-function which would be generalized by starting with our formula (2).

The summation formulas given by P. Anandani [3] are not included in these three types. They can be obtained, and without the restrictions on the second parameters of the pairs, but the coefficients are extremely messy and no simplifying notation is obvious to us at this time. In order to do this we consider formula (1) written in the format

$$d(b_1, a_p-1)H[b_1, a_p-1]=\alpha_p H[b_1+1, a_p]-\beta_1 H[b_1,a_p-1] \quad .$$

If we now iterate by expanding each term on the right by use of this same relation, we obtain

$$d(b_1+1, a_p-1)d(b_1, a_p-1)d(b_1, a_p-2)H[b_1, a_p]=$$

$$=\alpha_p^2 d(b_1, a_p-2)H[b_1+2, a_p]-$$

$$-\alpha_p\beta_1\Big(d(b_1+1, a_p-1)+d(b_1, a_p-2)\Big)H[b_1+1, a_p-1]+$$

$$+\ \beta_1^2 d(b_1+1, a_p-1)H[b_1, a_p-2] \ .$$

Further iterations produce expressions of the form

$$(1d)\qquad A_0 H[b_1, a_p]= \Sigma_{k=0}^{n} A_k H[b_1+n-k, a_p-k]$$

in which the A's involve sums and products of the determinants. For the special case in which $\beta_1=\alpha_p$ the coefficients are greatly simplified and we can write

$$\Big(\Gamma(b_1-a_p+n)/\Gamma(b_1-a_p+1)\Big)H[b_1, a_p]=$$
$$= \Sigma_{k=0}^{n}(-1)^k \binom{n}{k} H[b_1+n-k, a_p-k]$$

which is thus of the same type as certain of the summations in [3].

Related finite series for the G-function and in suitable cases for the E and ${}_pF_q$ functions can now be obtained by specializing the parameters.

In various papers recurrences have appeared in which the indices m,n,p,q have not been the same throughout; we have here intentionally omitted the discussion of such formulas.

REFERENCES

[1] B. M. Agarwal, On generalized Meijer 'H' functions satisfying the Truesdell F-equations, Proc. Nat. Acad. Sci. India Sect. A 38 (1968), 259-264.

[2] P. Anandani, On some recurrence formulae for the H-function, Ann. Polon. Math. 21 (1969), 113-117.

[3] P. Anandni, On finite summation, recurrence relations, and identities of H-functions, Ann. Polon. Math. 21 (1969), 125—137.

[4] A. Erdelyi, et al. Higher Transcendental Functions, Vol. I. McGraw-Hill, New York, 1953.

[5] Charles Fox, The G and H functions as symmetric Fourier kernels, Trans. Amer. Math. Soc. 98 (1961), 395—429.

[6] P. C. Golas, On a generalized Stieltjes transform, Proc. Nat. Acad. Sci. India Sect. A 39 (1969), 42—48.

[7] K. C. Gupta, On the H-function, Ann. Soc. Sci. Bruxelles, Ser. I 79 (1965), 97—106.

[8] K. C. Gupta and U. C. Jain, The H-function II, Proc. Nat. Acad. Sci. India Sect. A 36 (1966), 594—609.

[9] K. C. Gupta and U. C Jain, On the derivative of the H-function, Proc. Nat. Acad. Sci. India Sect. A 38 (1968), 189—192.

[10] U. C. Jain, Certain recurrence relations for the H-function, Proc. Nat. Inst. Sci. India Sect A 33 (1967), 19—24.

[11] Yudell L. Luke, The Special Functions and their Approximations, Vol.I, Academic Press, New York, 1969.

[12] S. L. Mathur, Certain recurrences for the H-function, Math. Education 4 (1970), 132—136.

[13] P. N. Rathie, Some finite and infinite series for F_C, F_4, ψ_2 and G-functions, Math. Nachr. 35 (1967), 125—136.

[14] P. Skibinski, Some expansion theorems for the H-function, Ann. Polon. Math. 23 (1970/71), 125—138.

[15] Aruna Srivastava and K. C. Gupta, On certain recurrence relations, Math. Nachr. 46 (1970), 13—23.

Jñānabha, Sect. A, Vol. 2, July 1972.

GENERALIZED LAGUERRE POLYNOMIALS AND THE POLYNOMIALS RELATED TO THEM, III

By

R. C. SINGH CHANDEL

Department of Mathematics, D. V. (Post-graduate) College, Orai, U. P., India

(Received on 7th July, 1970)

1. INTRODUCTION.

In our recent papers [1, 2, 3], we studied the generalized Laguerre polynomials which are defined by

$$(1.1)\quad (1-t)^{-c}\ \exp.\left\{-\left(\frac{r}{1-t}\right)^{r} x\, t\right\}=\sum_{n=0}^{\infty} f_{n}^{c}(x,r)\, t^{n},$$

and also their related polynomials defined by

$$(1.2)\quad \sum_{k=0}^{n} A_{k}^{c}(x,r) f_{n-k}^{c+k}=0 \quad n\geqslant 1,$$

$$(1.3)\quad A_{0}^{c}(x,r)=1,$$

$$(1.4)\quad R_{n}^{(c,\,b)}(y,x,r)=\sum_{k=0}^{n} f_{k}^{c}(y,r)\, A_{n-k}^{b+k}(x,r),$$

and

$$(1.5)\quad \mathcal{N}_{n}^{(c,\,b)}(x,y,r)=\sum_{k=0}^{n} A_{k}^{c}(x,r) f_{n-k}^{b+k}(y,r).$$

For a ready reference here, we write the following generating relations [1, 2], which will be used in our investigations :

$$(1.6)\quad (1+t)^{c-1} \exp. \{-x\, t\, r^r (1+t)^{r-1}\} = \sum_{n=0}^{\infty} f_n^{c-n}(x, r)\, t^n,$$

$$(1.7)\quad (1+t)^{-c} \exp. \{r^r (1+t)^{r-1} x\, t\} = \sum_{n=0}^{\infty} A_n^c(x, r)\, t^n,$$

$$(1.8)\quad (1-t)^{c-1} \exp. \{r^r x\, t\, (1-t)^{-r}\} = \sum_{n=0}^{\infty} A_n^{c-n}(x, r)\, t^n.$$

In this paper, we discuss the nature of $f_n^c(x, r)$ and $A_n^c(x, r)$, and establish some recurrence relations for them. We also obtain Rodrigues' formulae and some summation formulae for the above polynomials. Some of the summation formulae are the generalizations of [3, (4.4), (4.5)]. Finally, we derive two new identities associated with these polynomial systems.

2. NATURE OF $f_n^c(x, r)$ and $A_n^c(x, r)$.

In this section, we notice that the polonomials $f_n^c(x, r)$ are non-orthogonal except when $r=1$, and the polynomials $A_n^c(x, r)$ are non-orthogonal for all values of r.

Again by definitions (1.1) and (1.7), it is clear that $f_n^c(x, r)$ and $A_n^c(x, r)$ are both of Sheffer A-type zero [9]. All orthogonal polynomials which are of Sheffer A-type zero have been determined by several authors e. g. Meixner and Sheffer etc. Our polynomials $f_n^c(x, r)$ are not among them except for $r=1$, and similarly $A_n^c(x, r)$ are also not included in them for all values of r.

Using the above property of these polynomials, we establish the following recurrence relations which appear to be new :

$$(2.1)\quad \sum_{k=0}^{n-1} \left(c - \frac{x\,(k+1)\, r^r\, (r)_k}{k\,!}\right) f_{n-1-k}^c(x, r) = n f_n^c(x, r) \qquad n \geqslant 1,$$

$$(2.2)\quad x\frac{d}{dx}\, f_n^c(x, r) - n f_n^c(x, r) = \sum_{k=0}^{n-1} \left(\frac{x\, r^r\, (r)_{n-1-k}}{(n-k-2)\,!} - c \right) f_k^c(x, r) \quad n \geqslant 1,$$

$$(2.3)\quad \sum_{k=0}^{n-1} \left\{ (-1)^{k+1}\, c + r^r\, (k+1) \binom{r-1}{k}\, x \right\} A_{n-1-k}^c(x, r) = n A_n^c(x, r) \quad n \geqslant 1,$$

$$(2.4)\quad \sum_{k=0}^{\min(n-1,\, r-1)} r^r \binom{r-1}{k} A_{n-1-k}^c(x, r) = \frac{d}{dx} A_n^c(x, r) \quad n \geq 1,$$

and

$$(2.5)\quad x\frac{d}{dx} A_n^c(x, r) - n\, A_n^c(x, r) = -\sum_{k=0}^{n-1} \left\{ (-1)^{k+1}\, c + \binom{r-1}{k} r^r k\, x \right\} A_{n-1-k}^c(x, r) \qquad n \geqslant 1$$

3. SOME OTHER RECURRENCE RELATIONS.

Starting from the generating relation (1.1), we also obtain the following recurrence relations :

$$(3.1)\quad (n+1) f_{n+1}^c(x, r) + [r^r x - c - (r+1)\, n] f_n^c(x, r) + \left[\binom{r+1}{2} (n-1) - r^r x + r^{r+1} x + c\, r \right] \times f_{n-1}^c(x, r)$$

$$= \sum_{j=2}^{\min(r,\, n)} (-1)^j \left[\binom{r+1}{j+1} (n-j) + c \binom{r}{j} \right] f_{n-j}^c(x, r),$$

$$(3.2)\quad \left[1 - c - rn + \binom{r+1}{2} (n-1) + r^{r+1} x + c\, r \right] \frac{d}{dx} f_n^c(x, r) - (n+1)\, r^r f_n^{c+r-1}(x, r)$$

$$+ r^r \left[\binom{r+1}{2} (n-1) - r^r x + r^{r+1} x + c\, r \right] f_{n-1}^{c+r-1}(x, r) + r^r f_n^c(x, r)$$

$$+ r^r (r-1)\, f_{n-1}^c(x, r) = \sum_{j=2}^{\min(r,\, n)} (-1)^j \left[\binom{r+1}{j+1} + c \binom{r}{j} \right] \frac{d}{dx} f_{n-j}^c(x, r),$$

and

$$(3.3)\quad (1-c-rn+r^r x)\frac{d}{dx} f_n^c(x,r)+\left[\binom{r+1}{2}(n-1)-r^r x+r^{r+1}x+cr\right]\frac{d}{dx} f_{n-1}^c(x,r)$$

$$+r^r\left[f_n^c(x,r)+(r-1)\, f_{n-1}^c(x,r)-(n+1)\, f_n^{c+r-1}(x,r)\right]$$

$$=\sum_{j=2}^{\min(r,n)}(-1)^j\left[\binom{r+1}{j+1}(n-j)+c\binom{r}{j}\right]\frac{d}{dx} f_{n-j}^c(x,r)\quad .$$

The relations (3.1), (3.2) and (3.3) are generalizations of [5, p. 297, (7), p. 299, (12), p. 300, (13)] respectively, and they reduce to them for $r=1$, $c=a+1$.

Now making an appeal to the relationship [1, (3.8)]

$$A_k^c(x,r)=f_k^{-(c+k-1)}(-x,r),$$

we derive the following results from (3.1), (3.2) and (3.3) respectively :

$$(3.4)\quad (n+1)\, A_{n+1}^{c-1}(x,r)+\left[-r^r x+c-1-rn\right]A_n^c(x,r)$$

$$+\left[\binom{r+1}{2}(n-1)+r^r x-r^{r+1}x-(c+n-1)\,r\right]A_{n-1}^{c+1}(x,r)$$

$$=\sum_{j=2}^{\min(r,\,n)}(-1)^j\left[\binom{r+1}{j+1}(n-j)-(c+n-1)\binom{r}{j}\right]A_{n-j}^{c+j}(x,r)\ ,$$

$$(3.5)\quad \left[(c+n)(1-r)-rn+\binom{r+1}{2}(n-1)-r^{r+1}x+r\right]\frac{d}{dx} A_n^c(x,r)$$

$$+(1+n)r^r A_n^{c-r+1}(x,r)$$

$$-r^r\left[\binom{r+1}{2}(n-1)+r^r x-r^{r+1}x-(c+n-1)r\right]A_{n-1}^{c-r+2}(x,r)$$

$$-r^r\left[A_n^c(x,r)+(r-1)\ A_{n-1}^{c+1}(x,r)\right]$$

$$= \sum_{j=2}^{\min(r,n)} (-1)^j \left[\binom{r+1}{j+1} (n-j) - (c+n-1) \binom{r}{j} \right] \frac{d}{dx} A_{n-j}^{c+j} (x,r)$$

and

$$(3.6) \quad [c+n(1-r)-r^r x] \frac{d}{dx} A_n^c (x,r) + \left[\binom{r+1}{2} (n-1) + r^r x - r^{r+1} x - (c+n-1) r \right]$$

$$\frac{d}{dx} A_n^{c+1} (x,r) + r^r \left[(1+n) A_n^{c-r+1} (x,r) - A_n^c (x,r) - (r-1) A_{n-1}^{c+1} (x,r) \right]$$

$$= \sum_{j=2}^{\min(r,n)} (-1)^j \left[\binom{r+1}{j+1} (n-j) - (c+n-1) \binom{r}{j} \right] \frac{d}{dx} A_{n-j}^{c+j} (x,r) .$$

Particularly for $r=1$, $c=\alpha+1$, the relation (3.4) reduces to [6, (2.10)].

4. SOME SUMMATION FORMULAS.

Starting with the generating relations (1.1), (1.6), (1.7) and (1.8), we derive the following summation formulas respectively :

$$(4.1) \quad f_n^{c_1+c_2+\ldots+c_m} (x_1+x_2+\ldots+x_m, r)$$

$$= \sum_{n_1+n_2+\ldots+n_m=n} \prod_{j=1}^{m} f_{n_j}^{c_j} (x_j, r),$$

$$(4.2) \quad f_n^{c_1+c_2+\ldots+c_m-n+1} (x_1+x_2+\ldots+x_m, r)$$

$$= \sum_{n_1+n_2+\ldots+n_m=n} \prod_{j=1}^{m} f_{n_j}^{c_j-n_j+1} (x_j, r),$$

$$(4.3) \quad A_n^{c_1+c_2+\ldots+c_m} (x_1+x_2+\ldots+x_m, r)$$

$$= \sum_{n_1+n_2+\ldots+n_m=n} \prod_{j=1}^{m} A_{n_j}^{c_j} (x_j, r),$$

and

$$(4.4)\quad A_n^{c_1+c_2+\ldots+c_m-n+1}(x_1+x_2+\ldots+x_m, r)$$

$$= \sum_{n_1+n_2+\ldots+n_m=n} \prod_{j=1}^{m} A_{n_j}^{c_j-n_j+1}(x_j, r) .$$

The relation (4.1) is a generalization of [3, (4.4). Indeed for $x_1=x_2=\ldots=x_m=kx$; $c_1=c_2=\ldots=c_m=kr+1$ and $m=c$, the relation (4.1) reduces to [3, (4.4)]. Similarly, (4.3) is a generalization of [3, (4.5)] and it can be obtained from (4.3) by taking $x_1=x_2=\ldots=x_m=kx$, $c_1=c_2=\ldots=c_m=s$ and $m=c$.

Again, for $x_1=x_2=\ldots=x_m=x$, the relation (4.1) reduces to

$$(4.5)\quad f_n^{c_1+c_2+\ldots+c_m}(mx, r) = \sum_{n_1+n_2+\ldots+n_m=n} \prod_{j=1}^{m} f_{n_j}^{c_j}(x, r) ,$$

which, on equating the coefficients of x^n on both sides, gives us the identity

$$(4.6)\quad \frac{m^n}{n!} = \sum_{n_1+n_2+\ldots+n_m=n} \prod_{j=1}^{m} \frac{1}{(n_j)!} .$$

The relations (4.5) and (4.6) further suggest that the relation [8, (3.3)]

$$\frac{d^{\lambda k}}{dx^{\lambda k}} \left\{ C_{n+k}^{\lambda}(x) \right\} = 2^{\lambda k} (\lambda)_{\lambda k} \sum_{i_1+i_2+\ldots+i_{k+1}=n} \prod_{j=1}^{k+1} C_{i_j}^{\lambda}(x)$$

will give us another identity :

$$(4.7)\quad \frac{(\lambda k)_n}{n!} = \sum_{i_1+i_2+\ldots\ldots i_k=n} \prod_{j=1}^{k} \frac{(\lambda)_{i_j}}{(i_j)!} .$$

Now making an application to the definitions of the polynomials, we can also derive the following results :

$$(4.8)\quad R_n^{(c,b)}(x, y, r) = R_n^{(-b,-c)}(x, y, r) = R_n^{(c,b)}(-y, -x, r) ,$$

$$(4.9)\quad N_n^{(c,\,b)}(x,y,r)=N_n^{(-b,\,-c)}(x,y,r)=R_n^{(c,\,b)}(-y,-x,r)$$

$$(4.10)\quad R_n^{(c+c',\,b+b')}(x+x',y+y',r)\sum_{m=0}^{n}R_m^{(c,\,b)}(x,y,r)\,R_{n-m}^{(c',\,b')}(x',y',r)$$

and

$$(4.11)\quad N_n^{(c+c',\,b+b')}(x+x',y+y',r)$$

$$=\sum_{m=0}^{n}N_m^{(c,\,b)}(x,y,r)\,N_{n-m}^{(c',b')}(x',y',r).$$

The last two results may further be generalized in the following way :

$$(4.12)\quad R_n^{(c_1+c_2+\ldots+c_m,\;b_1+b_2+\ldots+b_m)}(y_1+y_2+\ldots+y_m,\;x_1+x_2+\ldots+x_m,\;r)$$

$$=\sum_{k_1+k_2+\ldots+k_m=n}\prod_{j=1}^{m}R_{k_j}^{(c_j,\,b_j)}(y_j,x_j,r),$$

and

$$(4.13)\quad N_n^{(c_1+c_2+\ldots+c_m,\;b_1+b_2+\ldots+b_m)}(x_1+x_2+\ldots+x_m,\;y_1+y_2+\ldots+y_m,\;r)$$

$$=\sum_{k_1+k_1+\ldots+k_m=n}\prod_{j=1}^{m}N_{k_j}^{(c_j,\,b_j)}(x_j,y_j,r)\quad .$$

Rainville [4] has obtained a series for Legendre polynomials and Singh [7] has obtained a series for Srivastava's polynomials [6]. Here we shall obtain a similar expression for $A_n^c(x,r)$.

From (1.7), we have

$$\sum_{n=0}^{\infty}A_n^c(\tan\beta,r)\,t^n=(1+t)^{-c}\exp.\;\{r^r(1+t)^{r-1}\,t\tan\beta\},$$

and

$$\sum_{n=0}^{\infty}A_n^c(\tan\delta,r)\,t^n=(1+t)^{-c}\exp.\;\{r^r(1+t)^{r-1}\,t\tan\delta\}$$

$$= (1+t)^{-c} \exp.\{r^r (1+t)^{r-1} t \tan\beta\} \exp.\{r^r (1+t)^{r-1} t (\tan\delta - \tan\beta)\}.$$

combining the above two results and equating the coefficients of t^n, on both sides, we get

$$(4.14)\quad A_n^c (\tan\delta, r) = \sum_{m=0}^{n} \frac{r^{rm}}{m!} \left[\frac{\operatorname{Sin}(\delta-\beta)}{\operatorname{Cos}\delta \operatorname{Cos}\beta} \right]^m A_{n-m}^{c-(r-1)m} (\tan\beta, r)$$

As a particular case, the above result reduces to [7, (4.1)] for $c=\alpha+1$, $r=1$.

5. RODRIGUES' FORMULAE.

The Rodrigues' formula for $f_n^c(x, r)$ are given by

$$(5.1)\quad f_n^c (x, r) = \frac{x^{-c+1}}{n!} \frac{d^n}{dx^n} \left\{ x^{c+n-1} {}_{r+1}F_{r+1} \left[\begin{matrix} -n, c, \Delta(r-1, c+n); \\ c+n, \Delta(r, c); \end{matrix} (r-1)^{r-1} x \right] \right\},$$

and

$$(5.2)\quad f_n^c (x, r) = \frac{(c)_n}{(n!)^2} \frac{d^n}{dx^n} \left\{ x^n {}_{r+1}F_{r+1} \left[\begin{matrix} -n, 1, \Delta(r-1, c+n); \\ n+1, \Delta(r, c); \end{matrix} (r-1)^{r-1} x \right] \right\},$$

where $\Delta(r, c)$ stands for the set of r parameters

$$\frac{c}{r}, \frac{c+1}{r}, \ldots\ldots, \frac{c+r-1}{r}, r \geqq 1.$$

In particular, for $c=\alpha+1$, $r=1$, we derive the following results from (5.1) and 5.2) :

$$(5.3)\quad L_n^{(\alpha)}(x) = \frac{x^{-\alpha}}{n!} \frac{d^n}{dx^n} \left\{ x^{\alpha+n} {}_1F_1 \left[\begin{matrix} -n; \\ \alpha+n+1; \end{matrix} x \right] \right\},$$

$$(5.4)\quad L_n^{(\alpha-n)}(x) = \frac{x^{-\alpha+n}}{(1+\alpha)_n} \frac{d^n}{dx^n} \left\{ x^\alpha L_n^{(\alpha)}(x) \right\}$$

and

$$(5.5)\quad L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{(n!)^2} \frac{d^n}{dx^n} \left\{ x^n {}_2F_2 \left[\begin{matrix} -n, 1; \\ n+1, \alpha+1; \end{matrix} x \right] \right\}$$

Again, for $a=0$, (5.5) reduces to

$$(5.6) \qquad L_n(x)=\frac{1}{n!}\frac{d^n}{dx^n}\left\{x^n {}_1F_1\,(-n;\,n+1;\,x)\right\} \, .$$

6. SOME RELATIONS WITH WELL KNOWN POLYNOMIALS.

In this section, we obtain some relations of $A_n^c\,(x, r)$ with well known polynomials. In our previous paper [1, (3.5)], we have

$$(6.1) \qquad A_k^c\,(x, r) = \sum_{j=0}^{k} \frac{x^j\, r^{rj}}{j\,!} \binom{(r-1)\,j-c}{k-j} \, .$$

Also for Laguerre polynomials, we have the relation [4a, p. 207]

$$\frac{x^n}{n\,!} = \sum_{k=0}^{n} \frac{(-1)^k\;(1+a)_n\;L_k^{(a)}\,(x)}{(n-k)\,!\,(1+a)_k}$$

From the above results, we establish

$$(6.2) \qquad A_n^c\,(x, r) = \sum_{k=0}^{n} {}_{r+1}F_r\left[\begin{matrix}-n+k,\ \triangle\,(r,\ c-(r-1)\,n);\\ -a-n,\ \triangle\,(r-1,\ c-(r-1)n);\end{matrix}\ -(r-1)^{r-1}\right]\times$$

$$\times\frac{r^{rn}\,(-1)^k(1+a)_n\,L_k^{(a)}(x)}{(1+a)_k\,(n-k)\,!} \, .$$

Now using (6.1) and the relation [4a, p. 194]

$$\frac{x^n}{n!} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{H_{n-2k}(x)}{2^n k!(n-2k)!} \, ,$$

for Hermite polynomials, we derive

$$(6.3) \qquad A_n^c\,(x, r) =$$

$$= \sum_{m=0}^{n} {}_{2r}F_{2r}\left[\begin{matrix}-\frac{m}{2},\ -\frac{m+1}{2},\ \triangle(2r-2,\ -c+1+(r-1)\ (n-m));\\ \triangle\,(2r,\ -c+1-m+(r-1)\ (n-m));\end{matrix}\ \frac{(r-1)^{2r-2}}{4}\right]$$

$$\times \frac{\left((r-1)(n-m)-c-m+1\right)_m \, r^{r(n-m)}}{2^{n-m}\,(n-m)! \qquad m!} H_{n-m}(x).$$

Similarly, we can also express $A_n^c(x,r)$ in terms of $Z_n(x)$, $C_n^v(x)$, $H_n(s, p, x)$, $P_n(x)$ Sister Celine's polynomials and Bernoulli polynomials etc. Similar results for $f_n^c(x,r)$ can also be obtained.

REFERENCES

[1] Chandel, R. C. S., "Generalized Laguerre polynomials and the polynomials related to them" : Indian J. Math., **11** (1969), 57—66.

[2] Chandel, R. C. S., "A short note on generalized Laguerre polynomials and the polynomials related to them" : Indian J. Math., **13** (1971), 25—27.

[3] Chandel, R. C. S., Generalized Laguerre polynomials and the polynomials related to them II" : Indian J. Math. **14** (1972), 149—155.

[4] Rainville, E. D., Bull. Amer. Math. Soc., **51** (1945), 268—271.

[4a] Rainville, E. D., "Special Functions" : Macmillan company, New York (1960).

[5] Sansone, G., "Orthoganal functions" : Direttore Dell' istituto Matematico Della Universita, Florence, Italy (1959).

[6] Srivastava, K. N., "Some polynomials related to Laguerre polynomials": J. Indian Math. Soc. (N. S.) **28** (1964), 43— 0.

[7] Singh, R. P., "Some polynomials of Srivastava related to Laguerre polynomials" : Math. Japon., **9** (1964), 5—9.

[8] Singh, R. P., "A note on Gegenbauer and Laguerre polynomials" : Math. Japon., **9** (1964), 1—4.

[9] Sheffer, I. M., Some properties of polynomials set of type zero" : Duke Math. J., **5** (1939), 590—622.

J n̆ ā n a b h a, Sect. A, Vol. 2, July 1972.

CN TWO MULTIPLE HYPERGEOMETRIC FUNCTIONS RELATED TO LAURICELLA'S F_D.

By

Harold Exton

University of Salford, Salford, Lancashire, U. K.

(Communicated by Professor H. M. Srivastava, F.N.A.Sc.)

(*Received on 19th September, 1972*)

1. INTRODUCTION AND DEFINITIONS : REGIONS OF CONVERGENCE.

In some recent investigations of quadruple hypergeometric functions, I encountered certain instances which are special cases of the two functions under consideration in this paper. These new functions are defined as follows :

$$\begin{matrix}(k)\\(1)\end{matrix} E_D^{(n)}(\alpha, \beta_1, .., \beta_n; \gamma, \gamma'; x_1, ..., x_n) =$$

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(\alpha)_{m_1+\ldots+m_n}(\beta_1)_{m_1}\ldots(\beta_n)_{m_n}}{(\gamma)_{m_1+\ldots+m_k}(\gamma')_{m_{k+1}+\ldots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \quad \ldots\ldots(1.1)$$

and

$$\begin{matrix}(k)\\(2)\end{matrix} E_D^{(n)}(\alpha, \alpha', \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) =$$

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(\alpha)_{m_1+\ldots+m_k}(\alpha')_{m_{k+1}+\ldots+m_n}(\beta_1)_{m_1}\ldots(\beta_n)_{m_n}}{(\gamma)_{m_1+\ldots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n} \quad \ldots\ldots(1.2)$$

and it is evident that they are closely related to Lauricella's function $F_D^{(n)}$ ([4], p. 113), defined by

$$F_D^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) =$$

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(\alpha)_{m_1+\ldots+m_n} (\beta_1)_{m_1} \ldots (\beta_n)_{m_n}}{(\gamma)_{m_1+\ldots+m_n}} \frac{x_1^{m_n}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} \quad \ldots (1.3)$$

We note the following special cases* :

$$\begin{Bmatrix}(o)\\(1)\end{Bmatrix} E_D^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) =$$

$$\begin{Bmatrix}(o)\\(2)\end{Bmatrix} E_D^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) =$$

$$F_D^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n)$$

$$\begin{Bmatrix}(1)\\(1)\end{Bmatrix} E_D^{(2)}(\alpha, \beta_1, \beta_2; \gamma, \gamma'; x_1, x_2) = F_2(\alpha, \beta_1, \beta_2; \gamma, \gamma'; x_1, x_2)$$

$$\begin{Bmatrix}(1)\\(1)\end{Bmatrix} E_D^{(3)}(\alpha, \beta_1, \beta_2, \beta_3; \gamma, \gamma'; x_1, x_2, x_3) = F_G(\alpha, \alpha, \alpha, \beta_1, \beta_2, \beta_3; \gamma, \gamma', \gamma'; x_1, x_2, x_3) [6]$$

$$\begin{Bmatrix}(3)\\(1)\end{Bmatrix} E_D^{(4)}(\alpha, \beta_1, \beta_2, \beta_3, \beta_4; \gamma, \gamma'; x_1, x_2, x_3, x_4) =$$

$$K_{11}(\alpha, \alpha, \alpha, \alpha, \beta_1, \beta_2, \beta_3, \beta_4; \gamma, \gamma, \gamma, \gamma'; x_1, x_2, x_3, x_4) [2]$$

$$\begin{Bmatrix}(2)\\(1)\end{Bmatrix} E_D^{(4)}(\alpha, \beta_1, \beta_2, \beta_3, \beta_4; \gamma, \gamma'; x_1, x_2, x_3, x_4) =$$

$$K_{12}(\alpha, \alpha, \alpha, \alpha, \beta_1, \beta_2, \beta_3, \beta_4; \gamma, \gamma, \gamma', \gamma'; x_1, x_2, x_3, x_4) [2]$$

$$\begin{Bmatrix}(1)\\(2)\end{Bmatrix} E_D^{(2)}(\alpha, \alpha' \beta_1, \beta_2; \gamma; x_1, x_2) = F_3(\alpha, \alpha', \beta_1, \beta_2; \gamma; x_1, x_2)$$

$$\begin{Bmatrix}(1)\\(2)\end{Bmatrix} E_D^{(3)}(\alpha, \alpha', \beta_1, \beta_2, \beta_3; \gamma; x_1, x_2, x_3) = F_S(\alpha, \alpha', \alpha', \beta_1, \beta_2, \beta_3; \gamma, \gamma, \gamma; x_1, x_2, x_3) [6]$$

$$\begin{Bmatrix}(3)\\(2)\end{Bmatrix} E_D^{(4)}(\alpha, \alpha' \beta_1, \beta_2, \beta_3, \beta_4; \gamma; x_1, x_2, x_3, x_4) =$$

$$K_{15}(\alpha, \alpha, \alpha, \alpha', \beta_1, \beta_2, \beta_3, \beta_4; \gamma, \gamma, \gamma, \gamma; x_1, x_2, x_3, x_4) [2]$$

---

* *Note that F_G and F_S are only alternative notations for Lauricella's triple hypergeometric functions F_8 and F_7 relatively (cf. [4], p. 114).*

$${}^{(2)}_{(2)}E_D^{(4)}\ (\alpha,\alpha',\ \beta_1,\beta_2,\beta_3,\beta_4;\gamma;\ x_1,x_2,x_3,x_4)=$$

$$K_{20}\ (\alpha,\alpha,\ \beta_3,\beta_4,\beta_1,\beta_2,\ \alpha',\alpha';\ \gamma,\gamma,\gamma,\gamma;\ x_1,x_2,x_3,x_4)[2].$$

The regions of convergence of ${}^{(k)}_{(1)}E_D^{(n)}$ and ${}^{(k)}_{(2)}E_D^{(n)}$ are investigated using the technique employed by Horn [5] and Srivastava ([7], [8]).

Let r_i , $i=1,2,3,\ldots$, be called the associated radii of convergence of the multiple power series

$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty} A_{m_1,\ldots,m_n}\, x_1^{m_1}\ldots x_n^{m_n}$$

if it converges absolutely when $|x_i|<r_i$ and diverges when $|x_i|>r_i$.

It is found that for the convergence of ${}^{(k)}_{(1)}E_D^{(n)}$,

$$r_1=r_2=\ldots=r_k,$$

$$r_{k+1}=r_{k+2}=\ldots=r_n\ ,$$

$$r_k+r_n=1,$$

and in the case of ${}^{(k)}_{(2)}E_D^{(n)}$,

$$r_1=r_2=\ldots=r_k,$$

$$r_{k+1}=r_{k+2}=\ldots=r_n\ ,$$

$$r_k+r_n=r_kr_n\ .$$

I omit the details of the working out.

2. INTEGRALS OF EULER TYPE.

From (1.1), ${}^{(k)}_{(1)}E_D^{(n)}(\alpha,\ \beta_1,\ldots,\beta_n;\ \gamma,\gamma';\ x_1,\ldots,x_n)=$

$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_k=0}^{\infty}\frac{(\alpha)_{m_1+\ldots+m_k}(\beta_1)_{m_1}\ldots(\beta_k)_{m_k}}{(\gamma)_{m_1+\ldots+m_k}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_k^{m_k}}{m_k!}\times$$

$$F_D^{(n-k)}\,(\alpha+m_1+\dots+m_k,\ \beta_{k+1},\dots,\beta_n;\ \gamma';\ x_{k+1},\dots,x_n) \tag{2.1}$$

Lauricella's formula, ([4], p. 147)

$$F_D^{(n)}\left(\alpha,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n\right)=\Gamma\left[\begin{matrix}\gamma\\ \beta_1,\dots,\beta_n,\gamma-\beta_1-\dots-\beta_n\end{matrix}\right]\times$$

$$\int\dots(n)\dots\int u_1^{\beta_1-1}\dots u_n^{\beta_n-1}\,(1-u_1-\dots-u_n)^{\gamma-\beta_1-\dots-\beta_n-1}$$

$$(1-u_1x_1-\dots-u_nx_n)^{-\alpha}\,du_1\dots du_n \tag{2.2}$$

$$u_1\geqslant 0,\dots,u_n\geqslant 0,\ u_1+\dots+u_n\leqslant 1,$$

$$Re(\gamma)>0,\ Re(\beta_1)>0,\ \ ,Re(\beta_n)>0, Re(\gamma-\beta_1-\dots-\beta_n)>0$$

is now applied to the inner $F_D^{(n-k)}$ series of the right-hand member of (2.1), which becomes

$$\Gamma\left[\begin{matrix}\gamma'\\ \beta_{k+1},\dots,\beta_n,\gamma'-\beta_{k+1}-\dots-\beta_n\end{matrix}\right]\times$$

$$\sum_{m_1=0}^{\infty}\dots\sum_{m_k=0}^{\infty}\int\dots(n-k)\dots\int u_{k+1}^{\beta_{k+1}-1}\dots u_n^{\beta_n-1}$$

$$(1-u_{k+1}-\dots-u_n)^{\gamma'-\beta_{k+1}-\dots-\beta_n-1}\times$$

$$(1-u_{k+1}x_{k+1}-\dots-u_nx_n)^{-\alpha}\ \frac{(\alpha)_{m_1+\dots+m_k}(\beta_1)_{m_1}\dots(\beta_k)_{m_k}}{(\alpha)_{m_1+\dots+m_k}\,m_1!\ m_k!}\times$$

$$\left\{\frac{x_1}{1-u_{k+1}x_{k+1}-\dots-u_nx_n}\right\}^{m_1}\dots\left\{\frac{x_k}{1-u_{k+1}x_{k+1}-\dots-u_nx_n}\right\}^{m_k}du_{k+1}\dots du_n \tag{2.3}$$

if $|x_i|\leqslant\xi_i,\ 1\leqslant i\leqslant k,$ then

$$\left|\frac{x_i}{1-u_{k+1}x_{k+1}-\dots-u_nx_n}\right|\leqslant\frac{\xi_i}{1-\xi_{k+1}-\dots-\xi_n},$$

which is less than unity if

$$\xi_i+\xi_{k+1}+\dots+\xi_n<1.$$

In this domain of $x_1,\dots,x_n$, the above series converges uniformly over the region of integration, so that the order of integration and summation may be reversed. Hence, (2.3) becomes

$$\Gamma\left[\begin{matrix} & \gamma' & \\ \beta_{k+1}, \ldots, \beta_n, & \gamma'-\beta_{k+1}-\ldots-\beta_n \end{matrix}\right]\times$$

$$\int\cdots(n-k)\cdots\int \frac{u_{k+1}^{\beta_{k+1}-1}\, u_n^{\beta_n-1}(1-u_{k+1}-\ldots-u_n)^{\gamma'-\beta_{k+1}-\ldots-\beta_n-1}}{(1-u_{k+1}x_{k+1}-\ldots-u_n x_n)^{-\alpha}}\times$$

$$F_D^{(k)}\left(\alpha, \beta_1,\ldots, \beta_k; \gamma; \frac{x_1}{1-u_{k+1}x_{k+1}-\ldots-u_n x_n},\ldots, \frac{x_k}{1-u_{k+1}x_{k+1}-\ldots-u_n x_n}\right) du_{k+1}\ldots du_n \qquad (2.4)$$

(2.2) is now applied to the inner $F_D^{(k)}$ series of (2.4), when we have the result

$$\begin{matrix}(k)\\(1)\end{matrix} E_D^{(n)}(\alpha, \beta_1,\ldots, \beta_n; \gamma,\gamma'; x_1,\ldots,x_n)=$$

$$\Gamma\left[\begin{matrix} & \gamma, & \gamma' \\ \beta_1,\ldots\beta_n, & \gamma-\beta_1-\ldots-\beta_k, & \gamma'-\beta_{k+1}-\ldots-\beta_n \end{matrix}\right]\times$$

$$\int\cdots(n)\cdots\int u_1^{\beta_1-1}\ldots u_n^{\beta_n-1}\,(1-u_1-\ldots-u_k)^{\gamma-\beta_1-\ldots-\beta_k-1}$$

$$(1-u_{k+1}-\ldots-u_n)^{\gamma'-\beta_{k+1}-\ldots-\beta_n-1}\times\ (1-u_1x_1-\ldots-u_nx_n)^{-\alpha}du_1\ldots du_n \qquad (2.5)$$

$$u_1\geqslant 0,\ldots,u_n\geqslant 0,\quad u_1+\ldots+u_k\leqslant 1,\quad u_{k+1}+\ldots+u_n\leqslant 1$$
$$Re(\gamma)>0,\ Re(\gamma')>0,\ Re(\beta_1)>0,\ \ldots,\ Re(\beta_n)>0,$$
$$Re(\gamma-\beta_1-\ldots-\beta_k)>0,\quad Re(\gamma'-\beta_{k+1}-\ldots-\beta_n)>0.$$
$$\xi_i+\xi_{k+1}+\cdots+\xi_n<1 \qquad 1\leqslant i\leqslant k$$

The above technique was used by Srivastava [7] .

The inner $F_D^{(k)}$ series of (2.4) may also be replaced by Lauricella's single integral representation of $F_D^{(n)}$ ([4], p. 146)

$$F_D^{(n)}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1,\ldots, x_n)=$$

$$\Gamma\left[\begin{matrix} & \gamma \\ \alpha, & \gamma-\alpha\end{matrix}\right]\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux_1)^{-\beta_1}\ldots(1-ux_n)^{-\beta_n}du \qquad (2.6)$$

This gives the following result :

$$\overset{(k)}{\underset{(1)}{}} E_D^{(n)} (\alpha, \beta_1, \ldots, \beta_n; \gamma, \gamma'; x_1, \ldots, x_n) =$$

$$\Gamma \left[\begin{matrix} \gamma, & \gamma' \\ \alpha, \beta_{k+1}, \ldots, \beta_n, & \gamma - \alpha, \gamma' - \beta_{k+1} - \ldots - \beta_n \end{matrix} \right] \times$$

$$\int \cdot\cdot(n-k+1)\cdot\cdot \int \frac{v^{\alpha-1} u_{k+1}^{\beta_{k+1}-1} \ldots u_n^{\beta_n-1} (1-v)^{\gamma-\alpha-1} \left(1-u_{k+1}-\ldots-u_n\right)^{\gamma'-\beta_{k+1}-\ldots-\beta_n-1}}{\left(1-vx_1-u_{k+1}x_{k+1}-\ldots-u_nx_n\right)^{-\beta_1} \ldots \left(1-vx_k-u_{k+1}x_{k+1}-\ldots-u_nx_n\right)^{-\beta_k}} \times$$

$$\times \left(1-u_{k+1}x_{k+1}-\ldots-u_nx_n\right)^{-\alpha-\beta_1-\ldots-\beta_k} dv\, du_{k+1} \ldots du_n \qquad (2.7)$$

$0 \leqslant v \leqslant 1, \qquad u_{k+1} \geq 0, \ldots, u_n \geq 0,$

$u_{k+1} + \ldots + u_n \leqslant 1$

$Re(\gamma) > 0, \quad Re(\gamma') > 0, \quad Re(\alpha) > 0, \quad Re(\gamma - \alpha) > 0$

$Re\left(\beta_{k+1}\right) > 0, \ldots, Re\ (\beta_n) > 0, \quad Re\left(\gamma' - \beta_{k+1} - \ldots - \beta_n\right) > 0$

$\xi_i + \xi_{k+1} + \ldots + \xi_n < 1 \qquad\qquad 1 \leqslant i \leqslant k$.

The following integral representations of $\overset{(k)}{\underset{(2)}{}} E_D^{(n)}(\alpha, \alpha' \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n)$ may be obtained by similar methods :

$$\Gamma \left[\begin{matrix} \gamma \\ \alpha, \alpha', \gamma - \alpha - \alpha' \end{matrix} \right] \times$$

$$\int_0^1 \int_0^1 u^{\alpha-1} v^{\alpha'-1} (1-u)^{\gamma-\alpha-\alpha'-1} (1-v)^{\gamma-\alpha'-1} \left\{1 - x_1^u (1-v)\right\}^{-\beta_1} \ldots$$

$$\left\{1 - x_k^u (1-v)\right\}^{-\beta_k} (1 - x_{k+1} v)^{-\beta_{k+1}} \ldots (1 - x_n v)^{-\beta_n} du\, dv \qquad (2.8)$$

$Re(\gamma) > 0, Re(\alpha) > 0, Re(\alpha') > 0, Re(\alpha - \alpha - \alpha') > 0,$

$|x_i| < 1, i = 1, 2, \ldots, n$

$$\Gamma \left[\begin{matrix} \gamma \\ \beta_1, \ldots, \beta_n, \gamma - \beta_1 - \ldots - \beta_n \end{matrix} \right] \times$$

$$\int \ldots (n) \ldots \int u_1^{\beta_1 - 1} \ldots u_n^{\beta_n - 1} (1 - u_1 - \ldots - u_k)^{\gamma - \beta_1 - \ldots - \beta_n - 1} (1 - u_{k+1} - \ldots - u_n)^{\gamma - \beta_{k+1} - \ldots - \beta_n - 1} \times$$

$$(1 - u_{k+1} x_{k+1} - \ldots - x_n u_n)^{-a'} \times$$

$$\left\{ 1 - (u_1 x_1 + \ldots + u_k x_k)(1 - u_{k+1} - \ldots - u_n) \right\}^{-a'} dn_1 \ldots du_n \tag{2.9}$$

$Re(\gamma) > 0$, $Re(\beta_1) > 0, \ldots, Re(\beta_n) > 0$,

$Re(\gamma - \beta_1 - \ldots - \beta_n) > 0$

$u_1 \geqslant 0, \ldots, u_n \geqslant 0$, $u_1 + \ldots + u_k \leqslant 1$, $u_{k+1} + \ldots + u_n \leqslant 1$

$|x_i| < 1$, $i = 1, 2, \ldots, n$.

$$\Gamma\left[\begin{matrix} \gamma \\ a, \beta_{k+1}, \ldots, \beta_n, \ \gamma - a - \beta_{k+1} - \ldots - \beta_n \end{matrix} \right] \times$$

$$\int \ldots (n-k+1) \ldots \int v^{a-1} u_{k+1}^{\beta_{k+1} - 1} \ldots u_n^{\beta_n - 1} (1 - v)^{\gamma - a - \beta_{k+1} - \ldots - \beta_n - 1} (1 - u_{k+1} - \ldots - u_n)^{\gamma - \beta_{k+1} - \ldots - \beta_n - 1} \times$$

$$\left\{ 1 - v x_1 (1 - u_{k+1} - \ldots - u_n) \right\}^{-\beta_1} \ldots \left\{ 1 - v x_k (1 - u_{k+1} - \ldots - u_n) \right\}^{-\beta_k} \times$$

$$(1 - u_{k+1} x_{k+1} - \ldots - u_n x_n)^{-a'} dv \, du_{k+1} \ldots du_n \tag{2.10}$$

$Re(\gamma) > 0$, $Re(a) > 0$, $Re(\beta_{k+1}) > 0, \ldots, Re(\beta_n) > 0$

$Re(\gamma - a - \beta_{k+1} - \ldots - \beta_n) > 0 \qquad 0 \leqslant v \leqslant 1$,

$u_{k+1} \geqslant 0, \ldots, u_n \geqslant 0$, $u_{k+1} + \ldots + u_n \leqslant 1$.

Various integral representation of similar type may also be obtained by the same methods.

3. POCHHAMMER INTEGRALS.

Consider the integral

$$I = \int_C (-t)^{-\rho} (t-1)^{-\rho'} F_D^{(k)}\left(a, \beta_1, \ldots, \beta_k; \gamma; \frac{x_1}{t}, \ldots, \frac{x_k}{t} \right) \times F_D^{(n-k)}\left(a', \beta_{k+1}, \ldots, \beta_n; \gamma'; \frac{x_{k+1}}{1-t}, \ldots, \frac{x_n}{t} \right) dt \tag{3.1}$$

where C is a pochhammer double-loop slung around the points O and 1.
If the integrand is expanded, supposing that integration and summation may be interchanged, we have

$$I=\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(\alpha)_{m_1+\dots+m_k}(\alpha')_{m_{k+1}+\dots+m_n}(\beta_1)_{m_1}\dots(\beta_n)_{m_n}}{(\gamma)_{m_1+\dots+m_k}(\gamma')_{m_{k+1}+\dots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\times$$

$$(-1)^{m_1+\dots+m_n}\int_C(-t)^{-\rho-m_1-\dots-m_k}(t-1)^{-\rho'-m_{k+1}-\dots-m_n}dt\ \dots\quad(3.2)$$

The inner integral on the right-hand side of (3.2) may be evaluated by means of the formula

$$\int_C(-t)^{\alpha-1}(t-1)^{\beta-1}dt=\frac{(2\pi i)^2}{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(\alpha+\beta)}\qquad(3.3)$$

([1], p. 378).

Hence, $$I=\frac{(2\pi i)^2}{\Gamma(\rho)\Gamma(\rho')\Gamma(2-\rho-\rho')}\times$$

$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(\rho+\rho'-1)_{m_1+\dots+m_n}(\alpha)_{m_1+\dots+m_k}(\alpha')_{m_{k+1}+\dots+m_n}}{(\gamma)_{m_1+\dots+m_k}(\gamma')_{m_{k+1}+\dots+m_n}}\times$$

$$\frac{(\beta)_{m_1}\dots(\beta_n)_{m_n}}{(\rho)_{m_1+\dots+m_k}(\rho')_{m_{k+1}+\dots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\qquad(3.4)$$

so that if $\alpha=\rho$, $\alpha'=\rho'$, we have the result

$$\begin{matrix}(k)\\(1)\end{matrix}E_D^{(n)}(\rho+\rho'-1,\beta_1,\dots,\beta_n;\gamma,\gamma';x_1,\dots,x_n)=$$

$$\frac{\Gamma(\rho)\Gamma(\rho')\Gamma(2-\rho-\rho')}{(2\pi i)^2}\times$$

$$\int_C(-t)^{-\rho}(t-1)^{-\rho'}F_D^{(k)}\left(\rho,\beta_1,\dots,\beta_k;\gamma;\frac{x_1}{t},\dots,\frac{x_k}{t}\right)\times$$

$$F_D^{(n-k)}\left(\rho',\beta_{k+1}\dots,\beta_n;\gamma';\frac{x_{k+1}}{1-t},\dots,\frac{x_n}{1-t}\right)dt\qquad(3.5)$$

Similarly, $\begin{matrix}(k)\\(2)\end{matrix}E_D^{(n)}(\alpha,\alpha',\beta_1,\dots,\beta_n;\rho+\rho';x_1,\dots,x_n)=$

$$\frac{\Gamma(1-\rho)\Gamma(1-\rho')\Gamma(\rho+\rho')}{(2\pi i)^2}\times$$

$$\int_C (-t)^{\rho-1}(t-1)^{\rho'-1} F_D^{(k)}(\alpha,\beta_1,\dots,\beta_k;\rho;x_1t,\dots,x_kt)\times$$

$$F_D^{(n-k)}\left(\alpha',\beta_{k+1},\dots,\beta_n;\rho';x_{k+1}(1-t),\dots,x_n(1-t)\right)dt \qquad (3.6)$$

4. SIMPLE TRANSFORMATIONS AND REDUCTIONS.

The following transformations are obtained by the elementary manipulation of series.

Lauricella's transformations ([4], p. 149)

$$F_D^{(n)}(\alpha,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n)$$

$$=(1-x_1)^{-\alpha} F_D^{(n)}\left(\alpha,\gamma-\beta_1-\dots-\beta_n,\beta_2,\dots,\beta_n;\gamma;\frac{x_1}{x_1-1},\frac{x_1-x_2}{x_1-1},\dots,\frac{x_1-x_n}{x_1-1}\right)$$

... ...

$$=(1-x_n)^{-\alpha} F_D^{(n)}\left(\alpha,\beta_1,\dots,\beta_{n-1},\gamma-\beta_1-\dots-\beta_n;\gamma;\frac{x_n-x_1}{x_n-1},\dots,\frac{x_n-x_{n-1}}{x_n-1},\frac{x_n}{x_n-1}\right) \qquad (4.1)$$

may be applied to the inner $F_D^{(n-k)}$ series of (2.1) to give the transformations

$${}^{(k)}_{(1)}E_D^{(n)}(\alpha,\beta_1,\dots,\beta_n;\gamma,\gamma';x_1,\dots,x_n)$$

$$=(1-x_{k+1})^{-\alpha}\,{}^{(k)}_{(1)}E_D^{(n)}\left(\alpha,\beta_1,\dots,\beta_k,\gamma'-\beta_{k+1}-\dots-\beta_n,\beta_{k+2},\dots,\beta_n;\gamma,\gamma';\right.$$

$$\frac{x_1}{1-x_{k+1}},\dots,\frac{x_k}{1-x_{k+1}},\frac{x_{k+1}}{x_{k+1}-1},\frac{x_{k+1}-x_{k+2}}{x_{k+1}-1},\dots,$$

$$\left.\frac{x_{k+1}-x_n}{x_{k+1}-1}\right)$$

... ...

$$=(1-x_n)^{-\alpha}\,{}^{(k)}_{(1)}E_D^{(n)}\left(\alpha,\beta_1,\dots,\beta_{n-1},\gamma'-\beta_{k+1}-\dots-\beta_n;\gamma,\gamma';\right.$$

$$\frac{x_1}{1-x_n},\dots,\frac{x_k}{1-x_n},\frac{x_n-x_{k+1}}{x_n-1},\dots,\frac{x_n-x_{n-1}}{x_n-1},$$

$$\left.\frac{x_n}{x_n-1}\right) \qquad (4.2)$$

Similarly, we also have

$$\overset{(k)}{\underset{(1)}{}} E_D^{(n)} (\alpha, \beta_1, \ldots, \beta_n; \gamma, \gamma'; x_1, \ldots, x_n)$$

$$=(1-x_1)^{-\alpha} \overset{(k)}{\underset{(1)}{}} E_D^{(n)}\Big(\alpha, \gamma-\beta_1-\ldots-\beta_k, \beta_2, \ldots, \beta_n; \gamma, \gamma';$$

$$\frac{x_1}{x_1-1}, \frac{x_1-x_2}{x_1-1}, \ldots, \frac{x_1-x_k}{x_1-1}, \frac{x_{k+1}}{1-x_1}, \ldots, \frac{x_n}{1-x_1}\Big)$$

... ...

$$=(1-x_k)^{-\alpha} \overset{(k)}{\underset{(1)}{}} E_D^{(n)}\Big(\alpha, \beta_1, \ldots, \beta_{k-1}, \gamma-\beta_1-\ldots-\beta_k; \gamma, \gamma';$$

$$\frac{x_k-x_1}{x_k-1}, \ldots, \frac{x_k-x_{k-1}}{x_k-1}, \frac{x_k}{x_k-1}, \frac{x_{k+1}}{1-x_k}, \ldots, \frac{x_n}{1-x_k}\Big)$$

$$=(1-x_1-x_{k+1})^{-\alpha} \overset{(k)}{\underset{(1)}{}} E_D^{(n)}\Big(\alpha, \gamma-\beta_1-\ldots-\beta_k, \beta_2, \ldots, \beta_k, \gamma'-\beta_{k+1}-\ldots-\beta_n,$$

$$\beta_{k+2}, \ldots, \beta_n; \gamma, \gamma'; \frac{x_1}{1-x_1-x_{k+1}}, \frac{x_1-x_2}{1-x_1-x_{k+1}}, \ldots, \frac{x_1-x_k}{1-x_1-x_{k+1}}$$

$$\frac{x_{k+1}}{1-x_1-x_{k+1}}, \frac{x_{k+2}-x_{k+2}}{1-x_1-x_{k+1}}, \ldots, \frac{x_{k+1}-x_n}{1-x_1-x_{k+1}}\Big)$$

... ...

$$=(1-x_1-x_n)^{-\alpha} \overset{(k)}{\underset{(1)}{}} E_D^{(n)}\Big(\alpha, \gamma-\beta_1-\ldots-\beta_k, \beta_2, \ldots, \beta_k, \beta_{k+1}, \ldots, \beta_{n-1},$$

$$\gamma'-\beta_{k+1}-\ldots-\beta_n; \gamma, \gamma'; \frac{x_1}{1-x_1-x_n}, \frac{x_1-x_2}{1-x_1-x_n}, \ldots, \frac{x_1-x_k}{1-x_1-x_n},$$

$$\frac{x_n-x_{k+1}}{1-x_1-x_n}, \ldots, \frac{x_n-x_{n-1}}{1-x_1-x_n}, \frac{x_n}{1-x_1-x_n}\Big)$$

$$=(1-x_2-x_{k+1})^{-(\alpha)} \overset{(k)}{\underset{(1)}{}} E_D^{(n)}\Big(\alpha, \beta_1, \gamma-\beta_1-\ldots-\beta_k, \beta_3, \ldots, \beta_k, \gamma'-\beta_{k+1}-\ldots-$$

$$\beta_n, \beta_{k+2}, \ldots, \beta_n; \gamma, \gamma'; \frac{x_2-x_1}{1-x_2-x_{k+1}}, \frac{x_2}{1-x_2-x_{k+1}}, \frac{x_2-x_3}{1-x_2-x_{k+1}}, ,$$

$$\frac{x_2-x_k}{1-x_2-x_{k+1}}, \frac{x_{k+1}}{1-x_2-x_{k+1}}, \frac{x_{k+1}-x_{k+2}}{1-x_2-x_{k+1}}, \ldots, \frac{x_{k+1}-x_n}{1-x_2-x_{k+1}}$$

... ...

$$=(1-x_k-x_n)^{-\alpha} \overset{(k)}{\underset{(1)}{}} E_D^{(n)}(\alpha, \beta_1, \ldots, \beta_{k-1}, \gamma-\beta_1-\ldots-\beta_k, \beta_{k+1}, \ldots, \beta_{n-1},$$

$$\gamma'-\beta_{k+1}-\dots-\beta_n;\gamma,\gamma'; \qquad \frac{x_k-x_1}{1-x_k-x_n},\dots,$$

$$\left.\frac{x_k-x_{k-1}}{1-x_k-x_n},\frac{x_k}{1-x_k-x_n},\frac{x_n-x_{k+1}}{1-x_k-x_n},\dots,\frac{x_n-x_{n-1}}{1-x_k-x_n},\frac{x_n}{1-x_k-x_n}\right) \tag{4.3}$$

Simple transformations of the above type do not appear to exist for the function ${}^{(k)}_{(2)}E^{(n)}_{D}$

If Lauricella's reduction formula ([4] p. 150)

$$F^{(n)}_{D}(\alpha,\beta_1,\dots,\beta_n;\gamma;x,\dots,x)= {}_2F_1(\alpha,\beta_1+\dots+\beta_n;\gamma;x) \tag{4.4}$$

where all the variables are made equal, is employed, it may readily be established that

$$\begin{aligned}&{}^{(k)}_{(1)}E^{(n)}_{D}(\alpha,\beta_1,\dots,\beta_n;\gamma,\gamma';x_1,x_i,x_i,\dots,x_i,x_{k+1},\dots,x_j,x_j,\dots,x_j)\\ &= {}^{(i)}_{(1)}E^{(j-k+i)}_{D}(\alpha,\beta_1,\dots,\beta_i-1,\beta_i+\dots+\beta_k,\beta_{k+1},\dots,\beta_j-1,\beta_j+\dots+\beta_n;\gamma,\gamma';\\ &\qquad x_1,\dots,x_i,x_{k+1},\dots,x_j)\\ &\qquad i\leqslant k, j>k,\end{aligned} \tag{4.5}$$

giving finally

$$\begin{aligned}&{}^{(k)}_{(1)}E^{(n)}_{D}(\alpha,\beta_1,\dots,\beta_n;\gamma,\gamma';x_1,x_1,\dots,x_1,x_{k+1},x_{k+1},\dots,x_{k+1})=\\ &F_2(\alpha\ \beta,+\dots+\beta_k.\beta_{k+1}+\dots+\beta_n;\gamma,\gamma';x_1,x_{k+1})\end{aligned} \tag{4.6}$$

together with a similar reduction for ${}^{(k)}_{(2)}E^{(n)}_{D}$

This section is concluded with an interesting reduction of ${}^{(k)}_{(2)}E^{(n)}_{D}$.

From the definition of $F^{(n)}_{D}$, it is evident that

$$F^{(n)}_{D}(\alpha,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n) = \sum_{m_1=0}^{\infty}\dots\sum_{m_k=0}^{\infty}\frac{(\alpha)_{m_1+\dots+m_k}(\beta_1)_{m_1}\cdots(\beta_k)_{m_k}}{(\gamma)_{m_1+\dots+m_k}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_k^{m_k}}{m_k!}$$

$$\gamma' - \beta_{k+1} - \ldots - \beta_n; \gamma, \gamma'; \qquad \frac{x_k - x_1}{1 - x_k - x_n}, \ldots,$$

$$\frac{x_k - x_{k-1}}{1 - x_k - x_n}, \frac{x_k}{1 - x_k - x_n}, \frac{x_n - x_{k+1}}{1 - x_k - x_n}, \ldots, \frac{x_n - x_{n-1}}{1 - x_k - x_n}, \frac{x_n}{1 - x_k - x_n}\Big) \tag{4.3}$$

Simple transformations of the above type do not appear to exist for the function ${}^{(k)}_{(2)}E^{(n)}_{D}$

If Lauricella's reduction formula ([4] p. 150)

$$F^{(n)}_{D}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x, \ldots, x) = {}_2F_1(\alpha, \beta_1 + \ldots + \beta_n; \gamma; x) \tag{4.4}$$

where all the variables are made equal, is employed, it may readily be established that

$$\begin{aligned} &{}^{(k)}_{(1)}E^{(n)}_{D}(\alpha, \beta_1, \ldots, \beta_n; \gamma, \gamma'; x_1, x_i, x_i, \ldots, x_i, x_{k+1}, \ldots, x_j, x_j, \ldots, x_j) \\ &= {}^{(i)}_{(1)}E^{(j-k+i)}_{D}(\alpha, \beta_1, \ldots, \beta_i - 1, \beta_i + \ldots + \beta_k, \beta_{k+1}, \ldots, \beta_j - 1, \beta_j + \ldots + \beta_n; \gamma, \gamma'; \\ &\qquad x_1, \ldots, x_i, x_{k+1}, \ldots, x_j) \qquad (4.5) \\ &\qquad i \leqslant k, j > k, \end{aligned}$$

giving finally

$$\begin{aligned} &{}^{(k)}_{(1)}E^{(n)}_{D}(\alpha, \beta_1, \ldots, \beta_n; \gamma, \gamma'; x_1, x_1, \ldots, x_1, x_{k+1}, x_{k+1}, \ldots, x_{k+1}) = \\ &F_2(\alpha\, \beta, + \ldots + \beta_k, \beta_{k+1} + \ldots + \beta_n; \gamma, \gamma'; x_1, x_{k+1}) \qquad (4.6) \end{aligned}$$

together with a similar reduction for ${}^{(k)}_{(2)}E^{(n)}_{D}$

This section is concluded with an interesting reduction of ${}^{(k)}_{(2)}E^{(n)}_{D}$.

From the definition of $F^{(n)}_{D}$, it is evident that

$$F^{(n)}_{D}(\alpha, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) =$$

$$\sum_{m_1=0}^{\infty} \ldots \sum_{m_k=0}^{\infty} \frac{(\alpha)_{m_1+\ldots+m_k} (\beta_1)_{m_1} \cdots (\beta_k)_{m_k}}{(\gamma)_{m_1+\ldots+m_k}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_k^{m_k}}{m_k!}$$

$$F_D^{(n-k)}(a+m_1+\ldots+m_k,\beta_{k+1},\ldots,\beta_n;\gamma+m_1+\ldots+m_k;x_{k+1},\ldots,x_n) \tag{4.7}$$

The transformation due to Lauricella ([4], p. 148),

$$F_D^{(n)}(a,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=$$

$$(1-x_1)^{-\beta_1}\ldots(1-x_n)^{-\beta_n}\,F_D^{(n)}\left(\gamma-a,\beta_1,\ldots,\beta_n;\gamma;\ \frac{x_1}{x_1-1},\ldots,\frac{x_n}{x_n-1}\right) \tag{4.8}$$

is now applied to the inner $E_D^{(n-k)}$ series on the right of (4.7), which yields after slight re-arrangement, the result

$${}^{(l)}_{(2)}E_D^{(n)}(a,\ \gamma-a,\ \beta_1,\ldots\beta_n;\ \gamma;\ x_1,\ldots,x_n)=$$

$$\left(1-x_{k+1}\right)^{-\beta_{k+1}}\ldots\left(1-x_n\right)^{-\beta_n}\times$$

$$F_D^{(n)}\left(a,\ \beta_1,\ldots\beta_n;\ \gamma;\ x_1,\ldots,x_k,\frac{x_{k+1}}{x_{k+1}-1},\ldots,\frac{x_n}{x_n-1}\right) \tag{4.9}$$

which generalises the well-known result

$$F_3\ (a,\ \gamma-a,\ \beta,\ \beta';\ \gamma;\ x,\ y)=$$

$$\left(1-y\right)^{-\beta'}F_1\left(a,\beta,\beta';\ \gamma;\ x,\ \frac{y}{y-1}\right).$$

5. TRANSFORMATION ARISING FROM THE POCHHAMMER INTEGRALS.

In the integral formula (3.5), let $\rho=\gamma$, $\rho'=\gamma'$ when we have

$${}^{(k)}_{(1)}E_D^{(n)}(\gamma+\gamma'-1,\ \beta_1,\ldots,\ \beta_n;\ \gamma,\ \gamma';\ x',\ldots,\ x_n)=$$

$$\frac{\Gamma(\gamma)\Gamma(\gamma')\Gamma(2-\gamma-\gamma')}{(2\pi_i)^2}\times$$

$$\int_C (-t)^{-\gamma} (t-1)^{-\gamma'}\left(1-\frac{x_1}{t}\right)^{-\beta_1}\ldots\left(1-\frac{x_k}{t}\right)^{-\beta_k}\left(1-\frac{x_{k+1}}{1-t}\right)^{-\beta_{k+1}}$$

$$\ldots\left(1-\frac{x_n}{1-t}\right)^{-\beta_n} dt \qquad (5.1)$$

The factors in the integrand may now be expanded using the formulae

$$\left(1-\frac{x}{1-t}\right)^{-\beta}=(1-x)^{-\beta}\sum_{m=0}^{\infty}\frac{(\beta)_m}{m!}\left(\frac{x}{1-x}\cdot\frac{1-t}{t}\right)^m \qquad (5.2)$$

and

$$\left(1-\frac{x}{1-t}\right)^{-\beta}=(1-x)^{-\beta}\sum_{m=0}^{\infty}\frac{(\beta)_m}{m!}\left(\frac{x}{1-x}\cdot\frac{t}{1-t}\right)^m$$

whereby, various transformations involving genneralised functions of Horn type may be obtained by the application of (3.3). The most interesting is as follows :

$$\overset{(k)}{\underset{(1)}{}}E\overset{(n)}{\underset{D}{}}(\gamma+\gamma'-1, \beta_1,\ldots,\beta_n; \gamma, \gamma'; x_1,\ldots, x_n)=$$

$$(1-x_1)^{-\beta_1}\ldots(1-x_n)^{-\beta_n} C_n^{(k)}\left(\beta_1, \ldots, \beta_n; 1-\gamma, 1-\gamma'; \frac{x_1}{1-x_1}, \ldots, \frac{x_n}{1-x_n}\right) \qquad (5.3)$$

where

$$C_n^{(k)}(a_1, \ldots, a_n; \beta_1, \beta_2; x_1, \ldots, x_n)=$$

$$\sum_{m_1=0}^{\infty}\ldots\sum_{m_n=0}^{\infty}$$

$$\frac{(a_1)_{m_1}\cdots(a_n)_{m_n}(\beta_1)_{m_{k+1}+\ldots+m_n-m_1-\ldots-m_k}(\beta_2)_{m_1+\ldots+m_k-m_{k+1}-\ldots-m_n}}{m_1!\ldots m_n!}$$

$$\times x_1^{m_1}\ldots x_n^{m_n} \qquad (5.4)$$

[3] p.86.

If the formulae (4.1) and (4.8) are applied to the E_D series in the integrand of (6.5) and (6.6), various transformations involving new types of multiple hypergeometric series of more variables and of higher order are obtainable. For example, if (4.8) is applied to the $F_D^{(k)}$ series in the integrand of (6.6), we have

$$\begin{aligned}
{}^{(k)}_{(2)} E_D^{(n)}(\alpha, \alpha', \beta_1; \rho+\rho'; x_1, \ldots, x_n) &= \frac{\Gamma(1-\rho)\Gamma(1-\rho')\Gamma(\rho+\rho')}{(2\pi_i)^2} \times \\
&\int_C (-t)^{\rho-1} (t-1)^{\rho'-1} (1-x_1 t)^{-\beta_1} \ldots\ldots (1-x_k t)^{-\beta_k} \times \\
&F_D^{(k)}\left(\rho-\alpha, \beta_1, \ldots, \beta_k; \rho; \frac{x_1 t}{x_1 t-1}, \ldots, \frac{x_k t}{x_k t-1}\right) \times \\
&F_D^{(n-k)}\left(\alpha', \beta_{k+1}, \ldots, \beta_n; \rho'; x_{k+1}(1-t), \ldots, x_n(1-t)\right) dt
\end{aligned} \tag{5.5}$$

After considerable reduction, it may be shown that the right-hand member of (5.5) may be written

$$\sum_{m_1=0}^{\infty} \ldots \sum_{m_{n+k}=0}^{\infty} \frac{(\rho-\alpha)_{m_1+\ldots\ldots+m_k} (\alpha')_{m_{k+1}+\ldots\ldots+m_n} (\beta_1)_{m_1+\ldots+m_{n+1}} \cdot \ldots (\beta_k)_{m_k+\ldots+m_{n+k}}}{(\rho+\rho')_{m_1+\ldots+m_{n+k}}}$$

$$\times \frac{(\rho)_{m_1+\ldots+m_k+m_{n+1}+\ldots+m_{n+k}}}{(\rho)_{m_1+\ldots+m_k}} \frac{(-x_1)^{m_1}}{m_1!} \ldots \frac{(-x_k)^{m_k}}{m_k!} \frac{x_{k+1}}{m_{k+1}!} \ldots \frac{x_n^{m_n}}{m_n!} \times \frac{x_1^{m_{n+1}}}{m_{n+1}!} \ldots \frac{x_k^{m_{n+k}}}{m_{n+k}!} \tag{5.6}$$

Acknowledgement.

I am very much indebted to Prof. H. M. Srivastava for his kind enlp and encouragement given during the preparation of this paper.

REFERENCES

[1] A. Erdelyi, Transformations of hypergeometric functions of two variables. Proc. Roy. Soc. Edin. Sect. A **42** (1948), 378—385.

[2] H. Exton, O certain hypergeometric functions of four variables. (To appear).

[3] H. Exton, On a certain hypergeometric differential system. Funkcialaj Ekvacioj **14** (1971), 79—87.

[4] G. Lauricella, Sulle funzioni ipergeometriche a piu variabili. Rend. Circ. Mat. Palermo **7** (1893), 111—158.

[5] J. Horn, Hypergeometrische funktionen zweier Veranderlichen. Math. Ann. **34** (1889), 544—600.

[6] S. Saran, Hypergeometric functions of three variables. Ganita **5** (1954), 77—91.

[7] H. M. Srivastava, Hypergeometric functions of three variables. Ganita **15** (1964), 97—108.

[8] H. M. Srivastava

and M.C. Daoust, A note on the convergence of Kampé de Fériet's double hypergeometric series, Math. Nachr. **53** (1972), 151—15.9

Jñānabha, Sect. A, Vol. 2, July 1972.

APPLICATION OF GEGENBAUER POLYNOMIALS TO NONLINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS AND THEIR STABILITY

By

R. K. SHUKLA

[Department of Mathematics Government Engineering College, Jabalpur (M. P.)]

(*Received on 29th July, 1972*)

1. Introduction. The nonlinear differential equations with periodic coefficients arise in certain physical problems like Melde's experiment on the vibrations of a thread [1, p.120] or the simple pendulum [1, p.132] with oscillating support. In each case we have a nonlinear differential equation with periodic coefficients. First and second order approximations of these are obtained in [1] and analysis though not difficult is somewhat involved. We in this note shall show that similar results, at times even better, can be obtained employing an equivalent linear differential equation to the above. We shall be employing certain ultraspherical polynomials here to obtain the linearisation. This technique of using ultraspherical polynomials has been first used by Denman and Liu [2], and Garde [3] and other quite successfully on the nonlinear differential equations with constant coefficients. The application of this technique to the nonlinear differential equation with periodic coefficients turns out equally successful since the equivalent linear equation obtained is very close to the original differential equation. The study of the stability conditions leads to amplitude dependent criteria.

2. GEGENBAUER POLYNOMIALS AND APPROXIMATION :

The Gegenbauer polynomials $C_n^{\lambda}(x)$ on the interval $(-1, 1)$ are the sets of polynomials orthogonal on this interval with respect to the weight factor $(1-x^2)^{\lambda-\frac{1}{2}}$, each set corresponding to a value of $\lambda > -\frac{1}{2}$, [4, p.276]

$$(1-2xt+t^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{\lambda}(x)t^n \tag{1}$$

These polynomials on interval (—A, A) are defined as the sets of polynomials orthogonal on this interval with respect to the weight factor $(1-x^2)/A^2)^{\lambda-\frac{1}{2}}$. This gives rise to the polynomials $C_n^{\lambda}(x/A)$.

Now for a function $f(x)$ expandable in these polynomials one gets,

$$f(x) = \sum_{n=0}^{\infty} a_n^{\lambda} P_n^{\lambda}(x/A) \tag{2}$$

where

$$a_n^{\lambda} = \frac{\int_{-1}^{1} f(Ax) P_n^{\lambda}(x)(1-x^2)^{\lambda-\frac{1}{2}} dx}{\int_{-1}^{1} \left[P_n^{\lambda}(x)\right]^2 (1-x^2)^{\lambda-\frac{1}{2}} dx} \tag{3}$$

The series in (2) may be terminated to get a linear or a cubic approximation according to the degree of accuracy desired and approximation for $f(x)$ takes the form,

$$\left[f(x)\right]_* = a_1^{\lambda} P_1^{\lambda}\left(\frac{x}{A}\right) + a_3^{\lambda} P_3^{\lambda}\left(\frac{x}{A}\right) \tag{4}$$

3. APPLICATION OF POLYNOMIAL LINEARISATION TO NONLINEAR DIFFERENTIAL EQUATION WITH DERIODIC COEFFICIENTS :

The differential equation,

$$y'' + (a - 2q \cos 2z) y + by^3 = 0, \quad (b > 0, z = wt) \tag{5}$$

arises in Melde's experiment [1, p. 120]. It is noted there that (i) the motion is periodic, the main component having half the frequency of the driving reed, (ii) the displacement was symmetrical about the equilibrium position of the thread. Due to the existence of the subharmonic of the order 2, the solution is taken as

$$y = A_1 \cos z + A_3 \cos 3z \tag{6}$$

This implies that a be unity or nearly unity. This ignores the effect of the nonlinearity on the frequency which should be taken into account. If we employ the polynomial linearisation to get an equivalent differential equation the effect of the nonlinearity on the frequency becomes distinct. The results obtained are closer to the actual solution since these reduce to the results in [1] on neglecting higher powers of q involved in them. If A be the amplitude for the oscillations produced by (5) the linear ultraspherical polynomial approximatin to $ay+by^3$ is on the interval $(-A, A)$

$$[\,ay+by^3\,]_* = \left[\,a+\frac{3bA^2}{2(\lambda+2)}\right]y=n^2y \tag{7}$$

and (5) reduces to the Mathieu equation,

$$y''+(n^2-2q\cos 2z)y=0 \tag{8}$$

To fulfill the experimental observations we must have n^2 nearly unity. This would mean that if a is nearly one, b should be very small. We observe that if b and A are suitably taken a need not be nearly one. It is enough that n^2 be so. Now substituting (6) into (7) we get,

$$\begin{aligned}&-[A_1\cos z+9\,A_3\cos 3z]+n^2[A_1\cos z+A_3\cos 3z]\\&-q[A_1\cos 3z+A_1\cos z+A_3\cos 5\,z+A_3\cos z]=0\end{aligned} \tag{9}$$

Now we must have coefficients of $\cos z$, $\cos 3z$, separately equal to zero.

$$\begin{aligned}&(n^2-1-q)\,A_1-q\,A_3=0\\&-q\,A_1+(n^2-9)\,A_3=0\end{aligned} \tag{10}$$

If A_1 and A_3 are non-zero, we get,

$$n^4-(10+q)n^2+9(1+q)-q^2=0 \tag{11}$$

which gives,

$$n^2=a+\frac{3bA^2}{2(\lambda+2)}=\frac{10+q\pm\sqrt{5q^2-16q+64}}{2} \tag{12}$$

A binomial expansion of (12) upto q^3 yields,

$$n^2=\left(9+\frac{q^2}{8}\right)\text{ or }\left(1+q-\frac{q^2}{8}-\frac{q^3}{64}\right) \tag{13}$$

the former being impossible sible since n^2 is nearly unity we accept the later. Now we get,

$$A^2 = \frac{2(\lambda+2)}{3b}\left(1+q-\frac{q^2}{8}-\frac{q^3}{64}-a\right) \tag{14}$$

Taking $\lambda=0$, and neglecting q^2 we get, (7) in [1, p. 124] which is obtained for $A_3=0$. Equation (10) gives the ratio

$$\left|\frac{A_3}{A_1}\right| = \left|\frac{n^2-1-q}{q}\right| = \left|\frac{q}{n^2-9}\right| = \frac{q}{8} \text{ nearly} \tag{15}$$

which gives, $A_1=\left(1+\frac{q}{8-q}\right)A$; $A_3=-qA/(8-q)$

Equation (15) gives same ratio for A_3/A_1 as in art. 2.17 in [5, p.20—21]. When $b<0$, the solution is taken as

$$y=B_1 \sin z+B_3 \sin 3z \tag{16}$$

Proceeding as before, pair (10) is reqlaced by

$$\left.\begin{aligned}(n_1{}^2-1+q)-qB_3&=0\\ -qB_1+(n^2-9)B_3&=0\end{aligned}\right\} \tag{17}$$

where $n_1{}^2=a-\dfrac{3bB^2}{2\lambda+4}$, and we have

$$n_1{}^4-(10-q)\, n_1{}^2+9(1-q)-q^2=0 \tag{18}$$

Roots for which are $9-q^2/8$, $1-q+q^2/8$. We accept the later being consistent with the experiment. The amplitude B is now given by

$$B^2=(B_1+B_3)^2=\frac{(4+2\lambda)}{3b}\,(-1+q+a-q^2/8) \tag{19}$$

We may compare with [1, p. 124], from where we have

$A_1=\pm 2\,(q/b)^{\frac{1}{2}}$. Now if we neglect q^2 and q^3 in (14) and let $a=1-2q$, $A=2\,(q/b)^{\frac{1}{2}}$ and for $a=1+2q$, $B=2\,(q/b)^{\frac{1}{2}}$ as in [1].

As shown in [1] art. 7. 231, the parametric point of (5) lies on the characteristic curve a_1 for Mathieu's equation (8). Since

$$a_1=n^2 = 1+q-\frac{q^2}{8}-\frac{q^3}{64}\dots \tag{20}$$

and parametric point (n^2, q) satisfies it. This is in agreement with a more accurate analysis made by taking larger number of terms of the Fourier series in (6) as a solution of (5). The second order approximation taken in [1, p.123] fails to bring out this property.

Since n^2 is nearly unity and the point (n^2, q) lies on the characteristic curve a_1 for (8) the solution becomes, in terms of Mathien functions,

$$y = C\, Ce_1\,(z, q) \quad \text{or} \quad C\, Se_1\,(z, q)$$

Constant C is determined from the initial values. It is clear that the initial value of the amplitude is to Satisfy (14) or (19) as the case be for the steady oscillations. Hence the initial values of y at $z=0$ are obtained from (14) or (19). For a more accurate value of the amplitudes one may use the property that $n^2=a_1$, hence

$$A^2 = \frac{2\ (\lambda+2)}{3b}\,(a_1-a)$$

a_1 being given from tables for values of q. A similar manipulation in case of B^2 also will hold. For small oscillations $\lambda=0$ and for large oscillations $\lambda=0\ 1$ yield good results.

4. PENDULUM WITH OSCILLATING SUPPORT :

Next we consider the pendulum with movable support. As shown in [1, p. 133], such a pendulum of length l, with a bob of mass m at the free end is suspended from a point on a long flexible cantilever with a large load m_1 at the free end. If the periodic time of the pendulum is twice that of the bar, the amplitude builds up and remains constant, apart from a slow decay due to damping. Introducing a damping term motion is given by

$$\theta'' + 2k\,\theta' + (a-2q\ \cos 2z)\sin\theta = 0 \tag{21}$$

McLachlan [1] solves this by approximating $\sin\theta$ as

$$\sin\theta = \theta - \theta^3/6. \tag{22}$$

We shall see that a cublic approximation employing ultraspherical polynomial leads to a better approximation. Now on the interval $(-A, A)$ where A is amplitude [2]

$$[\sin\ \theta]_*=\left(\frac{2}{A}\right)^{\lambda+1}\Gamma(\lambda+2)\Big[\ J_{(\lambda+1)}(A)+(\lambda+3)J_{\lambda+3}(A)\ \Big]\theta$$

$$-\left(\frac{2}{A}\right)^{\lambda+3}\Big[\frac{\Gamma(\lambda+4)}{6}\ J_{\lambda+3}(A)\ \Big]\theta^3 \qquad (23)$$

We expect to get better results from this approximation since (24) is closer to sin θ than (22). We put (23) as

$$(\sin\ \theta)_*=\alpha\theta-\beta\theta^3 \qquad (24)$$

The equation (21) now becomes,

$$\theta''+2k\theta'+(a-2q\ \cos\ 2z)\ (\alpha\theta-\beta\theta^3)=0 \qquad (25)$$

Now seeking a subharmonic of the order 2, we develop a first approximation,

$$\theta=A_1\ \sin z+B_1\ \cos z. \qquad (26)$$

Substituting into (25) and equating to zero coefficients of sin z and cos z, one obtains

$$A_1[\alpha(a+q)-1]-\frac{3a\ \beta\ A_1}{4}A^2-q\beta A_1{}^3-2kB_1=0$$

$$B_1[\alpha(a-q)-1]-\frac{3a\ \beta\ B_1}{4}A^2+q\beta B_1{}^3+2kB_1=0 \qquad (27)$$

Droping the cubic terms in A_1 and B_1, the pair becomes

$$\left.\begin{array}{l}\left[\left\{\alpha(a+q)-1\ \right\}-\frac{3a\ \beta\ A^2}{4}\right]A_1-2k\ B_1=0\\ \left[\left\{\alpha(a-q)-1\ \right\}-\frac{3a\ \beta\ A^2}{4}\right]B_1+2k\ A_1=0\end{array}\right\} \qquad (28)$$

which gives,

$$\left[\alpha(a+q)-1-\ \frac{3a\ \beta\ A^2}{4}\right]\left[\alpha(a-q)-1-\ \frac{3a\ \beta\ A^2}{4}\right]+4k^2=0 \qquad (29)$$

that is

$$\alpha a-1-\frac{3a\,\beta\,A^2}{4}=\pm\sqrt{\alpha^2q^2-4k^2} \tag{30}$$

which finally leads to,

$$A^2=\frac{3\,\alpha}{3a\,\beta}\left[\sqrt{q^2-\frac{4k^2}{\alpha}}+\left(a-\frac{1}{\alpha}\right)\right] \tag{31}$$

For A to be real, we must have $\alpha q^2>4k^2$. This becomes (8), [p. 134 in 1] for $\alpha=1$, and $\beta=\frac{1}{6}$. Getting A^2 from (31) is a matter of some difficulty since α and β both are functions of A^2. Their approximate forms being, for $\lambda=0$,

$$\alpha=\left(1-\frac{A^2}{384}\right)\text{ and }\beta=\tfrac{1}{6}\left(1-\frac{A^2}{16}\right) \tag{32}$$

First we get A^2 from (31) with $\alpha=1$. $\beta=\frac{1}{6}$ and find α and β from (32). Then use them at (31) to find A^2 more accurately. Now lctting $k=0$, with $\theta=A\sin z$, (25) gives,

$$\theta''+\left[a\alpha-\frac{\beta\,A^2}{2}(q+a)-2\left\{\alpha q-\frac{\beta\,A^2}{2}\left(q+\frac{a}{\alpha}\right)\right\}\cos 2z\right]\theta=0 \tag{33}$$

with $a=1$,

$$A^2=\frac{4\,\alpha}{3\,\beta}\left(q+1-\frac{1}{\alpha}\right) \tag{34}$$

So the parametric point, for (25) when k=0, becomes

$$a=\left[\frac{\alpha}{3}(1-4q)+\frac{2}{3}(q+1)\right],\quad q=\left[\frac{2q}{3}+\frac{\alpha-1}{2}\right] \tag{35}$$

For $\alpha=1$, this is $\left(1-\frac{2q}{3},\frac{2q}{3}\right)$ as in [1, p.135]

which lies on the characteristic curve b_1 for (8). But the point (a, q) will now be in the stable region between curves a_0, b_1 in diagram 36 in [1, p. 114] since the curve b_1 is given by

$$b_1=1+q \tag{36}$$

and $$b-1-q<0 \tag{37}$$

This removes the ambiguity in [1, p. 135, art. 7·413].

5. STABILITY ASPECTS OF THE EQUIVALENT EQUATION :

The stability behaviour of a close equivalent equation forecasts the stability behaviour of the original nonlinear differential equation. First we consider the nonlinear differential equation.

$$\theta''+(a-2q\cos 2z)\,\theta+f(\theta)=0 \tag{38}$$

The polynomial linearisation of $f(\theta)$ will give

$$[f(\theta)]_* = P\theta \tag{39}$$

and equivalent linearisation of (38) becomes,

$$\theta''+(a+P-2q\cos 2z)\theta=0 \tag{40}$$

Now letting,

$$\theta=A(z)\cos z+B(z)\sin z \tag{41}$$

assuming A and B to be slow variables in $z=wt$,

such that A'' and B'' may be negligible. We get on substituting (41) A and B as variables into (40),

$$(2B'-A)\cos z-(2A'+B)\sin z+(a+P)(A\cos z+B\sin z)$$
$$-q(A\cos z-B\sin z+A\cos 3z+B\sin 3z)=0 \tag{42}$$

Equating to zero coefficients of $\cos z$ and $\sin z$,

$$\left.\begin{aligned} 2B'-A+A(a+P)-qA &= 0 \\ -2A'-B+(a+P)B+qB &=0 \end{aligned}\right\} \tag{43}$$

which give

$$\frac{dA}{dB}=\frac{B(a+q-1+P)}{-A(a+P-1-q)} \tag{44}$$

Since (44) is an equation of the type

$$\frac{dy}{dv}=\frac{\alpha v+\beta y}{\lambda v+\delta y} \tag{45}$$

We apply criteria of stability at the singular points in 9.20 from [1, p. 189],

Now $\alpha=a+P+q-1$, $\beta=\gamma=0$, $\delta=-[a+P-1-q]$

We get $\quad \alpha\delta-\beta\gamma = -(a+P-1)^2-q^2]$ (46)

and $\quad D=4\,[q^2-(a+P-1)^2] = (\beta-\gamma)^2+4\alpha\delta$ (47)

Thus we observe,

(i) $D>O$, for $q>(a+P-1)$ or $q>1-a-P$, which also makes $\alpha\delta-\beta\gamma$ $+ve$. This gives that the singular point is a Col. Hence motion once started will be unstable. We may say that (40) will be unstable if

$$q+1-a>P \quad \text{or} \quad P>1-a-q. \tag{48}$$

(ii) $D<0$, $q<(a+P-1)$ and $q<-(a+P-1)$

which shows that for stable oscillations,

$$P>q+1-a \quad \text{or} \quad 1-q-a>P \tag{49}$$

In this case the motion dies out.

(iii) $D=0$, since $\beta+\gamma=0$, we have a neutral case, Oscillations have a period 2π in z. This happens when $P=q+1-a$ or $1-q-a$. (50)

Equations (43) yield the values of A and B as

$$A = A_1 e^{\mu z}+A_2\, e^{-\mu z} \text{ and} \tag{51}$$

$$B = B_1\, e^{\mu z}+B_2\, e^{-\mu z}$$

where $\quad \mu = \frac{1}{2}\,[q^2-(a+P-1)^2]^{\frac{1}{2}}$

Application of (iii) above shows that the amplitude becomes a constant and steady state is obtained.

These conditions are amplitude dependent and explain Melde's experiment where if amplitude is less than a certain A_0 the oscillation dies out or if greater than A_0 the string breaks. While first order analysis and application of Poincare's criteria leads only to amplitude independent results [1, Chap. 9]

Example :

The determination of the stability of the subharmonic solution $y=\left[\frac{4f}{\beta}\right]^{1/3}$ cos ωt, of the forced Duffing's equation, [1]

$$\ddot{y}+ay+\beta y^3=f\cos 3\,\omega t \tag{52}$$

where $a, \beta, f>0$ and $w^2=a+3\left(\frac{f^2\beta}{4}\right)^{1/3}$ leads to

Mathieu's equation (8) [1]. As a matter of fact it is of the type (38), but neglecting small quantities it reduces to (8). Instead of neglecting small terms completely we construct an equivalent linear equation as before. We substitute $y+v$ for y in (52), v being a small change in y. This leads to,

$$\ddot{v}+(a+3\beta y^2)\,v+\beta v^3+3\beta v^2 y=0 \tag{53}$$

Applying Chebyshev polynomials to approximate $\beta v^3+3\beta y v^2$ we get, A being the amplitude of v,

$$\ddot{v}\left[w^2+3\left(\frac{\beta f^2}{4}\right)^{1/3}+6\left(\frac{\beta f^2}{4}\right)^{1/3}\cos 2\,\omega t\right]v+\frac{3\,\beta\,A^2}{4}\,v=0$$

now letting $\omega t=z$ we have, (54)

$$v''+\left[1+\frac{3}{w^2}\left(\frac{\beta f^2}{4}\right)^{1/3}+\frac{3\beta A^2}{4w^2}+\frac{6}{w^2}\left(\frac{\beta f^2}{4}\right)^{1/3}\cos 2z\right]v=0 \tag{55}$$

with $q=\frac{3}{w^2}\left(\frac{\beta f^2}{4}\right)^{\frac{1}{3}}$. Comparing with (40)

$$a+P=1+q+\frac{w^4q^3A^2}{qf^2} \tag{56}$$

that is, $a+P>1+q$ which fulfills (49) for stability.

6. Having compared the polynomially approximated equation analytically with the Fourier series approximations we proceed to apply the procedure to any given case. The most general form being

$$y''+2ky'+(a-2q\cos 2z)\,y+f\,(y)=0 \tag{57}$$

becomes on approximation,

$$y''+2ky'+(a+P-2q\cos 2z)y=0 \tag{58}$$

and letting $.=e^{-kz}u\ (z)$ this becomes,

$$u''+(a+P-k^2-2q\cos 2z)u=0 \tag{59}$$

We take (41) as a solution for this where A and B are giuen by their values in (51) and $\mu=\frac{1}{2}[q^2-(a+P-k^2-1^2)]^{\frac{1}{2}}$.

This leads to

$$y=e^{-kz}\left[(A_1e^{\mu z}+A_2e^{-\mu z})\cos z+(B_1e^{\mu z}+B_2e^{-\mu z})\sin z\right] \tag{60}$$

For the system to acquire a steady state $\mu=k$ and coefficients of A_2 and $B_2\to 0$ as $z\to\infty$. If $\mu=k$, we have

$$P=\sqrt{q^2-4k^2}+1-a+k^2. \tag{61}$$

This gives the initial amplitude to start the system with. For the reality of this $q>4k^2$:

Example :

Now if we have $a=\cdot 9314$, $b=0\cdot 1$, $q=\cdot 16$, $k=\cdot 08$ and $f(y)=by^3$, we shall have, for $\lambda=0$,

$$P=\frac{0\cdot 3}{4}(1)^2 = \cdot 075$$

If we expect that the initial amplitude, $A=1$, remains constant since $k=q/2$ we take the interval as $(-1, 1)$.

$$a+P-k^2=\cdot 9314+\cdot 075-\cdot 0064=1$$

and (59) becomes,

$$u''+(1-\cdot 32\cos 2z)u=0 \tag{62}$$

The parametric point (1, 0·16) lies in the unstable region of Fig. 38, p. 114 in [1]. From [5, p. 122], the unstable solution of (62) is

$$u=Ce^{\cdot 08z}(\cos z-0\cdot 021\cos 3z+0\cdot 94\sin z-\cdot 0175\sin 3z) \tag{63}$$

and $y=e^{-kz}\,u(z)$ is,

$$y=C(\cos z-\cdot021\cos 3z+0\cdot94\sin z-\cdot0175\sin 3z) \tag{64}$$

with $y=A=1$ at $t=0$ and $\frac{dy}{dz}=0$, C is known $=\frac{1}{0.979}$

Discussion :

Ultraspherical polynomial approximations linearise the nonlinear $f(x)$ into a function of the amplitude A, giving equivalent linearised equation so that 'a' of the Mathieu's linearised equation is now replaced by $a+P$. This causes two things (i) it makes the period amplitude dependent which is verified by employing the perturbation method, (ii) it makes the parametric point amplitude dependent, so that the stability now depends on the amplitude also, as is clear from the stability behaviour of Melde's experiment [1. P. 120], If the amplitude of the prong is less than a certain value say A_0, the oscillations die away or when it is greater than A_0, the amplitude increases till the string breaks. In the discussions made above the amplitude was small and we have used $\lambda=0$ for linearising polynomial. We made comparisons of our results with those of [1] at various points and have shown that the later are only first approximations obtainable on dropping higher power terms involved. Since the analytical comparisons are good enough direct numerical applications are likely to be quite accurate. We feel it can be safely assumed that even if a were negative, we may choose b and A, so that $n^2=a+\frac{3bA^2}{2(\lambda+2)}$ may be positive so as to yield oscillatory motion.

ACKNOWLEDGEMENTS

I am thankful to Dr. R. M. Garde, Reader, Department of applied Mathematics, Government Eugineeiing College. Jabalpur for his useful suggestions and discussions in the preparation of this paper. I am also thankful to Dr. B. R. Bhonsle, D. Sc., Professor of mathematics, for his suggestions.

REFERENCES

1. N. W. Mclachlan, "Ordinary nonlinear differential equations in Engineering and Physical Sciences", Oxford, Second Edition, 1958.

2. H. H. Denman and Y. K. Liu, "Application of ultraspherical polynomials to nonlinear oscillations II—Free oscillations" Q. Appl. Maths., **22**, p. 273 to 292 (1965).

3. Garde, R. M., "Application of Gegenbauer polynomials to nonlinear oscillations—Forced and free oscillations without damping," The Indian Journal of Maths. 7 (2), p. 111-117 (1965).

4. Rainville, E. D., "Special functions", Mac millan Co., New York (1960).

5. N. W. Mc Lachlan, "Theory and application of Mathieu functions" (Oxford, 7951).

Jñānabha, Sect. A, Vol. 2, July 1972.

GENE REGULATION IN HIGHER ORGANISMS

By

H. K. SRIVASTAVA

(**Professor of Agricultural Botany, Institute of Advanced Studies, Meerut University, Meerut.**)

(*Received on July 3, 1973*)

1. INTRODUCTION :

It is generally considered that the genetic information in most of the complete cells of a complex metazoan organism is identical with that of every other cell. Thus the tremendous diversity of cell phenotypes found in an organism must be derived from; (a) cells expressing only a limited amount of its full genetic potential and (b) cells expressing different portions of their genome. Cell differentiation is based almost certainly on the regulation of gene activity, so that for each state of differentiation a certain set of genes is active in transcription (transfer of genetic information from DNA to RNA) while other genes remain inactive (Britten and Davidson, 1969). It has been established in bacteria and viruses that DNA is the primary genetic material and that genetic information is expressed through an intermediate messanger RNA, which acts as direct template for protein synthesis (Watson, 1969). A bacterial gene is "active" only when its corresponding messenger RNA is produced. Therefore, regulation of gene function depends on controlling the synthesis of specific messenger RNA's. Important regulatory processes occur at all levels of biological organisation. The genes controlling lactose metabolism in bacteriophage and *Escherichia coli* are activated by a specific "Inducer" that combines with the repressor, causing the latter to detach from the DNA and permitting the messenger RNA to be synthesized (Jacob and Monod, 1961; Ptashne, 1967). The formation of messenger RNA for the group of genes controlling lactose metabolism is further subjected to inhibition by the attachment of specific protein repressors to specific regulatory genes on the chromosome.

The elegance of these ideas and the clarity with which they have subsequently been verified in prokaryotes have led their widespread acceptance as an explanation for gene regulation in higher organisms. This acceptance has been bolstered by the demonstration that the fundamental mechanism of information flow (DNA————————→ RNA ————————→ Protein) in higher organism is virtually identical with that in micro-organism. Thus, in both cases, DNA is the primary genetic material, genetic information is expressed by transcription into RNA, and the codes assigning specific RNA triplets to specific amino acids of proteins are essentially indentical (Marshall *et al.*, 1967). However, certain features of the structure and function of the genetic apparatus of eukaryotic cells are very different from their bacterial counterparts. These differences raise the possibility that the mechanism which regulate gene expression in the two cases may also be significantly different. Some of the expremental evidences relating to these features are :

1. Change in state of differentiation in higher organism is often mediated by simple, external signals, as, for example, in the action of hormones or embryonic induction agents (Tomkins *et al* 1969; Tomkins and Martin, 1970).

2. A given state of differentiation tends to require the integrated activation of a very large number of noncontiguous genes (Epstein and Beckwith, 1968).

3. There exists a significant class of genomic sequences which are transcribed into RNA in the nuclei of higher organisms but appear to be absent from cytoplasmic RNA's or only a small fraction of these ever reaches the cytoplasm.

4. The genome present in higher organism is extremely large, compared to that in prokaryotes.

5. This genome differs strikingly from the bacterial genome due to the presence of a large fraction of repetitive nuclotide sequences which are scattered throughout the genome (Britten and Davidson, 1971).

6. Furthermore, these repetitive sequences are transcribed in differentiated cells according to cell type-specific patterns.

7. The cells of higher organisms also appear to use more complex mechanisms for processing nascent polypeptides than bacteria do. For example,

the initial product of translation of the poliovirus RNA seems to be a single, long polypeptide chain that is cleaved into the smaller viral components (Jacobson and Baltimore, 1968).

The extent of current interest in regulatory process is documented by the many articles and reviews on this subject. Recent work on protein synthesis and the genetic code has demonstrated the main outline of the pathway by which geneic information is translated in to the amino acid sequences of proteins. In contrast, the mechanisms which regulate gene expression and function are much less understood in higher organisms. It is, therefore, proposed to discuss current aspects of gene regulation in eukaryotes with particular regard to the operon. The reader's attention is also directed to other recent articles on regulation (Epstein and Beckwith 1968), Clever, 1968, Martin, 1969, Lewin, 1970).

II. THE OPERON CONCEPT :

One of the useful concepts in the study of biological regulation has been the concept of the operon, which was formulated in bacteria in three different kinds; 1). The clustering on the chromosome of genes that determine the enzymes of a biochemical pathway 2). Coordinate repression and depression of the enzymes controlled by these clusters 3). Specific regulatory mutations which affect the expression of the cluster of genes The crystalization and clear exposition of the operon theory were presented by Jacob and Monod (1961). The operon has also been reviewed by Ames and Martin (1964). Generally aceepted characteristics of the bacterial operon are :—

A. CONTROL OF A GENE CLUSTER

An operon consists of a cluster of genes that are activated or inactivated as a unit

B. INDUCTION AND REPRESSION

Certain specific small molecules (inducers or Co-repressors) that are substrates or products of the metabolic pathway-the enzymes of which are coded by the genes of the operon-can cause a rapid increase (induction) or decrease (repression) in the rate of synthesis of all the enzymes of the operon.

Jacob an Monod both of the Pasteur Institute in Paris proposed in 1961 an ingenious model to explain induction and repression. According to them there are, in general, two types of genes, the structural genes (SG) which determine the amino acid sequence in enzyme proteins and the regulator gene (RG) which produces a cytoplasmic protein called repressor (R). Operator (o) is considered to be the initiation point at a certain part of DNA adjacent to structural genes. The operator and structural gene unit is called an operon. The RG functions by forming a protein molecule called as repressors. Repressors by themselves are assumed to be nonfunctional. However, when a repressor attaches itself to an inducer usually a substrate molecule, the repressor is assumed to become inactivated and thus connot attach itself to the operator and hence does not block the messenger-RNA transcription. In a repressible system. the repressor attaches itself to a specific corepressor, usually a specific metabolite, and is assumed to become activated and attaches itself to its specific operator gene, thereby blocking the transcription of specific messenger RNA.

C. COORDINATE CONTROL

The ratio of the amount of an enzyme to that of any other enzyme of the same operon is constant for given growth conditions, regardless of the extent of repression or induction (Ames and Garry, 1959).

D. POLYGENIC MESSAGE

The operon is the chromosomal unit of transcription for the formation of a melecule of messenger RNA of the same length as the operon. Experiments have shown that transcription of DNA into RNA begins at the operator end of an operon (Alpers and Tomkins, 1966) and that translation of messenger RNA into protein also begins at the operator end of the message (Berberich, 1966). Since it is known that polypeptide chains are assembled sequentially from amino-to carboxyl terminal ends (Maughton and Zintzis, 1962), it appears that genes are oriented in sequential order so that the operon end of a cistron codes from the aminoterminal end of the corresponding protein molecule.

E. POLARITY

The polygenic message is translated unidirectionally by ribosomes that starts at the operator end. Certain mutations in the genes of an operon, besides, causing a loss of function of the affected gene, also cause polarity, a relative decrease in the amount of all enzymes specified by the genes located on the side of the mutation distal to the operator (Jacob and Monod, 1961). Polarity results from the occurrence of either "nonsense" mutations-the polypeptide-terminating triplets UAG or UAA (Yanofsky and Ito, 1966) or frame shift mutations (Banks, 1971) which give rise to nonsense triplets as a result of the shift in the reading frame during the translation of messenger RNA.

F. REGULATOR MUTATIONS

For most operons so far examined, several classes of mutations have been discovered which activate the operon in the absence of the low molecular weight inducer or co-repressor, usually required. These mutations are of two types: (i) The operator constitutive types (O), which maps at the starting end of the operon and affects only contiguous genes on the same chromosome; and (ii) the regulator gene type (e.g.i. in the lac operon) which lies on a different part of the chromosome and is recessive to the wild type. Because of these dominance relation-ships, it is generally concluded that the regulator genes produce a substance, the repressor, which inhibits operon function. The operator is to be the site of action of the repressor.

G. THE APO-REPRESSOR

The apo-repressor is a protein since it is able to exist in active or inactive forms as the results of its interaction with a small molecule co-inducer. This is most readily explained as an allosteric effect, the protein possessing a binding site for the operator which is affected by the binding of the co-inducer at a second, allosteric site. This can also most easily account for the altered behaviour of i gene regulatory mutants-is mutants for instance, are those which have lost this second, Co-inducer, bindiug site. Constitutive regulator mutants of both the alkaline phosphatase and lac systems of *E. Coli.* can be suppressed by an external nonsense condon suppressor; this phenomenon is known to be exerted at the level of translation into protein. If the apo-repressor protein interacts with DNA to prevent transcription, rather than with RNA to prevent translation, it should bind *in vitro* to DNA containing its operator, but not to DNA lacking the recognition site.

H. THE PROMOTOR

When Jacob and Monod proposed their theory of the operon in 1961, they suggested that a small segment of DNA, called the operator, located at one end of a series of genes, might control their function. This means that the oprator may possess the distinct properties of both comprising the recognition site of the apo-repressor and of initiating the transcription (or possibly translation) of the RNA. This would be the site where RNA polymerase might start transcribing the genes of the operon into a polygenic messanger RNA and where the repressor protein which controls the structural genes might act to stop transcription. However, recent experiments have revealed that the operator is not the site for the initiation of transcription, but that some separate locus must bear responsibility for this (Lewin, 1970). Later, more mutants of the lactose operon of *E. coli* were identified to divide this region into two parts; the operator at which repressor protein binds and the promotor at which RNA polymerase apparently binds and starts transcription. A third site has recently been added, for the promotor itself may comprise an RNA polymerase binding site and a region where the cyclc AMP system acts to switch on inducible operons. All these sites are adjacent, however, in an order which suggest simple model for the control of gene expression. The binding of apo-repressor to the operator at a site located between the promotor and the structurrl genes suggests that it blocks the progress of RNA polymerase into the operon, and thus prevents it from transcribing the structural genes (Arditti *et al,* 1968). It is difficult to investigase *in vivo* the interactions which take place between apo-repressor and operator, and between RNA polymerase and promotor, and this system should assist determination of the precise mode of action of the apo-repressor. Since this segment of DNA contains only a single promotor, it should also make possible further investigations into the action of the sigma factor and the initiation of transcription by RNA polymerase.

I. SIGMA FACTOR

Sigma factor is known to be a positive control element in the development of bacteriophase T_4, promoting transcription of a specific class of T_4 genes. The ability of *E. coli* RNA polymerase to transcribe native DNA depends on the presence of a protein factor (sigma factor) which is normally found in association with the enzyme. The sigma factor acts at the very first step in initation, namely the specific binding of the enzyme to the promotor site to form a "preinitiation complex" (Bantz and Bantz, 1970).

The operon concept as developed in bacteria is not easily adapted to higher organisms. In higher organisms, however, the uuits of transcription and translation are not always of the same length. It is not clear as to which aspects of the complex regulatory mechanisms the term "operon" should refer.

Transcriptional regulation in animal cells occurs at the level of the activation or inactivation of entire chromosomes, large chromosomal segments, and possibly smaller units. Besides, translational control at the level of messenger RNA, transfer RNA, ribosomal function, and protein synthesis also occurs. Selection of messengers, perhaps by their stabilization, could be regulated because more of the genome is transcribed than actually functions as messenger RNA. Biochemical arguments for the existence of operons in animal cells have been based on co-ordinate rises and falls in enzyme levels but this type of evidence in not very significant. Despite the difficulty of transferring the idea of an operon, it is important to evaluate the role of gene clusters in higher organisms. In the operon, genetic linkage doviously allows functionally related genes to be activated by a common mechanism. Several examples of operon-like behaviour have been discovered in eukaryotes. The 3 genes governing the enzymes of the galactose pathway in yeast are clustered and controlled together by several unlike regulatory elements (Douglas and Hawthorne, 1966). Three genes of the histidine biosynthetic pathway in yeast are clustered, as are 5 genes of the aromatic pathways in Neurospora (Giles, 1965).

III. REPETITIVE DNA SEQUENCES AND GENETIC COMPLEXITY:—

Experiments in recent years have demonstrated that the genome of higher organisms contains a large amount of order which is manifested as repeated nuclectide sequences in the DNA. The possible implications of repetitive DNA sequences for the mechanism of control of genetic activity in higher organism have been considered (Britten and Davidson, 1969). Repetitive DNA sequences involved in regulation have also been invoked by Georgiev (1969), who has suggested that messenger RNA is transcribed together with adjacent repetitive sequences. Large populations of repeated DNA sequences appear to be present universally in the genomes of eukaryotes sequences above the fungi. Evidence indicates that new families of repeated DNA requences have been incorporated throughout evolution (Rice, 1971). repeated sequence relationship have been detected between organisms whose ancestors diverged hundreds of millions of years ago. For example, 5 percent sequence homology exists between the repetitive DNA's of fish and primates, and 10 percent between the repetitive DNA's of birds and primates. Other lines of evidence also indicate that the incorporation of new families of repetitive sequences can occur suddenly on an evolutionary time scale. Such sudden replicative events are termed "Saltatory replications" (Britten and Kohne, 1968). Although the precise mechanism of saltatory replications is not known, the following processes seem necessary (Britten and Davidson, 1971). (1) Many copies are made of DNA sequences and appear in the germ cells of certain individuals. (2). The copies are some how integrated into the genome so that they are duplicated. (3) Over a sufficient period of time they are disseminated throughout the population and its evolutionary descendants. Dissemination could result from their association with a favourable genetic element or simply because of their multiplicity. (4) Individual sequences become scattered among many chromosomes and is transcribed intimately along the length of the DNA of the genome. (5) The growth of the family of repeated DNA is eventually terminated or controlled. Subsequently, individual members of the sequence families would diverge from each other through base substitutions. Also, the length of the recognisable related regions is presumably reduced by the events of rearrangement which led to their interspersion throughout the genome.

While this may be an adequate summary of the broad outlines of the history of repeated DNA families, there is subtle or no information on function and the resulting selection pressures of repeated DNA sequences in higher organisms. Britten and Davidson (1971) also commented on possible mechanism of saltatory replication, although there is again no useful experimental evidence. Four classes of events present themselves as possibilities:

(i) Erratic behaviour of a DNA polymerase perhaps caught in a closed short loop without adequate termination controls; (2) Geometric growth of a series of short duplicated sequence due to unequal crossing over; (3) the excessive replication of some nuclear element analogous to the episomes of bacteria (not yet observed in higher organisms); (4) the integration into the genome of many copies of a viral genome or viral-borne sequence.

IV. THE MODEL OF GENEREGULATION:

An elegant model of gene regulation in ukaryotes was first advanced by Britten and Davidson (1969). Five elements are used in the model. Genes which specify cellular products such as enzymes (i.e., are regulated rather than regulatory in nature) are termed "Producer" genes. A set of producer genes whose products carry out a closely related set of functions is termed a "battery". An example of a battery would be the producer genes coding for the group of liver enzymes required for purine synthesis, which might well be activated simultaneously. The activity of the producer genes of a battery is controlled through the interaction of particular diffusible regulatory molecules with a DNA sequence termed the "receptor" gene contigous to each producer gene of the battery. Therefore, the genes of a battery may be located at a distance from each other, or even occur on different chromosomes. The diffusible regulatory molecules of the model are derived from "integrator" genes. These regulatory molecules may ba the RNA transcribed from the integrator sequences or they may be proteins derived by translation of such RNA molecules. In either case the RNA transcribed from the integrator genes is reffered as "activator" RNA, since it bears information to be used for gene activation.

There are integrator genes whlch are transcribed in response to substances arising elsewhere. such as those hormones which affect gene activity. These substances are postulated to interact with a "sensor" structure adjacent to the integrator genes and thereby initiate transcription. Since such substances in goneral do not have an affinity for particular DNA sequence, the sensor structures must include macromolecules serving as specific binding sites.

GENE

The gene is the basic unit of life, capable of self reproduction which is the prime criterion. The word "gene" remains into use since long time though most geneticists do not like using this word any more, but its concept has entirely been changing during the last two or three decades. The classical concept of a gene assumed it to be a anitany particle by criteria involving each of the three kinds of observations: (i) a gene is a unit of chromosomal structure not subdivisible by chromosomal breakage or crossing over, (ii) A gene is a unit of physiological function or expression, and (iii) a gene is a unit of mutation. The smallest gene element that is interchangeable (but not divisable) by genetic recombination is termed as "Recon". The "mution" is defined as the smallest gene element that, when altered, can give rise to a mutant form of the organism. The "cistron" is considered to be a genetic unit of function sub-divisable into ultimate units of recombination or recon. However, the more recent difinition of the gene considers it to be a region of the genome with a narrowly definable or elementary function. It need not contain information for specifying the primary structure of a protein.

PRODUCER GENE

A region of the genome transcribed to yield a template RNA molecule or other species of RNA molecules (except those engaged directly in genomic regulation) is known as producer gene. This term is used in a manner analogous to that in which the term "structural gene" had been used in the context of certain bacterial regulation systems. Products of the producer gene include all RNA's other than those exclusively performing genomic regulation by recognition of a specific sequence. Among producer genes, for example are the genes on which the messenger RNA template for a hemoglobin subunit is synthesized, and also the genes on which transfer RNA molecules are synthesized.

RECEPTOR GENE

A DNA sequence linked to a producer gene which causes transcription of the producer gene to occur when a sequence spicific complex is formed between the receptor sequence and an RNA molecule (called an activator RNA). A receptor complex may include the DNA, histones, polymerases, and so forth. This model is concerned primarily with interrelations among the DNA sequences present in the genome.

ACTIVATOR RNA

The RNA molecules which form a sequence-specific complex with the receptor genes linked to producer genes. The complex suggested here is between native double stranded DNA and a single-stranded RNA molecule. The role proposed for activator RNA could well be carried out by protein molecules coded by these RNA's without charging the formal structure of the model.

INTEGRATOR GENES

The function of this gene is to synthesize an activator-RNA. The term integrator is intended to emphasize the role of these genes in leading, by way of their activator KNA's, to the co-ordinated activity of a number of producer genes. A set of linked integrator genes is activated together in response to a specific initiating event, resulting in the concerted activity of a number of producer genes not sharing a given receptor gene sequence.

SENSOR GENE

A gene sequence serving as a binding site for agents which induce the occurrence of specific patterns of activity in the genome is called sensor gene. Binding of these inducing agents is a sequence specific phenomenon dependent on the sensor gene sequence, and it results in the activation of the integrator gene or genes linked to the sensor gene. Such agents include, for example, hormones and other molecules active in intercellular relations as well as in intracellular control. Most of them will not bind to sensor gene DNA, and an intermediary structure such as a specific protein molecule will be required. This structure must complex with the inducing agent and must bind to the sensor gene DNA in a sequence specific way.

BATTERY OF GENES

These are the set of producer genes whose products carry out a closely related set of functions. A particular cell state will usually require the operation of many batteries. Batteries are vital to gene function and gene regulation.

An important function for repeated DNA sequences in higher organisms has been emphsized by this model. It appears that the existence of gene control functions in the cell (which can hardly be denied) logically requires the existence of repeated DNA sequence in the genome. The only way in which this implication could be totally avoided would be, to propose that all gene regulation operates through a kind of "falling domino" process in which each gene activates or represses a single other gene (Britten and Davidson, 1969). If, on the other hand, one regulatory gene is assumed to affect more than one regulated gene, then regions of sequence similarity which provide the molecular basis for the recognition process are implied among the regulated genes. This argument in itself suggests that some posts of the transcription from unrepeted sequences which is experimentally observed is likely to be involved in cellular control process.

NON-REPETITIVE DNA AND GENETIC COMPLEXITY :—

The term "complexity" is useful to express the amount of diverse DNA sequence in a given preparation and it is defined as the number of nuclotide pairs present in a single set of all the diverse sequences. In other words, the complexity is equal to the genome size as long as repeated sequences are absent. If repeated sequences are present an abvious extension is to count all of the nucleotide pairs in a single copy of each of the repeated sequences. This is an apparently simple definition. However, the repeated sequences are rarely identical to each other. A very wide range of degrees of similarity or difference occurs among the members of most sets of related sequences. No direct evidence is available to show that the relationships among the members of the set of sequences are actually used be the organism, although there are theoritical reasons (Hood *et al.*, 1970; Britten and Davidson, 1969) for believing that they may be. In the cases of salmon (Britten and Kohne, 1968), Wheat (Bendich and McCarthy, 1970) and the urodele *Necturus* (Strauss, 1971), more then 80 percent of the genome appears to be repetitive at the customary experimental criteria. However, other conditions particularly less stringent criteria, may supply unexpected insight into the history and patterns of occurrence of repeated DNA (Britten and Davidson. 1971).

The most direct approach to wards an assessment of the function of non-repeated DNA is through measurements of transcription of RNA from these sequence fractions. It is known that repeated sequences are transcribed into RNA, although it is not known how many of the individual member sequences of a set are transcribed. The complexity of the DNA which is observed to be transcribed yields a lower limit for the functional complexity of the genes expressed in any given circumstances. For newborn mice (Gelderman and Rake, 1971) it appears that more than 12 percent of the none-repetitive sequences in the genomes are transcribed, and thus the functional complexity of the mouse genome at this stage is at least 4 x 10^8 nucleotide pairs. This very large number is not yet interpretable, since if all of this RNA were messenger it would code for nearly a million different hemoglobin-sized proteins. Other measurements (Devidson and Hough, 1971) indicate that the RNA stored in the occytes of *Xenopus* represents at least 1.2 percent of the non-repetitive DNA. The complexity of this RNA is such that it could code for 40,000 different hemoglobin:sized proteins. Thus, it appears that a significant portion of the potential information content of these genomes is actually expressed. It is important to note that in the *Xenopus* occyte, many more copies are present of repetitive sequence transcripts than of non-repetitive sequence transcripts. This observation suggests a functional distinction between the repetitive and non-repetitive sequences and of course for the RNA transcribed for them.

HORMONES AND GENE EXPRESSION

Cell differentiation involves drastic changes in cell metabolism. The synthesis of many proteins is stopped and that of new ones is begun. This type of regulation could be called "Macroregulation" in contrast to "Microregulation", where the synthesis of a particular protein, or set of genes in an operon, is specifically regulated by induction (or derepression) and repression, according to Jacob and Monod (1961). There is no doubt that the principle of microregulation operates in cellular differentiation. A hormone is an effector molecule produced in low concentration by one cell which evokes a physiological response in another. Thus, hormones are extracellular compounds and they act at a different site other than the site of their production. In multicellular organisms a "second messenger" or adenosine 3'—5'— cyclic phosphate (cyclic AMP) is responsible for action of those hormones that stimulate its synthesis.

Cyclic AMP

Cyclic AMP acts as a positive allesteric effector for the synthesis of various proteins involved in phosphorylation (Tomkins and Martin. 1970). The hormones which influences cyclic AMP is called first messenger as it is transcribed by the genes. In vertebrates, numerous classes of chemically unrelated compounds such as polypeptides, amino acids, amines, fatty acid derivatives, and steroides have hormonal activity. Certain of these substances probablya re themselves the primary intracellular effectors wherea sothers clearly function at the cell surface where they activate the membrane-bound enzyme, adenyl cyclase, and thereby stimulate the production of cyclic AMP from ATP. (Sutherland *et al.* 1968 ; Robinson *et al.* 1968). A cyclic AMP-dependent kinase has been found in muscle, brain, and a number of vertebrate and invertebrate tissues (Miyamoto *et al.* 1969; Kuo and Greengard, 1969). The macromolecular substrates for the kinase reactions are either enzymes or structural proteins such as histones.Histone phosphorylation permits increased transcription specific genes,which accounts for the cyclic AMP-and glucagon-mediated induction of liver enzymes (Wicks *et al.*,1969). In addition to affecting gene transcription,cyclic AMP can also promote messenger RNA translation since its timulates the fromation of tryptophanase in *E. coli* (Pastan and perlman; 1969). Several lines of evidence using inhibitors of RNA synthesis or mutants in the promotor region suggest that cyclic AMP specifically stimulates the initiation of transcription of the operon. Cyclic AMP may also regulate the translation of messenger RNA's perhaps by stimulating the phosphorylation of certain components for example, ribosomes initiation factors etc , involved in protein synthesis. The regulatory role of cyclic AMP in prokaryotes, however, suggests that the cyclic AMP could have been the evolutionary precursor of hormonal control in higher organisms.

TRANSCRIPTIONAL CONTROL

It is generally agreed that the fundamental cellular action of the hormones is the stimulation of certain proteins, and it appears that a specific hormone-receptor protein complex is involved in this effect. Receptor proteins are found in each target tissue, associated both with the cytoplasm and the nucleus. They appear to be heat labile proteins with sedimentation coefficients of 8 to 10 s at low ionic strength and 3 to 5 s at ionic strength greater than that of 0.3 M Kcl, (Fang et al, 1969). Little is known about the mechanism of interaction of hormone with receptors. "Two step" mechanisms for hormone interaction with a target tissue has been proposed (Tomkins and Martin, 1970). The first step involves the interaction of hormone with its specific cytoplasmic receptor. The second step is the transfer of the hormone receptor complex to the nucleus. The hormones are supposed to act as inducers by antagonizing specific gene repressors (frequently identified as histones or other chromosomal proteins) thereby increasing the rates of synthesis of specific messenger RNA's. The arguments in favour of this view, aside from the analogy with prokaryotes are : (a) Steroid hormones administered *in vivo* cause increased rate of labelled precursor incorporation into RNA (Hamilton, 1968) (b) chromatin isolated from hormone treated tissues shows increased template activity when used to direct RNA synthesis in cell-free preparation. (c) Hormonal induction is inhibited by inhibitors of RNA synthesis. (d) New species, or increased concentrations of pre-existing species, of RNA appear in hormone-treated tissue (Church and MeCarthy, 1970). (e) Nuclear localisation of some steroid hormones and their binding to chromatin (Haussler and Norman, 1969) are frequently observed (f) Chromosomal "puffing" of specific loci occurs in polytene chromosomes from ecdysone-treated insect larvae. Taken together these results constitute a strong argument in favour of a primary action of the hormones in facilitating gene transcription. At present, most of the evidence suggests that the accumulation of specific RNA's accounts for enzyme induction.

POST-TRANSCRIPTIONAL CONTROL

In addition to the arguments that steroid hormones control DNA transcription directly, the possibility has also been considered that they may function by regulating post-transcriptional events in gene expression. Hormonal control of post-transcriptional processes was first suspected because of the "Paradoxical" stimulation of induced enzyme synthesis produced when inhibitors of RNA synthesis were adminstered to rats previously injected with cortisol (Garren et al., 1964). A model proposing that gene expression is controlled both at transcriptional and post-transcriptional levels has recently been advanced (Tomkins and Martin, 1970). This model involves two genes, a structural gene (G^s) and a regulatory gene (G^r) which in dividing cells have two possible states of activity (Fig. 1). During the inducible periods of the cell cycle both genes are transcribed actively whether or not the inducer is present. In the non-inducible periods, both genes are inactive and connot be activated by the hormone inducers. During the inducible periods (when both genes are active) transcription of the structural gene messenger RNA is inhibited by a labile post-transcriptional repressor (R), a product of the regulatory gene. The hormones are known to induce enzyme synthesis by antagonizing the post-trans criptional repressor, therby stabilizing active mssenger RNA and promoting its accumulation. The evidence for a harmone-insensitive control over gene expression comes from the fact that the steroids connot stimulate enzyme synthesis during the noninducible periods. The exact mechanism of this inducer-insensitive regulation is unknown, although it is presumed to be at the level of gene transcription and to be specific rather than general.

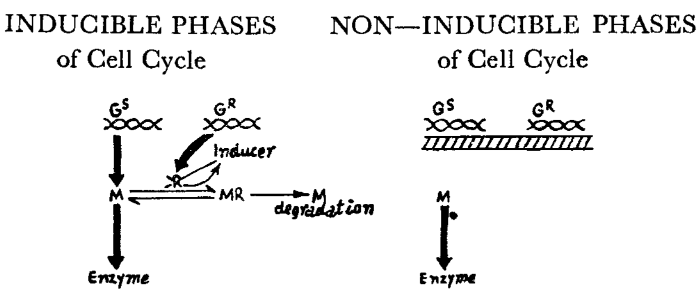

Fig. 1. Theory of homonal enzyme induction in higher organisms. The configuration shown on the left is assumed to exist during the inducible phases of the cell cycle (late G_1 and S), while that on the right, during the non-inducible phases. The G^s refers to the structural gene for the inducible enzyme, while G^r refers to the regulatory gene. During the inducible periods, G^s is transcribed and the resulting messenger RNA, M, can be translated to

form the enzyme. The G^r is likewise transcribed and its product is the labile post-transcriptional repressor, *R*. *R* combines with *M* to produce the inactive complex *MR* which leads to *M* degradation. *R* itself is labile, as shown by the arrow leading way from it. The figure shows that inducer some-how inactivates *R*. During the noninducible phases of the cycle, neither G^s nor G^r is transcribed, but pre-existing *M* can be translated. (after Tomkins et al., 1969).

VII. CONCLUSION :

In this article several questions on the nature of the mechanisms responsible for gene regulation in higher organisms have been discussed. The operon, defined in prokaryotes, is a group of contiguous genes which, on derepression, are transcribed into a single strand of messenger RNA. Both translation and transcription *proceed from the operator end of the operon. Recent results indicate that the operator is not the site for the initation of transcription, but a separate locus called the "promotor" must bear the responsibility for this. This locus could be the site where RNA polymerase is recognized and commences transcription. Furthermore, a protein factor more recently termed as "sigma" factor acts at the very first step in initiation of transcription. It helps in the binding of RNA polymerase to the promotor site to form a "preinitiation Complex" for transcription. Mutations to the triplets for chain termination in any of the genes of the operon cause polarity, a reduction in the activity of genes distal to the operator, as a consequence of the translation of the polygenic messenger RNA as a unit. Operon function is controlled hy two sorts of genes, operator genes and regulator genes. In higher orgnisms however, the units of transcription and of translation are not always of the same length as in microorganisms. Since thc chromosomal DNA is confined to the nucleus and protein synthesis occurs largely in the cytoplasm, the process of transcription and translation are physically separated. Recent experimental information claims that the genetic material in higher organisms contain repetitive and nonrepetitive DNA sequences. The significance of both repectitive and nonrepetitlve sequence with regard to their specific function has been considered. A new model of gene regulation consisting of five elements; producer gene, receptor gene, activator RNA, integrator gene, sensor gene, and battery of genes, in eukaryotic organisms has been described. The Major events in evolution may have required changes in patterns of gene regulation.These changes most likely consist of additions of novel patterns of regulation of the reorganisation of pre-existing patterns in prokaryotes. The appearance of new structural (producer) genes may represent a minor part of the changes involved. The model supports the motion of the theory of gene regulation, that gene activities may be co-ordinated in higher cell genomes, and provide a framework for considerations of the nature of change in the regulatory programmes The elements of the model taken together appear to have the potentiality of establishing a pattern of gene regulation which determine a particular cell state, and probably are sufficient to establish an orderly process of development leading to the full set of cell types and states of an orgnisms.

It is recognised that differential control of gene action accompanies the orderly sequence of events in the development of eukaryotic organisms. This control is accomplished throuth various cellular mechanisms. Basic to all of them, however, are those genetic systems that serve to initiate or programme the gene sequences. The manner in which hormones help in gene expression during the course of the development of an organism is discussed. The facts that hormones localize in specific organs and are concentrated and retained by these organs suggest the existence of specific "receptor" substances in target cells capable of recognizing a particular hormone. Certain of the hormones probably are themselves the primary intracellular offectors, whereas others function at the cell surface. The major question about the mechanism of action of these hormones has now become that of the intracellular action of cyclic AMP itself. In multicellular organisms cyclic AMP is a positive allosteric effector for the phosphorylation of various proteins by ATP.

VIII SUMMARY

Pertinent problems concerning the nature of the mechanisms responsible for gene regulation in higher oranisms have been discussed. The operon concept as applied to both prokaryotic and eukaryotic organisms for functioning of the gene has been elaborated. The significance of both repetitive and non-repetitive DNA sequences is enumerated. A new model of gene regulation consisting of five elements; producer gene, receptor gene, activator RNA, integrator gene, sensor gene, and battery of genes has been described. The element of this model possesses the potentiality of establishing a pattern of gene ragulation which determine a particular cell state and types. Some aspects of the mechanism of action of hormones through transcription or post-transcriptional regulation of gene expression have also been discussed.

REFERENCES

Alpers, D. H. and Tomkins, G. 1966. Sequential transcription of the genes of the lactose operon and its regulatin by protein synthesis. J. Biol. Chem. **241** : 4434-4443.

Ames, B. N., and Garry, B J. 1959. Co-ordinate repression of the synthesis of four histidine biosynthesis enzymes by histidine. Proce. Sci. U. S. **45** : 1453-1461.

Ames, B. N. and Martin, R. G. 1964. Biochemical aspects of genetics; the operon. Ann. Rev. Biochem. **38** : 235-258.

Aditil, R. R., Scaife, J. G., and Beckwith 1968. The nature of mutants in the lac promotor region. J, Mol. Biol. **38** : 421-426.

Banks, G. R. 1971. Mutagenesis a review of some molecular aspects. Science progress **59** : 475-504.

Bantz, E. K. F., and F. A. Bantz, 1970. Initiation of RNA synthesis The Function of sigma in the binding of RNA polymerase to promotor sites. Nature **226** : 1219-1222.

Bendich, A. J. and McCarthy, B. J. 1970. DNA comparisons among barley, oats, rye, and wheat. Genetics. 65 : 545-565.

Berberich, M. A. Venatianer, P. and Coldberger, R. F. 1966. Alternative models of derepression of the histidine operon observed in *Salmonella typhimurium*. J, Biol. chem. 241 : 4426-4433.

Britten, R. J. and Davidson, E. H., 1969. Gene regulation for higher cells : A theory. Science 165 : 349-357.

Britten, R. J. and Kohne, D. E. 1968. Repeated sequences in DNA, Science : 161 : 529-540.

Britten, R. J. and Davidson, E. H. 1971. Repetitive and non-repetitive DNA sequences and speculation on the origins of evolutionary novelty The Quart. Rev. Boil. 46 : 111-133.

Church, R. B. and McCarthy, B. J. 1970. Unstable nuclear RNA synthesis following estrogen stimulation. Biochem. Biophys. Acta **199** : 103-114.

Clever, U.1968. Regulation of chromosome function. Ann. Rev. Gent. **2** : 11-30.

Devidson, E. H. and Hough, B. R. 1971. Genetic information in oocyte RNA. J. Mol. Boil. **56** : 491-506.

Douglas, H. C., and Hewthorne, D. C. 1966. Regulation of genes controlling synthesis of the galactose pathway enzymes in yeast. Genetics 54 : 911-916.

Epstein, W. and Beckwith, J. R. 1968. Regulation of gene expression. Ann. Rev. Biochem. 37 : 411-436.

Fang, S., Anderson, K. M., Liao, S. 1969. Receptor proteins for androgens : On the role of specific proteins in selective retention of 17 β—hydroxy—5—Androst—an—3—one by Rat ventral prostate *in vivo* and *in vitro.* J. Biol. Chem. 244 : 6584-6595.

Garren, L. D., Howell, R. R., Tomkins, G. M., Grocco, R. M. 1964. A paradoxial effect of actionomycin D. The mechanism of regulation of enzyme synthesis by hydrocortisone. Proc. Nat. Acad. Sci. **52** : 1121-1129.

Gelderman, A. and Rake, A. 1971. Transcription of nonrepeated DNA sequences in mouse. Cornegie Ins. of Wash. Year book **67** : 320-325.

Giles, N. H. 1965. Symposium on structure and function of genes and chromosomes : A summation. Nat. Cancer. Inst. Monogra. **18** : 341-354.

Georgiev, G. P. 1969. On the structural organization of operon and the regulation of RNA synthesis in animal cells. J. Theor. Biol. **25** : 473-490

Hamilton, T. H. 1968. Control of estrogen of genetic transcription and translation. Science **161** : 449-661.

Haussler, M. R., Norman A. W. metabolite 1969. Chromosomal receptor for a vitamin D. proc. Nat. Acad. Sci. U. S. **62** : 155-162.

Hood, L., Eichman, K., Lackland, H., Krause, R. M., and Ohms, J. J. 1970. Rabbit antibody light chains and gene evolution. Nature **228** : 1040-1044

Jacob, F. and Manod, J. 1961. On the regulation of gene activity. Sympos. Quant. Biol. **26** : 193-211.

Jacobson, M. and Baltimore, D. 1968. Evidence for large precursor proteins in poliovirus synthesis. Proc. Nat. Acad. Sci. U. S. **59** : 966-971.

Kuo, J. F., Greengard, P. 1969. Cyclic nucleotide-dependent protein kinases, IV. Widespread occurenee of adenosine 3′—5′—monophosphate--dependent protein kinase in various tissues and phyla of the animal kingdom. Proc. Nat. Acad. Sci. U. S. **64** : 1349-1355.

Lewin, B. M., 1970. The molecular basis of gene expression (Ed. Wiley-Interscience, John-Wiley and Sons, Ltd., New York.)

Marshall, R. E.; Caskey, C. T. and Nirenberg, M. 1967. Fine Structure of RNA code words recognised by bacterial, Science 155 : 820 - 826.

Martin, R. C. Control of gene expression. Ann. Rev. Genet, 3 ; 181 - 216.

Miyamoto, E, Kuo, J. Greengard, P. 1969. Adenosine 3′, 5′- monophasphate dependent protein kinase from brain. Science 165 :63-65.

Naughton, M. A. and.Dintzis, H. M. 1962. Sequential biosynthesis. of the peptide chains of homoglobin. proc. Nat. Acad. Sci. 48 : 1822—1830

Ptashne, M. 1967. Specific binding of the phase repression to DNA. Nature 214 232—234.

Pastam, I. and Perlman, R. 1969. Stimulatian of Tryptophinase synthesis in Escherichia Coli by cyclic 3′, 5′ – adenosine monophoshate. Biol. chem. 244 : 2226–2232.

Rice, N. 1971. Thermal stability of reassociated repeated DNA from rodents, Carnegie Inst. of Wash. Yearbook 69 : 472—479.

Robinson, G. A., Butcher, R. W. Southerland, E. W. 1968. Cyclic AMP.Ann. Rev. Biochem. 37 : 149–165.

Strauss, N. 1971. Comparative DNA renaturation kinetics in amphibian. proc. Nat. Acad. Sci. U, S. 68 : 799—802.

Sutherland, E, W, , Robinson, G. A., Butcher, R. W. 1698. Cyclic AMP. Circulation, Res. 37 : 279—289

Tomkins, G. M., Gelehrter, T. D., Granner, D., Martin, D., Samuales, H. H. & Thompson, E. B.1969. Control of specific gene expression in higner organismı. Science 166 : 1474—1480.

Tomkins, G. M. and Martin, D, W. 1970. Hormones and gene expression. Ann. Rev. Genet. 4: 91—106.

Watson, J. D., 1969. The molecular Biology of the gene (Ed. W. A. Bengamin Inc. New York.)

Wicks, W. D. Kenney, F. T., Lee, K. L. 1969. Induction of hepatic enzyme synthesis in Vivo by Adenosine 3′—5′ monophosphate. J. Biol. Chem 244 : 6008—6013.

Yanofsky, C. and Ito, J. 1966. Nonsense codons and polarity in the tryptophane operon. J. Mol. Biol. 21 : 313—317.

J n ã n a b h a, Sect. A, Vol. 2, July 1972.

CHEMICAL CONSTITUTENTS OF FLOWERS OF BOMBAX - MALABARICUM

By

P. C. GUPTA
Chemistry Department
Allahabad Uuiversity
Allahabad (India).

G. S. NIRANJAN
Chemistry Department
D. V. Post-graduate College
Orai (U. P.) India.

(*Received on 18th May* 1972)

The flowers of Bombax-malabaricum are highly reputed for curing various ailments[1]. No work seems to have been done on this plant, except the recent report of a rhamno-arabinogalactan from the stamens of its flowers[2].

To the present communication we record the isolation and characterisation of two flavones and a phytosterolin from the flowers of Bombax-malabaricnm.

The 1% hydrochloric acid and cold methanolic extracts of petals of the flowers on concentration deposited a solid mass which was defatted with petrol and then extracted with ethyl acetate yielding two flavones, They were separated by thin layer chromatography and finally crystallised from acetone having m.p. 280° and 294° respectively. The structural studies of these flavones were performed by various colour reactions.[3–10], degradative and spectral studies [11–17]. Finally the structure of these two flavones were confirmed by the preparation of their acetylated derivatives, m.p. 118° and 216° respectively, superimposition of IR and cochromatography with authentic samples as 3,5,7,4′ tetra hydroxy flavone (Kaempferol) and 3,5,3′,4′ tetra hydroxy quercetin 7-methyl other (rhamnetin).

Similarly the ethanolic extract of the stamens of Bombax-malabaricum flowers on concentration deposited a solid substance which was defatted with light petrol and then extracted with Ethylacetate yielding a white compound. It was crystallised from methanol-pyridine as shinning needles, m.p. 286-288°. The structure of the compound was established on the basis of colour reactions and degradative studies as β-D-glucopyranoside of β-sitosterol.

Experimental

ISOLATION AND PURIFICATION OF COMPOUNDS

The 1% hydrochloric acid and cold methanolic extracts of fresh petals of the flowers on concentration deposited a solid mass which was defatted with petroleum ether (40-60°) and then extracted with ethylacetate yielding to flavones designated as A & B. They were separated by thin layer chromatography using silicagel-G and ethyl-acetate saturated with water as a solvent system system and finally crystallised from dry acetone having, m.p. 280° and 294° respectively.

Similarly the ethanolic extract of the stamens of Bombax-malabaricum flowers on concentration deposited a solid mass which was defatted with petroleum ether (40-60°) and then extracted with ethyl acetate yielding a white mass. It was found to be single entity which was crystallised from methanol-pyridine as shinning needles, m.p. 286-88°, $[\alpha]_D^{30}$ −128° in pyridine.

STUDY OF COMPOUND A :

$C_{15}H_{10}O_6$, m.p. 280°, crystallised from acetone as yellow crystalline substance, was found to belong to flavone group of colouring matters as it gave positive Shinoda test[3] and other colour reactions [4-10]. It yielded a tetra acetyl derivative, m.p. 118° and tetra methyl derivative, m.p. 216°. On oxidation it gave p-hydroxy benzoic acid, m.p. 21°, which showed that side phenyl ring contains only one hydroxyl group. The λ max (shift [11-17] and Rf values given in table 1 established the structure as 3,5,7,4′ tetra-hydroxy flavone (Kaempferol). Finally, it was confirmed by superimposition of IR and cochromatography with authentic sample.

STUDY OF COMPOUND B :

$C_{16}H_{12}P_7$, m.p. 294°, crystallised from dry acetone as golden yellow substance, was, also found to belong flavone group of colouring matters, it yielded a tetracetyl derivative, m.p. 182°. On demethylation it gave a compound which was found to be identical with quercetin, m.p. 315° on the basis of m.m.p. and co-chromatography with authentic sample. On oxidation it gave protocatechuic acid, m.p. 190° which showed that the phenyl ring contains two hydroxyl groups. The λ max (shift) [11-17] and Rf values given in table 1 established the structure as 3,5,3′, 4′ tetrahydroxy flavone 7-methyl ether (Rhamnetin). Finally, it was confirmed by IR and cochromatography with authentic sample.

TABLE 1

S. N.	COMPOUND	Rf in Solvent (a)	Rf in Solvent (b)	λ max in Et-OH	Et-OH/ $AlCl_3$ λ max	Et-OH/ $AlCl_3$ Shift	Et-OH/ NaOEt λ max	Et-OH/ NaOEt Shift	Et-OH/ NaOAc λ max	Et-OH/ NaOAc Shift	Et-OH/ H_3BO_3/ NaOAc λ max	Et-OH/ H_3BO_3/ NaOAc Shift	Et-OH/ Mg_+Hcl λ max	Et-OH/ Mg_+Hcl Shift
1.	A	0.81	0.61	267, 369	428	59	343	16	332	13	385	16	520	140
2.	B	0.71	0.38	257, 380	430	50	—	—	380	No	395	15	540	160
3.	Demethylated (B)	0.75	0.41	372	432	60	—	—	386	14	393	21	536	156

(a) Butanol : Acetic acid : water (4:1:5 v/v) system.

(b) Acetic acid : Hydrochloric acid : water (30:3:10 v/v) system.

Table 1

STUDY OF PHYTOSTEROLIN :

$C_{35}H_{60}O_6$' m.p. 286-288°, $[\alpha]_D^{30}$–in pyridine. It was crystallised from Me-OH-Pyridine as shinning needles. It formed a tetra-acetate, m.p. 138-140° (Me-OH-$CHcl_3$), $[\alpha]_D^{30}$–120° ($CHcl_3$) Hydrolysis of the phytosterolin with boiling 8% hydrochloric acid in methanol for 10 hr. give glucose (1 mole) which was characterised by PC in BAW (4:1:5 v/v) with an authentic sample and through osazone formation m.p. 202-204°. The aglycone was crystallised as shinning flakes m.p. 124-125°, $[\alpha]_D^{30}$–37° ($CHcl_3$). It was finally identified as β-sitosterol by direct comparision (m.m.p.,T.L.C.,IR.) with an tuthentic sample. Hydrolysis of the methyl ether of phytosterolin yielded β-sitosterol and 2,3,4,6 tetra-O-methyl-D-glucose (1 mole). Periodate oxidation studies also indicated the presence of 1 mole of glucose in the phytosterolin. The glycoside was hydrolysed with emulsin, thereby indicating a β-glucosidic linkage between glucose and sterol which is further supported by negative optical rotations of the phytosterolin and its derivatives.

ACKNOWLEDGEMENT :

The author (GSN) thanks to the Council of Scientific & Industrial Research, New Delhi for financial assistance.

REFERENCES :

[1] Kirtikar, K. R. and Basu, B. D., 'Indian Medicinal Plants', (Second edition), Vol. II, Lalit Mohan Basu, Allahabad PP. 860 (1933).

[2] Agarwal, G.D., Rizvi, S.A.I. Gupta, P. C. and Tiwari, J. D., Planta Medica, **21**, 293 (1972).

[3] Shinoda, J.J. Pharm. Soc. Japan, **48**, 214 (1928).

[4] Shimizu, M.J. Pharm. Soc. Japan, **71**, 1339 (1951).

[5] Asahina, Y. and Inubuse, M. Ber; **61** 1946 (1928).

[6] Geissman, T. A., 'In Modern Methods of Plants Analysis', edited by K. Peach and M.V. Tracey, Vol. III, p. 450, Julium Springer, Berlin (1955).

[7] Douglas, C.D., Morris, Q. L. and Wander, S. H. J. Amer. Chem. Soc. **73**,4023 (1951).

[8] Horowitz, R.M., J. Org. Chem,, **22**, 1733 (1957).

[9] Horhammer, L. and Hansel, R., Arch. Pharm., **288**, 315 (1955).

[10] Dean, F.M., In 'Naturally Occuring Oxygen Ring Compounds', p. 288, Butterworths, London (1963).

[11] Jurd, L., In 'Chemistry of Flavanoid compounds', edited by T. A. Geissman, p. 107, pergamon Press, Oxford (1962).

[12] Briggs, L.H. and Locker, R.H., J. Chem. Soc., 3136 (1951).

[13] Jurd, L. and Horowitz, R.M., J. Org. Chem., **21**, 1395 (1956).

[14] Jurd, L. and Rolle, L.A., J. Amer. Chem. Soc., **80**, 5527 (1958).

[15] Jurd, L. Arch. Biochem. Biphys., **63**, 376 (1956).

[16] Gage, T.B., Dougloss, C.D. and Wender, S.A., Analy. Chem., **23**, 1958 (1951).

[17] Manafield, G.H., Swain, T. and Nordstrom C. G., Nature (London), 23, **172** (1953).

[7] Douglas, C.D., Morris, Q. L. and Wander, S. H. J. Amer. Chem. Soc. **73**,4023 (1951).

[8] Horowitz, R.M., J. Org. Chem,, **22**, 1733 (1957).

[9] Horhammer, L. and Hansel, R., Arch. Pharm., **288**, 315 (1955).

[10] Dean, F.M., In 'Naturally Occuring Oxygen Ring Compounds', p. 288, Butterworths, London (1963).

[11] Jurd, L., In 'Chemistry of Flavanoid compounds', edited by T. A. Geissman, p. 107, pergamon Press, Oxford (1962).

[12] Briggs, L.H. and Locker, R.H., J. Chem. Soc., 3136 (1951).

[13] Jurd, L. and Horowitz, R.M., J. Org. Chem., **21**, 1395 (1956).

[14] Jurd, L. and Rolle, L.A., J. Amer. Chem. Soc., **80**, 5527 (1958).

[15] Jurd, L. Arch. Biochem. Biphys., **63**, 376 (1956).

[16] Gage, T.B., Dougloss, C.D. and Wender, S.A., Analy. Chem., **23**, 1958 (1951).

[17] Manafield, G.H., Swain, T. and Nordstrom C. G., Nature (London), 23, **172** (1953).

J ñā n a b h a, Sect. A, Vol. 2, July 1972.

THERMAL STRESSES IN AN INFINITE SOLID CONTAINING A CYLINDRICAL CAVITY AND TWO STRIP CRACKS

BY

K.N. SRIVASTAVA, M. KUMAR AND BAIJ NATH

(Department of Mathematics, Maulana Azad College of Technology, Bhopal M.P., India)

(Received on 27th July, 1972)

ABSTRACT

The thermoelastic equilibrium of an infinite solid containing a cylindrical cavity and two strip cracks situated symmetrically on a diametral plane of the cavity, is investigated. It is shown that the problem is equivalent to one finding distribution of thermal stress in an infinite two dimensional medium containing a circular hole and two Griffith cracks. By making suitable representations for the complex potentials occuring in the equatiuns of thermoelastic equilibrium, the problem is reduced to a set of triple integral equations with trigonometrical kernels. They are solved by using the technique of Finite Hilbert transform. Finally, the analytical expressions for quantities of physical interest are also found.

1. INTRODUCTION :

Recently the authors [1] have studied the Problem of determining the distribution of thermal stresses inside an infinite cylinder containing two strip cracks. It has been noticed that there is concentration of stresses and temperature at origin. The damages in the elastic material due to this phenomena can be avoided by creating a hole at the origin. The problem of a hole and two coplanar Griffith cracks is studied here. It is equivalent to the problem of determining the thermal stresses inside an infinite solid containing a cylindrical cavity and two strip cracks. The method of solution is a direct extension of the corresponding elastic problem [2].

2. FORMULATION OF PROBLEM :

We shall consider the temperature and displacement fields in an infinite elastic solid containing a cylindrical cavity and two strip cracks lying symmetrically on a diametral plane of cavity. The material of the solid is supposed to be homogeneous and isotropic with regard to both thermal & mechanical properties. The modulus of rigidity of the solid is denoted by μ and Poisson's ratio by η: The two dimensional problem is considered within the limits of the classical sheory of infinitesimal elasticity. If we use cylindrical polar coordinates, the cylindrical cavity Is supposed to occupy she region $r=\rho < a < b, 0 \leqslant \theta \leqslant 2\pi, -\infty < z < \infty$ with the axis of the cylinder being z — axis. The two strips cracks occupy the region $a < r < b, \theta = 0, \pi$ and $-\infty < z < \infty$. The state of stress in the solid is created by same temperature along the whole length of cracks so that the thermal stress field is the same in all the planes perpendicular to the axis of the cylinder. The above said problem is thus equivalent to one of finding the distribution of stress in an infinite two-dimensional medium containing a circular hole and two Griffith cracks situated symmetrically on a diameter of the hole. The displacement vector **U**, for symmetrical deformation, is taken to have components $(u_r, u_\theta, 0)$ and stress tensor $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta})$ The boundary conditions to be satisfied on $\theta=0$ are

$$\sigma_{\theta\theta}(r,\theta) = 0 \qquad , a < r < b \tag{2.1}$$

$$\sigma_{r\theta}(r,\theta) = 0 \qquad , \rho < r < \infty \tag{2.2}$$

$$u_\theta(r,\theta) = 0 \qquad , r > b, \rho < r < a. \tag{2.3}$$

Since temperature on the surface of the crack is prescribed we have on $\theta=0$,

$$T(r,\theta) = T(r) \qquad , a < r < b \tag{2.4}$$

$$\frac{\partial T(r,\theta)}{r\,\partial\theta} = 0 \qquad , \rho < r < a, \quad r > b. \tag{2.5}$$

Further, the lateral surface of cylindrical cavity is supposed to be free of traction and is kept at zero temperature. Hence

$$\sigma_{rr}(\rho,\theta) = \sigma_{r\theta}(\rho,\theta) = T(\rho,\theta)=0, 0 \leqslant \theta \leqslant \pi/2. \tag{2.6}$$

In addition, we have on $\theta = \pi/2$

$$\sigma_{r\theta}(r,\pi/2) = u_\theta(r,\pi/2) = T(r,\pi/2) = 0, \text{ for all } r. \tag{2.7}$$

3. THE EQUATIONS OF EQUILIBRIUM FOR THE THERMOELASTIC FIELD.

We quote from [1] the basic equations for two dimensional isotropic elasticity in presence of temperature field. If r,θ are polar coordinates and

$z = re^{i\theta}$, $\bar{z} = re^{-i\theta}$, then

$$\mu(u_r + iu_\theta) = \left[k\phi(z) - \overline{\phi(z)} - (z - \bar{z})\overline{\phi'(z)} - \overline{\omega(z)}\right]e^{-i\theta} + \left(v,_r + \frac{i}{r}v,_\theta\right) \tag{3.1}$$

$$\sigma_{rr} + \sigma_{\theta\theta} = 4\left[\phi'(z) + \overline{\psi'(z)}\right] - 2KT \tag{3.2}$$

$$\sigma_{rr} + i\sigma_{r\theta} = 2\left[\phi'(z) + \overline{\phi'(z)}\right] - 2\left[(z-\bar{z})\phi''(z) + \overline{\omega'(z)}\right]e^{-2i\theta} - KT + \left[\left(v,_{rr} - \frac{1}{r}v,_r - \frac{1}{r^2}v,_{\theta\theta}\right) - 2i\left(\frac{1}{r^2}v,_\theta - \frac{1}{r}v,_{r\theta}\right)\right] \tag{3.3}$$

$$\sigma_{\theta\theta} - i\sigma_{r\theta} = 2\left[\phi'(z) + \overline{\phi'(z)}\right] + 2\left[(z-\bar{z})\overline{\phi''(z)} + \overline{\omega'(z)}\right]e^{-2i\theta} - KT - \left[\left(v,_{rr} - \frac{1}{r}v,_r - \frac{1}{r^2}v,_{\theta\theta}\right) - 2i\left(\frac{1}{r^2}v,_\theta - \frac{1}{r}v,_{r\theta}\right)\right] \tag{3.4}$$

where μ is the modulus of rigidity, $k = (3 - 4\eta)$ for plane strain and $k = (3-\eta)/(1+\eta)$ for generalized plane stress; η being Poisson's ratio. $K = \dfrac{E_{at}}{1-\eta}$ for plane strain and $K = E_{at}$ for plane stress; E being Young's modulus and $_{at}$ the coefficient of linear expansion of the solid. $T(r,\theta)$ is the prescribed temperature field and $v(r,\theta)$ is a potential function satisfying the equation

$$\nabla^2 v = KT \tag{3.5}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$.

The temperature field in the solid is determined by the Laplace equation

$$\nabla^2 T = 0 \tag{3.6}$$

in the steady state and in the absence of heat source. The appropriate expression for T satisfying (3.6) in the region undcr consideration, i.e. an infinite medium with a circular hole, is given by

$$T(r,\theta) = \int_0^\infty \frac{A(\xi)}{\xi} e^{-\xi r \sin\theta} \cos(\xi r \cos\theta)\, d\xi + \sum_{n=0}^{\infty} a_n r^{-n} \cos n\theta. \tag{3.7}$$

With this choice for $T(r,\theta)$, the appropriate expressien for v satisfying (3.5) can be written as

$$v(r,\theta) = -\frac{K}{2}\int_0^\infty \frac{A(\xi)}{\xi^3}(1+\xi r \sin\theta)\, e^{-\xi r \sin\theta} \cos(\xi r \cos\theta)\, d\xi -$$

$$-\frac{K}{4}\sum_{n=0}^{\infty} \frac{a_n}{(n-1)} r^{-n+2} \cos n\theta. \tag{3.8}$$

The expressions for complex potentials $\phi(z)$ and $\omega(z)$ occuring in (3.1) to (3.4), we quote from the corresponding eleastic problems already solved by the first two authors [2]. These are

$$\phi(z) = \int_0^\infty i B(\xi)\, e^{i\xi z} d\xi + \sum_{n=0}^{\infty} \frac{C_n}{n+1} x^{-n-1} \tag{3.9}$$

$$\omega(z) = \sum_{n=0}^{\infty} \frac{d_n}{(n+1)} z^{-n-1}. \tag{3.10}$$

The expressions for components of nisplacement vector and stress tensor can be found by substituting the values $T(r,\theta)$, $v(r,\theta)$, $\phi(z)$, $\omega(z)$, and their corresponding derivatives in (3.1) to (3.4) and, then separating the real and imaginary parts.

THE TEMPERATURE FIELD :

The temperature field satisfies conditions (2.4) and (2.5) on $\theta=0$. These give rise to the following set of triple integral equations to desermine the unknown function $A(\xi)$ occuring in $T(r,\theta)$. The integral equations are :

$$\left.\begin{aligned}
&\int_0^\infty A(\xi)\cos\xi r\, d\xi = 0 \quad , \rho < r < a \\
&\int_0^\infty \frac{A(\xi)}{\xi}\cos\xi r\, d\xi = \frac{\pi}{2} P(r), \quad a < r < b \\
&\int_0^\infty A(\xi)\cos\xi r\, d\xi = 0 \quad , r > b
\end{aligned}\right\} \tag{4.1}$$

where

$$P(r) = \frac{2}{\pi}\left[T(r) - \sum_{n=0}^{\infty} a_n r^{-n} \right].$$

The solution for such a set of triple integral equations was found by Srivastava and Lowengrub [3] by utilizing the Finite Hilbert transform technique. Hence, quoting the result from [3] we obtain

$$A(\xi) = \int_a^b \frac{h(t^2)}{t}(1-\cos\xi t)\, dt \tag{4.2}$$

where the unknown function $h(t^2)$ is determined from the condition

$$h(t^2) = \mathrm{H}\left[rP'(r) \right] + C' / \left\{ (t^2-a^2)(b^2-t^2) \right\}^{\frac{1}{2}}. \tag{4.3}$$

Here C' is an arbitrary constant to be found from the condition

$$\int_a^b \frac{h(t^2)}{t}\, dt = \frac{2}{\log\left\{\frac{b-a}{b+a}\right\}} \int_a^b \frac{rP(r)\, dr}{\left\{(r^2-a^2)(b^2-a^2)\right\}^{\frac{1}{2}}}. \tag{4.4}$$

To complete the solution, we satisfy the boundary condition $T(\rho,\theta) = 0$, $0 \leqslant \theta \leqslant \pi/2$. This gives us a relationship between the unknown function $h(t^2)$ and the coefficient a_n.

Thus, $T(\rho,\theta) = 0$ yields

$$-\int_a^b \left[\log(\rho/t) + \sum_{n=1}^{\infty} \frac{1}{2n} (\rho/t)^{2n} \cos 2n\theta \right] \frac{h(t^2)}{t} dt +$$

$$\sum_{n=0}^{\infty} a_{2n} \rho^{-2n} \cos 2n\theta = 0.$$

The above equation tells us that all odd coefficients a_{2n+1} are zero and even coefficients are given by

$$a_0 = \int_a^b \frac{h(t^2)}{t} \log(\rho/t)\, dt \tag{4.5}$$

and

$$a_{2n} = \int_a^b \frac{1}{2n} \left\{ \frac{\rho^2}{t} \right\}^{2n} \frac{h(t^2)}{t}\, dt, \quad n \geqslant 1. \tag{4.6}$$

Substituting the value of the coefficient in the expression for $P(r)$, we find that

$$P(r) = \frac{2}{\pi} \left[T(r) + \int_a^b N(x,r)\, N(x,r) \frac{h(x^2)}{x}\, dx \right] \tag{4.7}$$

where

$$N(x,r) = -\log(\rho/x) - \sum_{n=1}^{\infty} \frac{1}{2n} \left(\frac{\rho^2}{rx} \right)^{2n}$$

or

$$N(x,r) = -\left[\sum_{n=1}^{\infty} \frac{(1-\rho/x)^n}{n} + \sum_{n=1}^{\infty} \frac{y^{2n}}{2n} \right].$$

$$M(x^2, t^2) = \frac{1}{\pi s(t^2)} \left[\frac{\alpha}{x} \log(\rho/x) + \left(\beta + \frac{2ab}{t^2}\right) \frac{\rho^4}{x^3} + 0(\rho^8) \right]$$

$$\alpha = \frac{4ab}{\log \left\{ \frac{b-a}{b+a} \right\}}, \beta = \frac{-1}{2ab} \left[\alpha + 2(a^2+b^2) \right].$$

The equation (4.10) is a Fredholm integral equation of the second kind. If we observe the kernel $M(x^2, t^2)$ carefully, we find that it has got weak sigularities at the end points. Inspite of this, all fundamental Fredholm theorems are applicable if they are stated in a suitable manner. This will be shown by the reduction of (4.10) to a Fredholm integral equation with a bounded $L^2(a, b)$ kernel. Writing (4.10) as

$$H(t^2) = \psi(t^2) S(t^2) + \int_a^b H(x^2) \frac{R(x^2, t^2)}{S(x^2)} dx \tag{4.11}$$

where

$$H(t^2) = h(t^2) S(t^2), R(x^2, t^2) = M(x^2, t^2) S(t^2).$$

The first term on the right hand side of (4.11) is easily shown to be bounded. Further $R(x^2, t^2)$ is also bounded with respect to the variable t. The variable x will now be replaced by θ, by writing $x^2 = a^2 \sin^2 \theta + b^2 \cos^2 \theta$. With this transformation (4.11) reduces to

$$H(t^2) = \psi(t^2) S(t^2) + \int_0^{\pi/2} H(x^2) \frac{R(x^2, t^2)}{x^2} d\theta. \tag{4.12}$$

This is a Fredholm integral equation with bounded $L^2(0, \pi/2)$ kernel Thus a solution of (4.10) is equivalent to the solution of (4.12) in ordinary sense. Consequently we conclude that the Fredholm theorems are valid for (4.10).

We now solve the Fredholm integral equation (4.10) by the method of successive approximations. Let the first approvimation be

$$h_0(t^2) = \psi(t^2). \tag{4.13}$$

The second approximation is obtained by substitut ng (4·13) on the right sibe of (4·10) and so on. We shall now consider the physically important case when the crackes are opened by constant temperature i. e. $T(r) = T$. In this case, we obtain

$$h_0(t^2) = \frac{P}{S\ (t^2)}, \quad P = \frac{\alpha\, T}{\pi} \tag{4.14}$$

$$h_1(t^2) = \psi(t^2) - \frac{P}{\pi S(t^2)} \int_a^b \frac{\alpha \log\ (\rho/x) + (\ \beta +\ 2ab/t^2)\ \rho^4/x^2 + O(\rho^8)}{x\ \{\ (x^2 - a^2)\ (b^2 - x^2)\ \}^{\frac{1}{2}}}\, dx$$

$$= \frac{\rho}{S\ (t^2)} \left[\alpha_1 + \alpha_2\ \left(\beta + \frac{2ab}{t^2} \right)\ \rho^4 + O(\rho^8)\ \right] \tag{4.15}$$

where

$$\alpha_1 = 1 + \frac{\alpha}{2ab}\ \log\ \frac{\rho\ (a+b)}{2ab}, \quad \alpha_2 = \frac{(a^2+b^2)}{4a^3\ b^3}.$$

The third approximation is

$$h_2\ (\ t^2\ ) = \frac{P}{S(\ t^2\ )} \left[\beta_1 + \left\{ \beta_2 + \alpha_1 \alpha_2 \left(\beta + \frac{2ab}{t^2} \right) \right\}\ \rho^4 + O\ (\rho^8)\ \right], \tag{4·16}$$

where $\beta_1 = 1 - \alpha_1 + \alpha_1{}^2$, $\beta_2 = \alpha_2 \left[(\alpha_1 - 1)\ (\beta + 4a^2b^2\alpha_2) + \alpha \left(2ab\ \alpha_2 - \frac{1}{2ab} \right) \right]$

The higher approximations can be obtained by continuing the process. In deriving (4·15) and (4·16) we have used results (12) and (13) of the appendix.

The expressions for the temperature field can be easily derived, they are

$$T\ (r, O) = \int_0^\infty \frac{A\ (\ \xi\ )}{\xi}\ \cos\ \xi r.\ d\xi + \sum_{n=0}^{\infty}\ a_n\ r^{-n}\ .$$

$$= \int_a^b \frac{\log\ |\ 1 - t^2/r^2\ |}{t}\ h\ (t^2)\ dt + \sum_{n=0}^{\infty}\ a_{2n}\ r^{-2n} \tag{4.17}$$

using results (14) and (15) of the appendix, we obtain

$$
T(r,0)=\begin{cases}
\frac{aT}{2ab}\Big[2\Big\{\beta_1+(\beta_2+\alpha_1\alpha_2\beta)\rho^4\Big\}\log\frac{a\sqrt{(b^2-r^2)}+b\sqrt{(a^2-r^2)}}{r(a+b)}+ \\
+\Big(2+\frac{2(R-ab)}{r^2}+\frac{a^2+b^2}{ab}\Big\{\log\Big(\frac{b-a}{b+a}\Big)+ \\
\log\frac{a^2(b^2-r^2)+b^2(a^2-r^2)+2abR}{r^2(b^2-a^2)}\Big\}\Big)\alpha_1\alpha_2\rho^4+O(\rho^8)+Q(r),\Big] \\
\qquad\qquad\qquad\qquad \rho<r\leqslant a \\
T(r)=T, \qquad\qquad a\leqslant r\leqslant b \\
\frac{aT}{2ab}\Big[2\Big\{\beta_1+(\beta_2+\alpha_1\alpha_2\beta)\rho^4\Big\}\log\frac{a\sqrt{(r^2-b^2)}+b\sqrt{(r^2-b^2)}}{r(a+b)}+ \\
+\Big(2-\frac{2(R_1+ab}{r^2}+\frac{(a^2+b^2)}{ab}+\Big\{\log\Big(\frac{b-a}{b+a}\Big)+ \\
\log\frac{a^2(r^2-b^2)+(r^2-a^2)+2abR_1}{r^2(b^2-a^2)}\Big\}\Big)\alpha_1\alpha_2\rho^4+ \\
+O(\rho)+Q(r),\Big] \qquad\qquad r>b
\end{cases} \tag{4.18}
$$

where

$$R=\{(a^2-r^2)(b^2-r^2)\}^{1/2},\ R_1=\{(r^2-a^2)(r^2-b^2)\}^{1/2}$$

and

$$Q(r)=aT\Big[d_0+(d_2-\frac{\alpha_2\beta_1)}{2r^2}\rho^4+O(\rho^8)\Big],$$

$$d_0=\frac{(\alpha_1-1)\beta_1}{a},\ d_2=(\beta_2+\alpha_1\alpha_2\beta)\frac{d_0}{\beta_1}+\frac{\alpha_1\alpha_2}{ab}\Big\{(a^2+b^2)\frac{d_0}{\beta_1}+$$

$$\frac{(a-b)^2}{4ab}\Big\}.$$

5. The Thermoelastic field :

We divide the solution of the problem into two parts.

Condition on cracks Face : In this part we consider the relations which exist between arbitrary functions and the unknown coefficients C_n and d_n, if boundary conditions on the line $\theta=O$ and $\theta=\pi/2$ are to be satisfied. The boundary conditions (2·7) on $\theta=\pi/2$ are statisfied if and only if the coefficients C_{2n+1} and d_{2n+1} are all zero. With this choice of C_{2n+1} and d_{2n+1}, we see that (2.2) is also satisfied. The conditions (2.1) and (2.3) lead to the triple integral equations

$$\left.\begin{aligned} &\int_0^\infty B(\xi)\cos\xi r\, d\xi = O, && O<r<a \\ &\int_0^\infty \xi B(\xi)\cos\xi r\, d\xi = \frac{\pi}{2}G(r), && a<r<b \\ &\int_0^\infty B(\xi)\cos\xi r\, d\xi = O, && r>b \end{aligned}\right\}$$

where

$$G(r)=-\frac{2}{\pi}\sum_{n=o}^{\infty}\left[(2C_n+d_n)r^{-n-2}+\frac{K}{2}(n+2)\,a_n r^{-n}+K\int_0^\infty\frac{A(\xi)}{\xi}\cos\xi r.\, d\xi\right]. \tag{5.2}$$

Also we have used the following definitions

$$B(\xi)=2B(\xi) \quad \text{and } A(\xi)=2A(\xi). \text{ in deriving} \tag{5.1}$$

The solution of the above set of triple integral equations as given by Srivastava and Lowengrub [3] is

$$B(\xi)=\frac{1}{\xi}\int_a^b g(t^2)\sin\xi t.\,dt \tag{5.3}$$

where $g(t^2)$ is determined by

$$g(t^2)=-H[G(r)]+C/\{(t^2-a^2)(b^2+t^2)\}^{1/2} \tag{5.4}$$

satisfying the condition $\int_a^b g\,(t^2)\,\mathrm{dt}=O$ and C is an arbitrary constant.

Thus, (5.4) gives the relation connecting $g\,(t^2)$ and C_{2n} and d_{2n}.

Conditions on the free boundary : We now complete the solution by satisfying the boundary conditions (2.6).

Replacing $\beta\ (\xi)^*$in (3.9) by (5.3) and noting that for $z=\rho\, e^{i\theta}$, $0 \leqslant \theta \leqslant \pi$, $\rho < t$, we obtain after some calculations.

$$\phi'\,(z) = -\sum_{n=0}^{\infty}\left[A_{2n}\, Z^{2n} + C_n\, Z^{-n-2} \right], \phi''\,(z) =$$

$$-\sum_{n=0}^{\infty}\left[(2n+2)\ A_{2n+2}\ Z^{2n+1} - (n+2)\ C_n\ Z^{-n-3} \right]$$

where

$$A_{2n} = \int_a^b \frac{g\,(t^2)}{t^{\,2n+1}}\ dt.$$

Also we denote B_{2n} by

$$B_{2n} = \int_a^b \frac{h\,(t^2)}{t\,2n+1}\ dt.$$

Now, the boundary conditions (2.6) lead to the expressions.

$$\sum_{n=0}^{\infty}\left\{ 2\,(n+1)\ C_{2n} + d_{2n} \right\}\ \rho^{-2n-2}\ \cos 2n\,\theta - \sum_{n=2}^{\infty} 2\,(n+1)$$

$$C_{2n-2}\ \rho^{-2n}\ \cos 2n\theta\ = 2\ A_0\ +$$

$$+\sum_{n=1}^{\infty}(2n-2)\ \left\{ A_{2n-2}\ \rho^{2n-3} - A_{2n}\ \rho^{2n} \right\}\ \cos 2n\theta\ +$$

$$KB_0\left\{ \tfrac{1}{2} - \log\,(\rho/t) \right\} -$$

$$- \frac{K}{2} \sum_{n=1}^{\infty} \left\{ B_{2n-2} \quad \rho^{2n-2} - \frac{(n-1)}{n} B_{2n} \rho^{2n} \right\} \cos 2n\,\theta +$$

$$+ K \sum_{n=0}^{\infty} (n+1)\; a_{2n}\, \rho^{-2n} \cos 2n\,\theta ,$$

$$\sum_{n=1}^{\infty} \left\{ 2\,(n+1)\, C_{2n} + d_{2n} \right\} \rho^{-2n-2} \quad \sin 2n\theta - \sum_{n=2}^{\infty} 2n\; C_{2n-2}\; \rho^{-2n} \quad \sin 2n\,\theta =$$

$$= - \sum_{n=1}^{\infty} \left\{ (2n-2)\; A_{2n-2} \quad \rho^{2n-2} - 2n\; A_{2n}\; \rho^{2n} \right\} \sin\; 2n\theta\; -$$

$$- \frac{K}{2} \sum_{n=1}^{\infty} \left\{ B_{2n}\, \rho^{2n} - B_{2n-2} \quad \rho^{2n-2} \right\} \sin\; 2n\,\theta +$$

$$+ K \sum_{n=1}^{\infty} n \quad a_{2n} \quad \rho^{-2n}\; ' \sin 2n\;\theta ,$$

for the determination of the coefficients C_{2n} and d_{2n}. In deriving these relations we have used the results (5—8) of the appendix. From these expressions, we observe thas

$$(\,2\;\; C_0 \;+\; d_0\,) \quad \rho^{-2} \;=\; 2A_0 + K \left[\; a_0 + B_0 \left\{ \tfrac{1}{2} - \log\,(\rho/t) \right\} \right] \qquad (5.6)$$

and for $n \geqslant 1$, we get

$$2\; C_{2n-2} \quad \rho^{-2n} \;= 2\;(2n-1)\;\; A_{2n} \;\; \rho^{2n} - 2\;(2n-2)\;\; A_{2n-2} \;\; \rho^{2n-2} \;\; +$$

$$+ \frac{K}{2} \left\{ 2\; B_{2n-2} \quad .\, \rho^{2n-2} - \frac{1(n-1)}{n}\; B_{2n}\, \rho^{2n} \right\} - K\; a_{2n}\, \rho^{-2n} , \qquad (5.6)$$

$$\left\{\; 2\;\;(n+1)\;\; C_{2n} + \; d_{2n} \right\} \rho^{-2n-2} \;=\; 4n^2\;\; A_{2n} \quad \rho^{2n} - 2\;(n-1)\;\;(2n+1)$$

$$A_{2n-2}\ \rho^{2n-2} + \frac{K}{2}\left\{ (2n+1)\ B_{2n-2}\ \rho^{2n-2} - 2n\ B_{2n}\ \rho^{2n} \right\}. \qquad (5.7)$$

Substituting these values of the coefficients in the expression (5.2) for $G(r)$, we obtain that

$$G(r) = -\frac{2}{\pi}\left[2\int_a^b \frac{g(x^2)}{x} K_1(x^2, r^2)\, dx + \frac{K}{2}\int_a^b \frac{h(x^2)}{x} K_2(x^2, r^2)\, dx \right] \qquad (5.8)$$

where

$$K_1(x^2, r^2) = \frac{\rho^2}{r^2} + \sum_{n=1}^{\infty} \frac{\rho^2}{r^2}\left(\frac{\rho^2}{rx}\right)^{2n}\left[4n^2 - n\,\rho^2\left(\frac{2n+3}{r^2} + \frac{2n+1}{x^2}\right)\right]$$

and

$$K_2(x^2, r^2) = -\log(\rho/x) + \frac{\rho^2}{r^2} + \sum_{n=1}^{\infty} \frac{x^2}{r^2}\left(\frac{\rho^2}{rx}\right)\left[(2n+1) - 4n\,(\rho^2/x^2) + \right.$$

$$\left. + \frac{2n^2+6n+1}{2(n+1)}\left(\rho^4/x^4\right)\right].$$

Whenever $\rho << b$, the kernels $K_1(x^2, r^2)$ and $K_2(x^2, r^2)$ can be approximated to

$$K_1(x^2, r^2) = \frac{\rho^2}{r^2} + \frac{4\rho^6}{r^4x^2} - \frac{\rho^8}{r^4x^2}\left(\frac{5}{r^2} + \frac{3}{x^2}\right) + 0(\rho^{10}) \qquad (5.9)$$

$$K_2(x^2, r^2) = -\log(\rho/x) + \frac{\rho^2}{r^2} + \frac{3\rho^4}{r^4} - \frac{4\rho^6}{r^4x^2} + 0(\rho^8). \qquad (5.10)$$

We now consider the equation (5.4). We evaluate the arbitrary constant C from the condition $\int_a^b g(t^2)\, dt = 0$. Its value is

$$C = \frac{1}{F} \int_a^b H\left[\quad G\ (r) \quad \right] \, dt$$

and

$$g\ (t^2) = -H\left[\quad G\ (r) \right] + \frac{1}{FS\ (t^2)} \int_a^b H\left[\quad G\ (r) \quad \right] \, dt \qquad (5.11)$$

where $F{=}F\ (\pi/2,\ q)$, the complete elliptic integral of the first kind and $S\ (t^2) = \{\ (t^2 - a^2) \quad (b^2 - t^2)\ \}^{\frac{1}{2}}$. Substituting the values of $G\ (r)$ from (5.8) in (5·11) and using the results (9, 10, 11) of the appendix, we get a Fredholm integral equation of the second kind.

$$g\ (t^2) = K \int_a^b M_1\ (x^2,\ t^2) \frac{h\ (x^2)}{x}\ dx + \int_a^b M_2\ (x^2,\ t^2) \frac{g\ (x^2)}{x}\ dx, \qquad (5.12)$$

where

$$M_1\ (x^2, t^2) = \frac{a\ b^3}{\pi\ S\ (t^2)} \left[\frac{1}{a^3 b} \left(t^2 - \frac{b^2 E}{F}\right) \log\ (\rho/x) + \left(\frac{1}{t^2} - \frac{E}{a^2 F}\right) \rho^2 - \right.$$

$$- \left(3\ \rho^4 - \frac{4\ \rho^6}{x^2} \right) \left\{ \left(\gamma + \frac{1}{a^2 t^2} - \frac{1}{t^4} \right) + \right.$$

$$\left. \left. + \ \delta \left(\frac{1}{t^2} - \frac{E}{a^2 F} \right) \right\} + 0\ (\rho^8) \right],$$

$$M_2\ (x^2, t^2) = \frac{4\ ab^3}{\pi S\ (t^2)} \left[\left(\frac{1}{t^2} - \frac{E}{a^2 F} \right) \rho^2 - \frac{4\ \rho^6}{x^2} \left\{ \left(\gamma + \frac{1}{a^2 t^2} - \frac{1}{t^4} \right) + \right. \right.$$

$$\left. \left. + \ \delta \left(\frac{1}{t^2} - \frac{E}{a^2 F} \right) \right\} + 0\ (\rho^8) \right]$$

and

$$\gamma = \frac{1}{3a^4} \left[\frac{a^2}{b^2} \left(\frac{2E}{F} - 1 \right) - \frac{E}{F} \right], \ \delta = \frac{3\ (a^2 - b^2)}{2a^2 b^2}.$$

The first term on the right hand side of (5.12) is the free term of the integral equation since $h(x^2)$ is a known function. The integral equation (5.12) can be easily seen to have weak singularities at the end points. Whatever has been said in section 4 regarding the solution of a similar Fredholm integral equation (4.10) will be valid for (5.12). To show $M_2(x^2, t^2)$ is bounded. it will be sufficient if we show that $K_1(x^2, t^2)$ is bounded. We express $K_1(x^2, t^2)$ as

$$K_1(x^2, t^2) = \frac{\rho^2}{t^2} + \frac{4\rho^2 z(1+z^2)}{t^2(1-z^2)^3} - \frac{\rho^4 z^2(5-z^2)}{t^4(1-z^2)^3} - \frac{z^4(3-z^2)}{(1-z^2)^2} \qquad (5.13)$$

where $(\rho^2/t x) = z < 1$. The series on the right of (5.13) is easily seen to be convergent. Hence the solution of the Fredholm integral equation (5.12) can be found by the method of successive approximations. Let the first approximation be

$$g_0(t^2) = K\int_a^b \frac{h(x^2)}{x} M_2(x^2, t^2)\,dx$$

$$= \frac{PK}{2a^2 S(t^2)}\Big[\left(t^2 - \frac{b^2E}{F}\right)(p_1 + p_2\rho^4) + \left(\frac{1}{t^2} - \frac{E}{a^2F}\right)(p_3\rho^2 + p_4\rho^4 +$$

$$+ p_5\rho^6) + \beta_1\left(\gamma + \frac{1}{a^2t^2} - \frac{1}{t^4}\right)(3a^2b^2\rho^4 - 2(a^2+b^2)\rho^6) + 0(\rho^8)\Big], \qquad (5.14)$$

where

$$p_1 = \frac{2(1-\alpha_1)b^2\beta_1}{a},\quad p_2 = \frac{2(\beta_2+\alpha_1\alpha_2\beta)p_1}{\beta_1} +$$

$$+ \frac{\alpha_1\alpha_2\beta\{4(1-\alpha_1)ab(a^2+b^2)+\alpha(a-b)^2\}}{2a^2\alpha}$$

$$p_3 = a^2b^2\beta_1,\ p_4 = 3a^2b\,\beta_1\,\delta,\ p_5 = a^2b^2(\beta_2+\alpha_1\alpha_2\beta) + ab(a^2+b^2)\alpha_1\alpha_2 -$$

$$- \frac{2(a^2+b^2)\delta\beta_1}{b}.$$

The second approximation is obtained by putting (5.14) on the right fo (5.12). Hence

$$g_1(t^2) = g_0(t^2) + \int_a^b \frac{g_0(x^2)}{x} M_1(x^2, t^2)\, dx$$

$$= \frac{KP}{2a^2 S(t^2)} \Bigg[\left(t^2 - \frac{b^2 E}{F} \right) (p_1 + p_2 \rho^4) + \left(\frac{1}{t^2} - \frac{E}{a^2 F} \right) +$$

$$+ \left\{ (p_3 + 2b^2 q_1) \rho^2 + (p_4 + 2b^2 q_2) \rho^4 + (p_5 + 2b^2 q_3) \rho^6 \right\} +$$

$$+ \left(\gamma + \frac{1}{a^2 t^2} - \frac{1}{t^4} \right) \left\{ 3p_3 \rho^4 + (2b^2 q_4 - 2\beta_1 (a^2+b^2) \rho^6 \right\} + 0(\rho^8) \Bigg], \tag{5.15}$$

where

$$q_1 = \left(ab - \frac{b^2 E}{F} \right) p_1, \quad q_2 = \left\{ \frac{a^2+b^2}{2a^2b^2} - \frac{E}{a^2 F} \right\} p_3$$

$$q_1 = \frac{q_1 p_2}{p_1} + \frac{q^2 p_4}{p_3} + \frac{3\beta_1}{4a^2b^2} \{4a^4 b^4 \gamma + 2b^2 (a^2 + b^2) -$$

$$-3(a^4 + b^4) - a^8 b^8\} - \frac{2p_1 \delta}{a^2 b} \{2a^2 - (a^2 + b^2) E/F\},$$

$$q_4 = \frac{2p_1}{a^2} \{(a^2 + b^2) E/F - 2a^2\} \quad .$$

The higher successive approximations can be continued similarly.

6- QUANTITIES OF PHYSICAL INTEREST :

Stress intensity factors :

The stress intensity factors at the two ends of a crack are given by

$$\left.\begin{aligned} N_a &= \underset{r\to a-}{Lt}\ (a-r)^{\frac{1}{2}}\left[\ \sigma_{\theta\theta}\,(r,0)\ \right]\ 0<r<a \\ N_b &= \underset{r\to b+}{Lt}\ (r-b)^{\frac{1}{2}}\left[\ \sigma_{\theta\theta}\,(r,0)\ \right]\ r>b. \end{aligned}\right\} \tag{6.1}$$

The equation (3.4) together with expressions for $\phi\,(z)$, $\omega\,(z)$, $T\,(r,\theta)$ and $V\,(r,\theta)$ and their corresponding derivatives, yields on separating the real part

$$\sigma_{\theta\theta}\,(r,0) = -2\int_a^b \frac{tg\,(t^2)}{t^2-r^2}\,dt - K\int_a^q \frac{\log|\,1-t^2/r^2\,|\,h\,(t^2)}{t}\,dt +$$

$$+\,2\int_a^b \frac{g\,(t^2)}{t}\,K_1\,(t^2,r^2)\,dt + \frac{K}{2}\int_a^b \frac{h\,(t^2)}{t}K_3\,(t^2,r^2)dt,$$

where $K_1\,(t^2,r^2)$ is given by (5.9) and

$$K_3\,(t^2,r^2) = -\log\,(\rho/t) + \rho^2/r^2 + \sum_{n=1}^{\infty}\left\{\frac{\rho^2}{rt}\right\}^{2n}\left[\ \frac{n+1}{2n} +\right.$$

$$\left. + (2n+1)\,\frac{t^2}{r^2} - 4n\,(\rho^2/r^2) + \frac{2n^2+5n+1}{2\,(n+1)}\ \frac{\rho^4}{r^2\,t^2}\right].$$

For $0<r\leqslant a$, we have

$$-2\int_a^b \frac{t\,g\,(t^2)}{t^2-r^2}\,dt = \frac{-KP\pi}{a^2}\left[\ (p_1+p_2\,\rho^4)\ \left\{1+\frac{r^2-b^2\,E/F}{R}\right\}+\right.$$

$$+\ \{(p_3+2b^2q_1)\,\rho^2 + (p_4+2b^2\,q_2)\,\rho^4 + (p_5+2b^2\,q_3)\,\rho^6\} +$$

$$+\ \left\{\frac{1}{R}\left(\frac{1}{r^2}-\frac{E}{a^2F}\right)-\frac{1}{abr^2}\right\} +$$

$$+ \{3p_3 \rho^4 + (2b^2 q_4 - 2 \beta_1 (a^2 + b^2) \rho^6\} \left\{ \frac{1}{R} (r + a^{-2} r^{-2} - r^{-4}) + \right.$$

$$\left. + \frac{1}{r^2 ab} \left(\frac{1}{r^2} + \frac{(a^2+b^2)}{2a^2 b^2} - \frac{1}{a^2} \right) \right\} + 0 (\rho^8) \Bigg], \qquad (6.2)$$

and

$$- K \int_a^b \frac{\log | 1 - t^2/r^2 | h (t^2)}{t} dt = \frac{-PK\pi}{2ab} \Bigg[2 \{\beta_1 + (\beta_2 + \alpha_1 \alpha_2 \beta) \rho^4\} \times$$

$$\times \log \frac{a\sqrt{(b^2 - r^2)} + b\sqrt{(a^2 - r^2)}}{r (a+b)} + \left(\frac{R}{r^2} - \frac{ab}{r^2} - \frac{b}{a} + 2 + \right.$$

$$\left. + \frac{(a^2+b^2)}{2ab} \left\{ 2 \log \left(\frac{b-a}{b+a} \right) + \log \frac{a^2 (b^2 - r^2) + b^2 (a^2 - r^2) + 2ab R}{r^2 (b^2 - a^2)} \right\} \right)$$

$$\alpha_1 \alpha_2 \rho^4 + 0 (\rho^8), \Bigg], \qquad (6.3)$$

where $R = \left\{ (a^2 - r^2) (b^2 - r^2) \right\}^{1/2}$. In deriving (6.2) and (6.3), we have used results (14, 15) of the appendix. Again, for $r \geqslant b$, we have

$$- 2 \int_a^b \frac{t g (t^2)}{t^2 - r^2} dt = - \frac{KP\pi}{2a^2} \Bigg[(p_1 + p_2 \rho^4) \left\{ 1 - \frac{r^2 - b^2 E/F}{R_1} \right\} +$$

$$+ \left\{ (p_3 + 2b^2 q_1) \rho^2 + (p_4 + 2b^2 q_2) \rho^4 + (p_5 + 2b^2 q_3) \rho^6 \right\}$$

$$\left\{ \frac{1}{R_1} \left(\frac{E}{a^2 F} - \frac{1}{r^2} \right) - \frac{1}{abr^2} \right\} +$$

$$+ \{3p_3\,\rho^4 + (2b^2 q_4 - 2\,\beta_1\,(a^2+b^2)\,\rho^6\} \left\{ \frac{1}{R_1} \quad (r - a^{-2}\,r^{-2} + r^{-4} \quad + \right.$$

$$+ \frac{1}{abr^2}\left(r^{-2} + \frac{a^2+b^2}{2a^2\,b^2} - a^{-2} \right) \Big\} + 0\,(\rho^8) \Big] , \tag{6,4}$$

and

$$-K \int_a^b \frac{\log|\,1-t^2/r^2\,|\,h\,(t^2)}{t}\,dt = -\frac{KP\pi}{2ab}\Big[\ 2\,\{\,\beta_1 + (\beta_2+\alpha_1\alpha_2\beta)\,\rho^6\}\ \times$$

$$\log \frac{a\sqrt{(r^2-b^2)} + b\sqrt{(r^2-a^2)}}{r\,(a+b)} + \left(2 - \frac{a}{b} - \frac{ab}{r^2} - \frac{R_1}{r^2} + \frac{(a^2+b^2)}{2ab} \right.$$

$$\left. \left\{ 2\log\left\{ \frac{b-a}{b+a} \right\} + \log\frac{a^2\,(r^2-b^2)+b^2\,(r^2-a^2)+2ab\,R_1}{r^2\,(b^2-a^2)} \right\} \right)$$

$$\alpha_1\,\alpha_2\,\rho^4 + 0\,(\,\rho^8\,)\ \Big] , \tag{6.5}$$

where $R_1 = \sqrt{\{\,(r^2-b^2)\,(r^2-a^2)\,\}}$.

It is easy to show with the help of results of the appendix that

$$2\int_a^b \frac{g\,(t^2)}{t}\,K_1\,(r^2,t^2)\,dt = \frac{KP\pi}{2a^2\,r^2}\Big[\ p_1\,c_0\,\rho^2 + (p_3+2b^2\,q_1)\ C^2\,\rho^4 +$$

$$+ \left\{ (p_2\,c_0 + 4\,p_1\,c_1\,r^{-2}) + c_2\,(p_4 + 2b^2\,q_2) + 3a^2b^2\beta_1\,c_3 \right\} \rho^6 + 0\,(\rho^8) \Big] \tag{6.6}$$

where

$$c_0 = 1 - \frac{bE}{aF},\ c_1 = \frac{1}{ab}\Big[\ 1 - (1+b^2/a^2)\ E/F\ \Big]$$

$$c_2 = \frac{1}{2a^3b}\left[\left(\frac{a^2}{b^2} + 1 \right) - \frac{2E}{F} \right], c_3 = \frac{1}{8a^5b^5}\left[+ \right.$$

$$\left. + 8\gamma a^4b^4 + 2a^2b^2 + b^4 - 3a^4 \right];$$

and

$$\frac{K}{2}\int_a^b K_3\,(t^2, r^2)\;\frac{h(t^2)}{t}\;dt = \frac{KP\,\pi}{2}\left[d_0 + d_1\;\frac{\rho^2}{r^2} + (d_2 + a_2\beta_1 r^{-2} + \right.$$

$$\left. + 3d_1\,r^{-4})\;\rho^4 + (d_3 r^{-2} - 4\,a_2\,\beta_1\,r^{-4})\;\rho^6 + 0\;(\rho^8) \right], \tag{6.7}$$

where

$$d_0 = \frac{(a_1 - 1)\,\beta_1}{a}, d_1 = \frac{\beta_1}{2ab}, d_2 = (\beta_2 + a_1\,a_2\,\beta)\frac{d_0}{\beta_1} +$$

$$+ \frac{a_1 a_2}{ab}\left\{ (a^2 + b^2))\;\frac{d_0}{\beta_1} + \frac{(a-b)^2}{4ab} \right\},$$

$$d_3 = \frac{1}{2a^2b^2}\;[a_1\,a_2\,(a^2+b^2) + ab\;(\beta_2 + a_1\,a_2\,\beta)\,].$$

Hence the stress intensity factors as estimated from (6.1) are

$$\mathcal{N}_a = \frac{KP\,\pi}{2a^2\sqrt{2a\,(b^2 - a^2)}}\left[(p_1 + p_2\,\rho^4)\left(\frac{b^2E}{F} - a^2 \right) + \right.$$

$$+ \{ (p_3 + 2b^2\,q_1)\,\rho^2 + (p_4 + 2b^2\,q_2)\,\rho^4 + (p_5 + 2b^2\,q_3)\,\rho^6 \} +$$

$$+ \left(\frac{E}{F} - 1 \right) a^{-2} - \gamma \left\{ 3\,\beta_1\,a^2b^2\,\rho^4 + \right.$$

$$\left. \left. + (2b^2\,q_4 - 2\,(a^2 + b^2)\beta_1\;\rho^6 \right\} + 0\;(\rho^8) \right] \tag{6.8}$$

$$\mathcal{N}_b = \frac{KP\pi}{2a^2\sqrt{2b\,(b^2-a^2)}}\Big[\; b^2\,(p_1 + p_2\,\rho^4)\Big(1 - \frac{E}{F}\Big) +$$

$$+ \{\,(p_3+2b^2\,q_1)\,\rho^2+(p_4+2b^2\,q_2)\,\rho^4 + (p_5+2b^2\,q_3)\,\rho^6\}\,\Big(\frac{1}{b^2}-\frac{E}{a^2F}\Big) +$$

$$+(\gamma - a^{-2}b^{-2} + b^{-4})\Big\{3\,\beta_1\,a^2\,b^2\,\rho^4 \;+$$

$$+ (2b^2\,q_4 - 2\;(a^4 + b^4)\,\beta_1)\,\rho^6\;\Big\} + 0\,(\rho^8)\Big]\,. \tag{6.9}$$

The normal displacement along the crack is given by

$$\mu\,u_\theta\;(r,\,0) \;=\; \frac{1+k}{2}\int_0^\infty B\,(\xi)\,\cos\,(\xi r)\;\;d\xi\,, \qquad\qquad a \leqslant r \leqslant b. \tag{6.10}$$

Substituting the value of $B\,(\xi)$ from (5.3) and performing the change of order of integration, we obtain

$$\mu\,u_\theta\;(r,\,0) = \frac{\pi\,(1+k)}{4}\int_r^b g\;(t^2)\;dt. \tag{6.11}$$

Hence

$$\mu u_\theta(r,\,0) \;=\; \frac{\pi PK\,(1+k)}{8a^4b}\Big[\Big\{E(\phi,\,q) - \frac{E}{F}\;F\,(\phi,q)\;\Big\}\Big\{a^2b^2p_1 +$$

$$(p_3+2b^2q_1)\,\rho^2+(a^2b^2p_2+p_4+2b^2q_2)\,\rho^4+(\,p_5+2b^2q_3)\rho^6\;\Big\}+$$

$$+\;\frac{1}{3a^2b^2}\Big\{\,(b^2-2a^2)\,E\,(\phi,\,q)\;+a^2(3a^2b^2\gamma+1)\;F\,(\phi,\,q)+$$

$$\frac{\{(r^2-a^2)\;(b^2-r^2)\}^{\frac{1}{2}}}{r^2}\Big\}\Big\{\;3\beta_1a^2b^2\rho^4+2(b^2q_4-(a^2+b^2)\beta_1)\rho^6\;\Big\}\;+0\;(\rho^8)\;\Big] \tag{6.12}$$

where

$$\sin\,\phi=\Big\{(b^2-r^2)\;/\;(b^2-a^2)\Big\}^{\frac{1}{2}}.$$

REFERENCES

1. K. N. Srivastava, M. Kumar & Baij Nath, Thermal stresses in an infinite cylinder containing two strip cracks (under communication.)

2. K. N. Srivastava & M. Kumar, Distribution of stresses in an infinite solid containing a cylindrical cavity and two strip cracks, Ind. Journ. Pure and Appl. Maths. (shortly appearing)

3. K. N. Srivastava & M. Lowengrub, Finite Hilbert Transform technique for lriple integral equations involving trigonometric Kernels, Proc. Roy. Soc. Edin. A68, pp. 309-21, 1970.

4. I. S. Gradshteyn and I. M. Ryzhik : Tables of Integrals, series and Products, Academic Press, 1965.

APPENDIX

It is simple to derive the following results.

$$\int_0^\infty e^{-\xi\rho\sin\theta}\left(\frac{1-\cos\xi t}{\xi}\right)\left\{\begin{matrix}\cos(\xi\rho\cos\theta)\\ \sin(\xi\rho\cos\theta)\end{matrix}\right\}d\xi=\begin{cases}-\log(\rho/t)-\sum_{n=1}^{\infty}\left(\frac{\rho}{t}\right)^{2n}\frac{\cos 2n\theta}{2n}\\ (\pi/2-\theta+\sum_{n=1}^{\infty}\left(\frac{\rho}{t}\right)^{2n}\frac{\sin 2n\theta}{2n}\end{cases}\tag{1}$$

$$\int_0^\infty e^{-\xi\rho\sin\theta}(1-\cos\xi t)\left\{\begin{matrix}\cos(\xi\rho\cos\theta)\\ \sin(\xi\rho\cos\theta)\end{matrix}\right\}d\xi=\sum_{n=0}^{\infty}\frac{\rho^{2n-1}}{t^{2n}}\left\{\begin{matrix}\sin(2n-1)\theta\\ \cos(2n-1)\theta\end{matrix}\right\}\tag{2}$$

$$\int_0^\infty e^{-\xi\rho\sin\theta}\sin\xi t\left\{\begin{matrix}\cos(\xi\rho\cos\theta)\\ \sin(\xi\rho\cos\theta)\end{matrix}\right\}d\xi=\sum_{n=0}^{\infty}\frac{\rho^{2n}}{t^{2n+1}}\left\{\begin{matrix}\cos 2n\theta\\ \sin 2n\theta\end{matrix}\right\}\tag{3}$$

$$\int_0^\infty e^{-\xi\rho\sin\theta}\xi\sin\xi t\left\{\begin{matrix}\cos(\xi\rho\cos\theta)\\ \sin(\xi\rho\cos\theta)\end{matrix}\right\}d\xi=\pm\sum_{n=0}^{\infty}\frac{2(n+1).\rho^{2n+1}}{t^{2n+3}}\left\{\begin{matrix}\sin(2n+1)\theta\\ \cos(2n+1)\theta\end{matrix}\right\}\tag{4}$$

From these results we can easily show that

$$\int_0^\infty e^{-\xi\rho\sin\theta}(1-\cos\xi t)\,\rho\sin\theta\sin(2\theta+\xi\rho\cos\theta)\,d\xi =$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\left\{\frac{\rho^{2n}}{t^{2n+1}}-\frac{\rho^{2n-1}}{t^{2n-1}}\right\}\sin 2n\theta \qquad (5)$$

$$\int_0^\infty e^{-\xi\rho\sin\theta}\left(\frac{1-\cos\xi t}{\xi}\right)\left[\xi\rho\sin\theta\cos(2\theta+\xi\rho\cos\theta)-\cos(\xi\rho\cos\theta)\right]d\xi =$$

$$t^{-1}\{\log(\rho/t)-\tfrac{1}{2}\}-\sum_{n=1}^{\infty}\left[\frac{n-1}{2n}\frac{\rho^{2n}}{t^{2n+1}}-\frac{1}{2}\frac{\rho^{2n-2}}{t^{2n-1}}\right]\cos 2n\theta, \qquad (6)$$

$$-\int_0^\infty \xi\rho e^{-\xi\rho\sin\theta}\sin\theta\sin(2\theta+\xi\rho\cos\theta)\sin\xi t\,d\xi =$$

$$=\sum_{n=1}^{\infty}\left[(2n-2)\frac{\rho^{2n-2}}{t^{2n-1}}-2n\frac{\rho^{2n}}{t^{2n+1}}\right]\sin 2n\theta \qquad (7)$$

$$-\int_0^\infty e^{-\xi\rho\sin\theta}\Big[(1-\xi\rho\sin\theta\cos 2\theta).\cos(\xi\rho\cos\theta)+\xi\rho\sin 2\theta\sin\theta$$

$$\sin(\xi\rho\cos\theta)\Big]\times\sin\xi t\,d\xi = -\frac{1}{t}-\sum_{n=1}^{\infty}(2n-2)\left[\frac{\rho^{2n-2}}{t^{2n-1}}-\frac{\rho^{2n}}{t^{2n+1}}\right].\cos 2n\theta \qquad (8)$$

Following results are well known and can be found in Gardshteyn and Ryzhik [4].

$$\int_a^b \{(t^2-a^2)(b^2-t^2)\}^{-\frac{1}{2}}\,dt = \frac{1}{b}F(\pi/2,q) = \frac{F}{b} \qquad (9)$$

$$\int_a^b t^2\{(t^2-a^2)(b^2-t^2)\}^{-\frac{1}{2}}\,dt = bE(\pi/2,q) = bE \qquad (10)$$

$$\int_a^b \frac{t\,dt}{\sqrt{\{(t^2-a^2)(b^2-t^2)\}}\,(t^2-r^2)} \begin{cases} \pi/_2\,\{(a^2-r^2)(b^2-r^2)\}^{-\frac{1}{2}}, & 0<r\leqslant a \\ 0 & , a<r<b \\ \pi/_2\,\{(r^2-a^2)(r^2-b^2)\}^{-\frac{1}{2}}, & r\geqslant b \end{cases} \tag{11}$$

Let $I_n = \displaystyle\int_0^{\pi/2} \frac{\log(a^2\sin^2\theta + b^2\cos^2\theta)}{(a^2\sin^2\theta+b^2\cos^2\theta)^n}\,d\theta$

Then

$$I_{n+1} = \frac{(a+b)}{nb}\int_0^{\pi/2}\frac{\sin^2\theta\,d\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^{n+1}} +$$

$$+\frac{(a+b)}{na}\int_0^{\pi/2}\frac{\cos^2\theta\,d\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^{n+1}} -$$

$$-\frac{I_n}{ab}-\frac{(a+b)}{2nab}\left(\frac{\partial I_n}{\partial a}+\frac{\partial I_n}{\partial b}\right), \tag{12}$$

and

$$\int_0^{\pi/2}\frac{\log(a^2\sin^2\theta+b^2\cos^2\theta)}{a^2\sin^2\theta+b^2\cos^2\theta}\,d\theta = \frac{\pi}{ab}\log\frac{2ab}{a+b}. \tag{13}$$

Following results can be easily derived from (12) and (13)

$$\int_a^b \frac{\log|1-t^2/r^2|}{t\sqrt{\{(t^2-a^2)(b^2-t^2)\}}} \begin{cases} \dfrac{\pi}{ab}\left[\log\dfrac{a\sqrt{(b^2-r^2)}+b\sqrt{(a^2-r^2)}}{r(a+b)}\right], & 0<r\leqslant a \\ \dfrac{\pi}{2ab}\log\dfrac{b-a}{b+a} & , a<r<b \\ \dfrac{\pi}{ab}\log\dfrac{a\sqrt{(r^2-b^2)}+b\sqrt{(r^2-a^2)}}{r(a+b)}, & r\geqslant b \end{cases} \tag{14}$$

$$\int_a^b \frac{\log|1-t^2/r^2|}{t^3\sqrt{\{(t^2-a^2)(b^2-t^2)\}}}dt = \begin{cases} \dfrac{\pi}{2a^2b^2}\left[1+\dfrac{R-ab}{r^2}\right]+\dfrac{\pi(a^2+b^2)}{4a^3b^3}+ \\ +\left[\log\dfrac{b-a}{b+v}+\right. \\ \left.+\log\dfrac{a^2(b^2-r^2)+b^2(a^2-r^2)+2abR}{r^2(b^2-a^2)}\right], \\ \qquad 0<r\leqslant a \\ \dfrac{\pi}{2a^2b^2}\left[1-\dfrac{ab}{r^2}\right]+\dfrac{\pi(a^2+b^2)}{4a^3b^3} \\ \log\dfrac{b-a}{b+a}, \qquad a<r<b \\ \dfrac{\pi}{2a^2b^2}\left[1-\dfrac{R_1+ab}{r^2}\right]+\dfrac{\pi(a^2+b^2)}{4a^3b^3} \\ \left[\log\dfrac{b-a}{b+a}+\right. \\ +\log\dfrac{a^2(r^2-b^2)+b^2(r^2-a^2)+2abR_1}{r^2(b^2-a^2)}, \\ \qquad r\geqslant b \end{cases} \tag{15}$$

where

$R=\sqrt{\{(a^2-r^2)(b^2-r^2)\}}$ and $R_1=\sqrt{\{(r^2-a^2)(r^2-b^2)\}}$.

Jñānabha, Sect. A, Vol. 2, July 1972.

m–SUBPARACOMPACT SPACES

BY

M. K. SINGAL AND PUSHPA JAIN

(*Received on 21st August, 1972*)

A space X is said to be subparacompact [3] if every open covering of X admits a σ -discrete closed refinement. Subparacompact spaces have been studied by Arhangel'skii [1], McAuley [17], Čoban [7], Burke ([3], [4], [5]), Burke and Stoltenberg [6], Singal and Jain ([23], [24]) and others. Burke [3] proved that subparacompactness of X is equivalent to each of the following three properties of X :

(i) Every open covering of X has a σ -locally finite closed refinement.

(ii) Every open covering of X has a σ -closure preserving closed refinment.

(iii) If C is an open covering of X, then there exists a sequence $\{V_n\}_{n=1}^{\infty}$ of open coverings of X such that for each point x of X there is a positive integer m (x) and a $U \in C$ such that st (x, V_m (x)) $\subset C$. (St(x, V_m (x) denotes the union of all those members of V_m (x) which contain the point x.)

In [13] and [23], countably subparacompact spaces are also studied. X is said to be countably subparacompact if every countable open covering of X admits a σ -discrete closed refinement. This definition is due to Hodel [13]. Further, it has been proved in [23] that countable subparacompactness of X is equivalent to each of the following properties of X :

(a) Every countable open covering of X has a σ -locally finite closed refinement.

(b) Every countable open covering of X has a σ - closure preserving closed refinement.

In the present paper, we generalize the concept of subparacompactness to m-subparacompactness, where m is an infinite cardinal. A space X will be called m-subparacompact if every open covering of X of cardinality $\leqslant m$ has a σ-discrete closed refinement. If $m = \aleph_0$, then m-subparacompact spaces are precisely the countably subparacompact spaces. For a space X, having an open base of cardinality $\leqslant m$, m subparacompactness is equivalent to subparacompactness. In section 1 of the present paper, some characterizations and relationship of m-subparacompactness with other covering properties are obtained. Section 2 deals with subsets of m-subparacompact spaces. In section 3, direct and inverse preservation of m-subparacompact spaces under certain types of mappings are studied, and in section 4, some sum theorems have been proved. Invertibility, simple extention and adjunction of m-subparacompact spaces are discussed in section 6.

1. Characterizations

Lemma 1.1 Let every open covering of X of cardinality $\leqslant m$ have a σ-closure preserving closed refinement. For each $n \in \mathcal{N}$ let $U(n) = \{U\alpha(n): \alpha \in \Omega\}$ be an open covering of X with $|\Omega| \leqslant m$ and $U\alpha(n+1) < U\alpha(n)$ for all $\alpha \in \Omega$. Then there is a sequence $\{v_n\}_{n=1}^{\infty}$ of closed coverings of X such that, for each $n \in \mathcal{N}$, $v_n = \bigcup_{m=1}^{\infty} v_m\ n)$ and the following conditions are satisfied :

(1). $v_m(n) = \{V\alpha,m(n) : \alpha \in \Omega\}$ and is closure preserving for each $m \in \mathcal{N}$.

(2) $V_{\alpha,m}(n) \subset U\alpha(n)$ for each $\alpha \in \Omega$, $m \in \mathcal{N}$.

(3) $V_{\alpha,m}(n) \subset V_{\alpha,m+1}(n)$ for each $\alpha \in \Omega$, $m \in \mathcal{N}$.

(4) $V_{\alpha,m}(n+1) \subset V_{\alpha,m}(n)$ for each $\alpha \in \Omega$, $m \in \mathcal{N}$.

Proof. Proof is on the same lines as the proof of Lemma 3.1 in [23 and is therefore omitted.

Using the above lemma, the following theorem can be proved in the same way as Theorem 3.1 in [23].

Theorem 1.1 The following properties of X are equivalent :

(a) Every open covering of X of cardinality $\leqslant m$ has a σ-discrete closed refinement.

(b) Every open covering of X of cardinality $\leqslant m$ has a σ-locally finite closed refinement.

(c) Every open covering of X of cardinality $\leqslant m$ has a σ -closure preserving closed refinement.

From the above theorem and Theorem 1.1 in [19] it follows that every m-paracompact normal space is m-subparacompact. Below we give an example (Example 1.1) to show that the converse is not true.

In the next theorem we obtain a condition which together with m-subparacompactness implies m-paracompactness. But before that, we give definitions of some of the concepts which have been recently introduced by Krajewski [15] and Smith and Krajewski [15].

A Space X is said to be **expandable** if corresponding to each locally finite family $\{F\alpha : \alpha \in \Omega\}$ of subsets of X there is a locally finite open family $\{G\alpha : \alpha \in \Omega\}$ of subsets of X such that $F\alpha \subset G\alpha$ for each $\alpha \in \Omega$. X is said to be **almost expandable** if for each locally finite family $\{F\alpha : \alpha \in \Omega\}$ of subsets of X there exists a point-finite open family $\{G\alpha : \alpha \in \Omega\}$ of subsets of X such that $F\alpha \subset G\alpha$ for each $\alpha \in \Omega$. X is **discretely expandable** if for every discrete family $\{F\alpha : \alpha \in \Omega\}$ of subsets of X there exists a locally finite family $\{G\alpha : \alpha \in \Omega\}$ of open subsets of X such that $F\alpha \subset G\alpha$ for each $\alpha \in \Omega$. X is called **almost-discretely expandable** if for every discrete family $\{F\alpha : \alpha \in \Omega\}$ of subsets of X there exists a point-finite open family $\{G\alpha : \alpha \in \Omega\}$ of subsets X such that $F\alpha \subset G\alpha$ for each $\alpha \in \Omega$.

For any cardinal number m, the cardinality dependent concepts of m—expandability, almost m—expandability, di^{s}crete m—expandability, almost discrete m—expandability can be difined in a natural manner.

Theorem 1.2 A space X is m—paracompact if X is m—expandable and m - subparacompact.

Proof. Let X be an m-expandable m-subparacompact space and let $u = \{U\alpha : \alpha \in \Omega\}$ be an open covering of X of cardinality $\leqslant m$. Since X is m-subparacompact, there exists a σ-locally finite closed refinement $v = \bigcup_{i=1}^{\infty} V_i$ of u. Without any loss of generality, we can take $V_i = \{V_{i,\alpha} : \alpha \in \Omega\}$. Since X

is m-expandable, therefore for each $i \in N$ there is a locally finite open family $w_i = \{W_{i,\alpha} : \alpha \in \Omega\}$ of subsets of X such that $V_{i,\alpha} \subset W_{i,\alpha}$. Also for each $V_{i,\alpha}$ there exists a $U_{i,\alpha} \in u$ such that $V_{i,\alpha} \subset U_{i,\alpha}$. Let $G_{i,\alpha} = U_{i,\alpha} \cap W_{i,\alpha}$, and let $C_i = \{G_{i,\alpha} : \alpha \in \Omega\}$. Then $C = \bigcup_{i=1}^{\infty} G_i$ is a σ-locally finite open refinement of u. For each $i \in N$, put $C_i = \bigcup \{G : G \in G_i\}$ Then $\{G_i : i \in N\}$ is a countable open covering of X. X is countably paracompact, since $\aleph_0$ expandability is equivalent to countable paracompactness [15, Theorem 2.5]. Let $\{H_\beta : \beta \in \Delta\}$ be a locally finite open refinement of $\{G_i : i \in N.\}$ For $\beta \in \Delta$, let $i(\beta) \in N$ such that $H_\beta \subset G_{i(\beta)}$. Then $\{H_\beta \cap G : G \in G_{i(\beta)}, \beta \in \Delta\}$ is a locally finite open refinement of u. Hence X is m-paracompact.

Corollary 1 1 A normal space X is m-paracompact if and only if X is m-expandable and m-subparacompact.

Example 1.1 An $\aleph_1$-subparacompact normal space which is not $\aleph_1$-paracompact.

Let F denote the normal but not collectionwise normal space constructed by Bing [2, Example G] where the underlying space P has cardinality $\aleph_1$. In [15] Krajewski proved that F is a normal metacompact Hausdorff space which is the countable union of closed paracompact subspaces (and hence subparacompact) but is not $\aleph_1$-expandable. Since every $\aleph$—paracompact space is $\aleph_1$-expandable [15, Theorem 2.4], therefore F is not even $\aleph_1$-paracompact.

We now give two theorems which exhibit relationship of m-subparacompact spaces with m-metacompact spaces. A space X is called m-**metacompact** if every open covering of X of cardinality $\leqslant m$ admits a point-finite open refinement.

Theorem 1.3 Every m-metacompact space in which every closed set is a G_δ - set is m-subparacompact.

Proof. Follows easily from Theorem 1 in [13].

Theorem 1.4 Every almost discretely m-expandable m-subparacompact space is m-metacompact.

Proof. Let $u = \{u_\alpha : \alpha \in \Omega\}$ be an open covering of X of cardinality $\leqslant m$. Since X is m-subparacompact, there is a σ-discrete closed refinement

$v=\bigcup_{i=1}^{\infty} v_i$. Without any loss of generality we can take $v_i=\{V(i,\alpha):\alpha\in\Omega\}$. Since X is almost discretely m-expandable there exists for each i, a point-finite open collection $k_i=\{k(i,\alpha):\alpha\in\Omega\}$ such that $V(i,\alpha)\subset k(i,\alpha)$ for all $\alpha\in\Omega$, For each $i\in\mathcal{N}$ and for each $\alpha\in\Omega$ there exists a member $U\alpha_i$ of u such that $V(i,\alpha)<U\alpha_i$. Take $G(i,\alpha)=k(i,\alpha)\cap U\alpha_i$. Then $C=\bigcup_{i=1}^{\infty} C_i$, where $C_i=\{G(i,\alpha):\alpha\in\Omega\}$ is a σ-point-finite open refinement of u. For each $i\in\mathcal{N}$ let $G_i=\bigcup\{G(i,\alpha):\alpha\in\Omega\}$. Then $\{G_i:i\in\mathcal{N}\}$ is a countable open covering of X. Since for every infinite cardinal m, an m-subparacompact space is countably metacompact [13] the covering $\{G_i:i\in\mathcal{N}\}$ admits a point finite open refinement $H=\{H_\beta:\beta\in\Delta\}$. For $\beta\in\Delta$, let $i(\beta)\in\mathcal{N}$ such that $H_\beta\subset G_{i(\beta)}$. Then $\{H_\beta\cap G:G\in C_{i(\beta)};\beta\in\Delta\}$ is a point finite open refinement of u. Hence X is m-metacompact.

Since every discrete family of subsets of a countably compact space is finite, the following result is immediate.

Theorem 1.5 Every m-subparacompact countably compact space is m-compact. (A space X is said to be m-compact if every open covering of X of cardinality $\leqslant m$ has a finite subcovering).

2. Subsets and m—subparacompactness

It can be easily verified that every closed subset of an m—subparacompact space is m—subparacompact. But, as with some other classes of topological spaces such as paracompact and collectionwise normality, a more general result holds.

Definition 2.1 A subset A of a space is said to be an $\mathbf{F}_\sigma$ **- subset** if it is a union of countable number of closed sets,

Definition 2.2 A subset A of X is said to be a **generalized $\mathbf{F}_\sigma$-subset** if for each open set U containing A there is an F_σ-subset B such that $A\subset B\subset U$.

Theorem 2.1 Every F_σ-subset of an m—subparacompact space is m—subparacompact.

Proof. Let A be an F_σ subset of an m—subparacompact space X. Let $U=\{u_\alpha:\alpha\in\Omega\}$, where $|\Omega|\leqslant m$ be a relatively open covering of A. There exists a collection U^* of open subsets of X such that $U^* = \{U\alpha = U^*_\alpha \cap A : U\alpha \in U\}$ Also there is a countable family $\{A_i\}$ of closed subsets of X such that $A = \bigcup_{i=1}^{\infty} A_i$.

For each i, let $w_i = U^* \cup \{X - A_i\}$. Then w_i is an open covering of X of cardinality $\leqslant m$. Let $v_i = \bigcup_{j=1}^{\infty} U_{i,j}$ be a σ-locally finite closed refinement of w_i. Let $V_{i,j}$ be the collection of all those members of $U_{i,j}$ which intersect A_i. Each $V_{i,j}$ is locally finite with respect to X. $\bigcup_{j=i}^{\infty} V_{i,j}$ is as dosed covering of A_i such that each member of $V_{i,j}$ is containfd in some $U\alpha^*$. Let $\omega_{i,j} = \{B \cap A : B \in U_{i,j}\}$ and let $w = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \omega_{i,j}$. Then w is a σ-locally finite (in A) closed (in A refinement of U. Hence A is m-subparacompact.

Theerem 2.2 If A is a subset of X such that every open subset of X which contains an m-subparacompact set that contains A, then A is m-subparacompact.

Proof. Let $U = \{U\alpha : \alpha \in \Omega\}$ be an open (in A) covering of A of cardinality $\leqslant m$. For each $\alpha \in \Omega$, let $V\alpha$ be an open subset of X such that $U\alpha = A \cap V\alpha$. Then $\bigcup_{\alpha \in \Omega} V\alpha$ is an open set containing A. Thus, by hypothesis there exists an m—subparacompact subset B of X such that $A \subset B \subset \bigcup_{\alpha \in \Omega} V\alpha$. Now $\{V\alpha \cap B : \alpha \in \Omega\}$ is an open (in B) covering of B of cardinality $\leqslant m$. Let $w = \bigcup_{i=1}^{\infty} w_i$ be a σ-locally finite (in B) closed (in B) refinement of $\{V\alpha \cap B : \alpha \in \Omega\}$. If we let $w_i = \{W_{ij} : j \in \Delta_i\}$ and $W_{ij} = V_{ij} \cap B$ where V_{ij} is closed in X. then each $V_{ij} \cap A$ is closed in A and the family $v_i = \{V_{ij} \cap A : j \in \Delta_i\}$ is locally finite in A for each i. Also $\bigcup_{i=1}^{\infty} v_i$ covers A. Hence A is m-subparacompact.

Corollary 2.1 Every generalized F_σ-subset of an m-subparacompact space is m-subparacompact.

Proof. Follows directly from the definition of generalized F_σ-subset and the above theorem.

Theorem 2.3 If every open subset of a space X is m—subparacompact then every subset of X is m—subparacompact.

Proof. Let A be any subset of X. Let $U=\{U_\alpha : \alpha \in \Omega\}$ be an open (in A) covering of A such that $|\Omega| \leqslant m$. For each $\alpha \in \Omega$, let U^*_α be an open subset of X such that $U_\alpha = A \cap U^*_\alpha$. Then $U^* = \{U^*_\alpha : \alpha \in \Omega\}$ is an open covering of $G = \bigcup_{\alpha \in \Omega} U^*_\alpha$ of cardinality $\leqslant m$. Since G is m—subparacompact, there exists a σ-locally finite closed (in G) refinemet $v^* = \bigcup_{n=1}^{\infty} v_n^*$ of U^*. For each $n \in N$ let $v_n = \{V \cap A : V \in v_n^*\}$. Then $v = \bigcup_{n=1}^{\infty} v_n$ is a σ-locally finite (in A) closed (in A) refinement of U. Hence A is m–subparacompact.

3. Mappings and m-subparacompactness

Theorem 3.1 Every closed continuous image of an m-subparacompact space is m-subparacompact.

Proof. Let $f : X \to Y$ be a closed continuous mapping of an m-subparacompact space X onto Y. Let $U = \{U_\alpha : \alpha \in \Omega\}$ be an open covering of Y of cardinality $\leqslant m$ Then $f^{-1}(U) = \{f^{-1}(U_\alpha) : U_\alpha \in U\}$ is an open covering of X of cardinality $\leqslant m$. Thus there is a σ-closure preserving closed refinement $v = \bigcup_{n=1}^{\infty} v_n$ of $f^{-1}(U)$. Since f is a closed and continuous mapping, each $f(v_n)$ is a closure preserving closed collecttion. Hence $f(v) = \bigcup_{n=1}^{\infty} f\, v_n)$ is a σ-closure preserving closed refinement of U. Hence Y is m subparacompact.

Definition 3.1 A mapping $f : X \to Y$ is said to be a **perfect mapping** if f is closed, continuous and $f^{-1}(y)$ is compact for each $y \in Y$.

Theorem 3.2 If X is a normal space and f is a perfect mapping from X onto Y, then X is m-subparacompact if Y is so.

Proof. Let $U = \{U_\alpha : \alpha \in \Omega\}$ be an open covering of X of cardinality $\leqslant m$. For each $y \in Y$ we can find a finite subcollection $U(y)$ of U such that $f^{-1}(y) \subset u(y) = \cup\{u : u \in U(y)\}$. Let $V(y) = Y - f(X - u(y))$. Then $v = \{V(y) : y \in Y\}$ is an open covering of Y Clearly, v is of cardinality $\leqslant m$. Since Y is m–subparacompact, therefore v has a σ-discrete closed refinement $v^* = \bigcup_{n=1}^{\infty} v_n^*$. Then $f^{-1}(v^*) = \{f^{-1}(V^*) : V^* \in v^*\}$ is a σ-discrete refinement of $\{u(y) : y \in Y\}$. Given $V^* \in v^*$, let $y(V^*)$ be a fixed element of Y such that $f^{-1}(V^*) \subset u(y(V^*))$. Let $u\{y(V^*)\} = \{u_1, \ldots, u_{k(V^*)}\}$, so that $u(y(V^*))$ is a finite open covering of the normal space $f^{-1}(V^*)$. Thus there is an open covering $\{u_1^*, u_2^*, \ldots u^*_{k(V^*)}\}$ of $f^{-1}(V^*)$ such that $u_i^* \subset u_i$ for all $i = 1, 2, \ldots, k(V^*)$ For each $n \in N$, let $w_n^* = \{f^{-1}(V^*) \cap u_i^* : V^* \in V_n^*, u_i \in U(y(V^*))\}$. We shall prove that each w_n^* is locally finite in X. Let $x \in X$. Since $f^{-1}(v_n^*)$ is a discrete collection in X, there exists a neighbourhood N_x of x which intersects at the most one member of $f^{-1}(v_n^*)$. Since each element of $f^{-1}(v_n^*)$ intersects at the most finitely many members of w_n^* and each member of w_n^* is contained in some element of $f^{-1}(v_n^*)$, it follews that N_x will intersect only finitely many members of w_n^*. Thus every open cover of $X$of cardinality $\leqslant m$ has a σ-locally finite closed refinement. Hence X is m-subparacompact.

4. Sum Theorems :

Sum theorems give conditions under which the union of topological spaces of given type are of the same type. Various sum theorems have been given for the class of paracompact, regular, completely normal, metrizable, m-paracompact and normal, subparacompact and other classes of spaces. In this section we shall obtian some sum theorems for the class of m-subparacompact spaces. It is easy to note that a countable union of closed m-subparacompact spaces is m-subparacompact. Throughout in the section, p will denote a topological property which is closed hereditary and which satisfies the locally finite sum theorem, which states the following :

'If $\{F_\alpha : \alpha \in \Omega\}$ is a locally fiinite closed covering of X such that each F_α possesses the property p, then X possesses p.'

Hodel [13] proved that p satisfies the following sum theorems.

Theorem 4.1 Let X be a topological space and let U be a σ-locally finite open covering of X such that the closure of each element of U possesses the property p. Then X possesses p.

Theorem 4.2 Let X be a topological space and let v be a σ-locally finite elementary covering (see Definition 4.1 below) of X. Then X possesses the property p if each $V \in v$ possesses p.

Theorem 4.3 Let X be a regular space and let v be a σ-locally finite open covering of X, each element of which possesses p and has the compact frontier. Then X has the property p.

Definition 4.1 A subset A of X is said to be **elementary** if it is open and is the union of a countable family of open subsets, the closure of each member of which is contained in A. A covering of X consisting of elementary sets is called an **elementary covering**,

Definition 4.2 (Y. Katuta, [14]). A family $\{A_\alpha : \alpha \in \Omega\}$ is said to be **order locally finite** if there is a linear ordering '$<$' on Ω such that for each $\alpha \in \Omega$, the family $\{ A_\lambda : \lambda < \alpha \}$ is locally finite at each point of A_α.

Every σ-locally finite family is order locally finite but not conversely .

In [22] Singal and Arya obtained the following sum theorems for p. These theorems generalise Theorems 4.1 and 4.3.

Theorem 4.4 Let v be an order locally finite open covering of a space X such that closure of each member of v possesses the property p. Then X possesses p.

Theorem 4.5 If X is a regular space, v is an order locally finite open covering of X each member of which possesses the property p and if the frontier of each member of v is compact, then X possesses the property p.

we shall prove that all these sum theorems hold for m-subparacompact spaces also. For that we have to first prove that the locally finite sum theorem holds for m-subparacompact spaces.

Theorem 4.6 If $\{F_\alpha : \alpha \in \Omega \}$ be a locally finite closed covering of X such that F_α is m-subparacompact, then X is m-subparacompact.

proof Let $u\{U_\beta : \beta \in \triangle \}$ be an open covering of X of cardlnality $\leq m$. For each $\alpha \in \Omega$, $\{U_\beta \cap F_\alpha : \beta \in \triangle\}$ is then an open (in F_α) covering of F_α of cardinality $\leq m$. Thus there exists a family $U^\alpha = \bigcup\limits_{i=1}^{\infty} V_i^\alpha$ of closed subsets (of F_α and hence of X) such that each V_i^α is a discrete (in F_α and hence in X) family of subsets of F_{α} such that V^{α} is a covering of F_{a}. For each i, let $w_i = \bigcup_{\alpha \in \Omega} V_i^{\alpha}$ and let $w = \bigcup_{i=1}^{\infty} w_i$. Then w is a closed covering of X which is a refinement of U. Also it can be proved that each w_i is locally finite. Thus w is a σ - locally finite closed refinement of U and hence X is m-subparacompact.

Theorem 4.1 Every disjoint topological sum of m-subparacompact spaces is m-subparacompact.

In view of the Theorem 4.6 and the fact that m-subparacompactness is a closed hereditary property, we have the following results.

Theorem 4.7 If v is an order locally finite open covering of a space X such that the closure of each member of v is m-subparacompact, then X is m-subparacompact.

Theorem 4.8 If X is regular and V is an order locally finite open covering of X such that each member of v is m-subparacompact and the frontier of each member of v is compact, then X is m-subparacompact.

Theorem 4.9 If v is a σ -locally finite elementary covering of X such that each element of v is m-subparacompact, then X is m-subparacompact.

We now obtain some other sum theorems as consequences of the locally finite sum theorem for m-subparacompact spaces. The proofs of all these theorems follow same arguments as the corresponding theorems for subparacompact spaces in [23] and [24], We, therefore, only state these theorems.

Theorem 4.10 Let X be a regular space and let v be a locally finite open covering of X such that for each $V \in v$, V is m-subparacompact and $Fr\, V$ is Lindelöf. Then X is m-subparacompact.

Theorem 4.11 Let $U = \{U_{\alpha} : \alpha \in \Omega\}$ be a locally finite open covering of a normal space X such that each U_{α} is m-subparacompact space. Then X is m-subparacompact.

Theorem 4.12 If $\{U_{\alpha} : \alpha \in \Omega\}$ is a point finite open covering of a collectionwise normal space such that each U_{α} is m-subparacompact, then X is m-subparacompact.

An open covering U of X is said to be **noamai** if there exists a sequence $\{U_n : n=1,2,\ldots\}$ of open coverings or X such that each U_{n+1} is a star refinement of U_n (that is, the covering $\{St(x, U_{n+1}) : x \in X\}$ refines U_n) and U_1 is a refinement of U.

Theorem 4.13 Let $\{U\alpha : \alpha \in \Omega\}$ be a normal open covering of a normal space X such that each $U\alpha$ is m-subparacompact. Then X is m-subparacompact.

Therem 4.14 Let $\{U\alpha : \alpha \in \Omega\}$ be a σ-locally finite open covering of a normal space X such that $U\alpha$ is an F_σ -subset of X. Then X is m-subparacompact if each $U\alpha$ is m-subparacompact.

Theorem 4.15 Let $\{U\alpha : \alpha \in \Omega\}$ be a σ-locally finite open covering of a countably paracompact normal space X such that each $U\alpha$ is m- subparacompact. Then X is m-subparacompact.

Definition 4 3 [Frolik, 11]. A space X is said to be **weakly regular** if every open subset of X contains a non-empty regularly closed set.

Theorem 4.16 Every space which contains a proper non-empty regularly closed set is m–subparacompact if and only if every regularly closed subset of X is m-subparacompact.

Corollary 4.2 A weakly regular space X is m-subparacompact if and only if every proper regularly closed subset of X is m-subparacompact.

Corollary 4.3 A semi-regular space X is m-subparacompact if and only if every proper regularly closed subset of X, is m-subparacompact.

5. Embedding of m-subparacompact spaces

In [21] Mrowka proved that a locally m-paracompact completely regular space can be embedded in an m-paracompact space as an open subspace. We wish to prove the same for m-subparacompact spaces.

We need the following lemma.

Lemma 5.1 Let X be a regular space and let $\mathcal{G}$ be an open basis of neighbourhoods of a point $x \in X$ such that $X—G$ is m-subparacompact for each $G \in \mathcal{G}$, then X is m-subparacompact.

Proof. Let $\mathrm{U}=\{U\alpha : \alpha \in \Omega\}$ be an open covering of X of cardinality $\leqslant m$. For $x \in X$ there is a $\alpha_x \in \Omega$ such that $x \in U_{\alpha_x}$. Let $G_x \in \mathcal{G}$ be such that $x \in G_x \subset \overline{G_x} \subset U_{\alpha_x}$. Since $X - G_x$ is m—subparacompact and $\{(X-G_x) \cap U\alpha : \alpha \in \Omega\}$ is an open (in $X-G_x$) covering of $X-G_x$ of cardinality $\leqslant m$, therefore there exists a σ-locally finite (in $X-G_x$ and hence in X) closed (in $X-G_x$ and hence in X) refinement $v=\bigcup_{n=1}^{\infty} v_n$ of $\{(X-G_x) \cap U_\alpha : \alpha \in \Omega\}$.

Let $v_0=\{\overline{G_x}\}$. Then $\bigcup_{n=0}^{\infty} v_n$ is a σ — locally finite closed refinement of U. Hence X is m—subparacompact.

Let us call a space locally m—**subparacompact** if each point of X has a neighbourhood whose closure is m—subparacompact.

Theorem 5.1 Every completely regular locally m-subparacompact space can be embedded in an m-subparacompact space as an open subspace.

Proof. Since m-subparacompactness is a closed hereditary property and is finitely additive for closed subsets, therefore Lemma 5.1 shows that m—subparacompactness satisfies the condition (w) of Mrowka [21]. Hence the result follows as in [21].

Corollary 5.1 Every completely regular locally subparacompact space can be embedded in a subparacompact space as an open subspace.

6. Invertibility, Simple Extensions and Adjunction of m-subparacompact Spaces.

Definition 6.1 [Doyle and Hocking, 9]. A space X is said to be **invertible** if for each open subset U of X there exists a homeomorphism $h : X \to X$ such that $h(X-U) \subset U$. h is called an **inverting homeomorphism** for U.

Theorem 6.1 Let X be an invertible space in one of its open sets U and let $\bar{u}$ be m-subparacompact. Then X is m-subparacompact.

Proof. Let $h : X \to X$ be an inverting homeomorphism for U. Then $h(X\text{-}U) \subset U$, and therefore $X=\bar{u} \cup h(\bar{u})$ where $\bar{u}$ and $h(\bar{u})$ are both m-subparacompact. Hence X is m-subparacompact, since a countable union of closed m-subparacompact subspaces is m-subparacompact.

Definition 6.2 [Levine, 16]. Let (X, T) be any topological space and let A be a subset of X such that $A \notin T$. Then the topology $T(A) = \{U \cup (V \cap A) : U, V \in T\}$ is called a **simple extension** of T.

It can easily be checked that $(A, T \cap A) = (A, T(A) \cap A)$ and $(X\text{-}A, T \cap (X\text{-}A)) = (X\text{-}A, T(A) \cap (X\text{-}A))$.

Theorem 6.2 If (X, T) is an m-subparacompact space and $A \subset X$ is such that $X\text{-}A \in T$, then $(X, T(A))$ is m-subparacompact if and only if $(X\text{-}A, T \cap (X\text{-}A))$ is m-subparacompact.

Proof. The 'only if' part follows from the facts that $(X\text{-}A, T \cap (X\text{-}A)) = (X\text{-}A, T(A) \cap (X\text{-}A))$ and every closed subspace of an m-subparacompact space is m-subparacompact. We shall now prove the 'if' part. Since (X, T) is m-subparacompact and A is a closed subspace of (X, T), it follows that $(A, T \cap A)$ is m-subparacompact. Thus if $(X\text{-}A, T \cap (X\text{-}A))$ is also m-subparacompact, X is the union of two closed m-subparacompact subspaces of $(X, T(A))$. Hence $(X, T(A))$ is m-subparacompact.

Theorem 6.3 If (X, T) is hereditarily m—subparacompact, and A is a subset of X such that $X\text{-}A \in T$, then $(X, T(A))$ is also hereditarily m—subparacompact.

Proof. Since (X, T) is hereditarily m-subparacompact, therefore $(X—A, T \cap (X—A))$ is also hereditarily m—subparacompact. The rest of the proof is on the same lines as the proof of Theorem 6.2.

Let X and Y be two topological spaces and let A be a closed subset of X. Let $f : A \to Y$ be a continuous mapping. Denote by $X+Y$ the disjoint topological sum of X and Y. Then an equivalence relation R may be defined as $(a, b) \in R$ if a=b or $a=f(b)$ or $b=f(a)$. The quotient space $(X+Y)/R$ is called the adjunction space obtained by joining X to Y by means of the mapping f. The adjunction space is denoted by $X \underset{f}{\cup} Y$ and f is called the attaching map.

Let p denote the natural mapping (or projection) of $X+Y$ to $X \underset{f}{\cup} Y$; that is p maps a point of $X+Y$ to the equivalence class containing that point. We paracompact. Hence X is m-subparacompact, since a countable union of closed m-subparacompact subspaces is m-subparacompact.

Definition 6.2 [Levine, 16]. Let (X, T) be any topological space and let A be a subset of X such that $A \notin T$. Then the topology $T(A) = \{U \cup (V \cap A) : U, V \in T\}$ is called a **simple extension** of T.

It can easily be checked that $(A, T \cap A) = (A, T(A) \cap A)$ and $(X\text{-}A, T \cap (X\text{-}A)) = (X\text{-}A, T(A) \cap (X\text{-}A))$.

Theorem 6.2 If (X, T) is an m-subparacompact space and $A \subset X$ is such that $X\text{-}A \in T$, then $(X, T(A))$ is m-subparacompact if and only if $(X\text{-}A, T \cap (X\text{-}A))$ is m-subparacompact.

Proof. The 'only if' part follows from the facts that $(X\text{-}A, T \cap (X\text{-}A)) = (X\text{-}A, T(A) \cap (X\text{-}A))$ and every closed subspace of an m-subparacompact space is m-subparacompact. We shall now prove the 'if' part. Since (X, T) is m-subparacompact and A is a closed subspace of (X, T), it follows that $(A, T \cap A)$ is m-subparacompact. Thus if $(X\text{-}A, T \cap (X\text{-}A))$ is also m-subparacompact, X is the union of two closed m-subparacompact subspaces of $(X, T(A))$. Hence $(X, T(A))$ is m-subparacompact.

Theorem 6.3 If (X, T) is hereditarily m—subparacompact, and A is a subset of X such that $X\text{-}A \in T$, then $(X, T(A))$ is also hereditarily m—subparacompact.

Proof. Since (X, T) is hereditarily m-subparacompact, therefore $(X—A, T \cap (X—A))$ is also hereditarily m—subparacompact. The rest of the proof is on the same lines as the proof of Theorem 6.2.

Let X and Y be two topological spaces and let A be a closed subset of X. Let $f : A \to Y$ be a continuous mapping. Denote by $X+Y$ the disjoint topological sum of X and Y. Then an equivalence relation R may be defined as $(a, b) \in R$ if $a=b$ or $a=f(b)$ or $b=f(a)$. The quotient space $(X+Y)/R$ is called the adjunction space obtained by joining X to Y by means of the mapping f. The adjunction space is denoted by $X \underset{f}{\cup} Y$ and f is called the attaching map.

Let p denote the natural mapping (or projection) of $X+Y$ to $X \underset{f}{\cup} Y$; that is p maps a point of $X+Y$ to the equivalence class containing that point. We shall now show that the adjunction space $X \underset{f}{\cup} Y$ is m-subparacompact if X is m-subparacompact and Y is m-subparacompact and Hausdorff and A is a compact subspace of X.

We need the following result due to Dugundji [10; page 128]

Lemma 6.1 Let $X \underset{f}{\cup} Y$ be an adjunction space and let $p : X + Y \to X \underset{f}{\cup} Y$ be the projection mapping. If $F \subset X+Y$ is such that $F \cap X$ is closed in X, then $p\,(F)$ is closed if and only if $(F \cap Y) \cup f\,(F \cap A)$ is closed in Y.

Theorem 6.4 Let *X be an m-subparacompact space, and Y be an m-subparacompact Hausdorff space. Then $X \underset{f}{\cup} Y$ is an m-subparacompact space if the domain of the attaching map f is compact.

Proof. If $A \subset X$ is compact and Y is Hausdorff then for a closed subset F of $X+Y$, $p\,(F)$ is closed since $F \cap Y$ and $f\,(F \cap A)$ are both closed. Thus the mapping $p : X+Y \to X \underset{f}{\cup} Y$ is a closed continuous mapping. Hence the result follows in view of Corollary 4.1 and Theorem 3.1.

REFERENCES

1 A. Arhangel'skii : Mappings and spaces; Russian Math. Surveys 21 (1966), 115-162.

2 R. H. Bing : Metrization of topological spaces, Canad. J. Math. 3 (1951), 175-186.

3 D. K. Burke : On subparacompact spaces, Proc. Amer. Math. Soc. 23 (1969), 655-663.

4 D K. Burke : Subparacompact spaces, Proceedings of the topology conference, Washington State University, 1967.

5 D. K. Burke : On p-spaces and $\omega\Delta$ - spaces, Pac. J. Math. 35 (1970), 285-296.

6 D. K. Burke, and R.H. Stoltenberg : A note on p-spaces and Moore spaces, Pac. J. Math. 30 (1969), 601-608.

7 M. M. Čoban : On σ -paracompact spaces, Vestnik Moskov Univ. Ser-I. Mat. Meh. (1969), 20-27.

8 C. H. Dowker : On a theorem of Hanner, Ark. Mat. 2 (1962), 307-313.

9 P. H. Doyle, and J. G. Hocking : Invertible spaces, Amer. Math. Monthly, 68 (1961), 959-965.

10 J. Dugundji : Topology, Allyn and Bacon, Boston (1968).

11 Z. Frolik : Generalizations of compact and Lindel of spaces, Czech. Math. J. 9 (1959), 172-217.

12 R. E. Hodel : Sum theorems for topological spaces, Pacific J. Math. 30 (1969), 59-65.

13 R. E. Hodel : A note on subparacompact spaces, Proc. Amer. Math. Soc. 25 (1970), 842-845.

14 Y. Katuta : A theorem on paracompactness of product spaces, Proc. Japan Acad. 43 (1967), 615-618.

15 L. L. Krajewski : On cxpanding locally finite collections, Canad. J. Math. 23 (1971), 58-68.

16 N. Levine : Simple extensions of topologies, Amer. Math. Monthly, 71 (1964) 22-25.,

17 L; F. McAuley : A note on complete collectionwise normality and paracompactness. Proc. Amer. Math. Soc. 9 (1958), 796-799.

18 E. A. Michael : Point finite and locally finite coverings, Canad. J. Math. 7 (1955), 275-279.

19 K. Morita : Paracompactness and product spaces, Fund. Math. 50 (1961), 223-236.

20 K. Morita : Products of normal spaces with metric spaces, Math. Annalen 154. (1964), 365-382.

21 S. Mrowka : On local topological properties, Bull Acad. Polon. Sci. 5 (1957), 951-956.

22 M.K. Singal, and Shashi Prabha Arya : Two sum theorems for topological spaces, Israel J. Math. 8 (1970), 155-158.

23 M.K. Singal, and Pushpa Jain : On subparacompact and countably subparacompact spaces, Bull. Aust. Math. Soc. 5 (1971), 289-304.

24 M.K. Singal, and Pushpa Jain : A note on subparacompact spaces. To appear.

25 J. C. Smith, and L.L. Krajewski : Expandability and collectionwise normality, Trans. Amer. Math. Soc. 160 (1971), 855-883.

Institute of Advanced Studies,
Meerut University,
Meerut.

and

Maitreyi College,
Netaji Nagar,
New-Delhi-23

INFORMATION FOR CONTRIBUTORS/SUBSCRIBERS

JÑĀNABHA is published annually in two sections: Section A (Sciences) and Section B (Humanities). Papers intended for publication in this section should be in typed form or offset-reproduced, (not dittoed), double spaced with generous margins, and they may be written in Hindi or English. Manuscripts, in duplicate if possible, may be submitted to either one of the Editors.

25 reprints are provided free for each article; additional reprints, in multiples of 25, may be obtained at cost price, if the authors give prior intimation while returning proofs.

Effective with volume 2 the price per volume (both sections) is Rs. 35.00 (or U.S. $15.00); each section: Rs. 20.00 (or U.S. $10.00). Individual members of the Vijnana Parishad are entitled to free subscriptions to the current issues of Section A. Individual membership: Rs. 15.00 (or U.S. $5.00) per calendar year; Life membership: Rs. 150.00 (or U S. $50.00). Back volumes are available at special price.

By a reciprocity agreement with the American Mathematical Society, an individual/life member of the Parishad residing outside the North American continent may join the Society by submitting an application for membership on the form that may be obtained from the office of the Society (P. O. Box 6248, Providence, Rhode Island 02940, U.S.A.) and by paying U. S. $12.00 per calendar year, which is only half of the current rate for annual dues for members whose annual professional income is below $15,000.00; the usual requirements that the applicant be endorsed by two members of the Society and that the candidate be elected by the Council are waived, but this reduction in dues to the Society does not apply to those individual/life members of the Parishad who reside, even on a temporary basis, in the North American area.

The mathematical content of this journal is indexed in the Society's biweekly CONTENTS OF CONTEMPORARY MATHEMATICAL JOURNALS AND NEW PUBLICATIONS and is reviewed, among others, by MATHEMATICAL REVIEWS (U.S.A.), ZENTRALBLATT FÜR MATHEMATIK (W. Germany) and APPLIED MECHANICS REVIEWS (U.S.A.).]

All communications regarding subscriptions, orders for back volumes, membership of the Vijnana Parishad, changes of address, etc., and all books for review, should be addressed to:

The Secretary,
VIJNANA PARISHAD,
D. V. Postgraduate College,
Orai, U.P., India.

CONTENTS

Printed at Ellora Printers Souterganj, Kanpur-1 Phone : 6 2 7 2 7