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APPLICATION OF GEGENBAUER POLYNOMIALS TO NONLINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS AND THEIR STABILITY

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1. Introduction. The nonlinear differential equations with periodic coefficients arise in certain physical problems like Melde's experiment on the vibrations of a thread [1, p.120] or the simple pendulum [1, p.132] with oscillating support. In each case we have a nonlinear differential equation with periodic coefficients. First and second order approximations of these are obtained in [1] and analysis though not difficult is somewhat involved. We in this note shall show that similar results, at times even better, can be obtained employing an equivalent linear differential equation to the above. We shall be employing certain ultraspherical polynomials here to obtain the linearisation. This technique of using ultraspherical polynomials has been first used by Denman and Liu [2], and Garde [3] and other quite successfully on the nonlinear differential equations with constant coefficients. The application of this technique to the nonlinear differential equation with periodic coefficients turns out equally successful since the equivalent linear equation obtained is very close to the original differential equation. The study of the stability conditions leads to amplitude dependent criteria.

2. GEGENBAUER POLYNOMIALS AND APPROXIMATION :

The Gegenbauer polynomials $C_n^{\lambda}(x)$ on the interval $(-1, 1)$ are the sets of polynomials orthogonal on this interval with respect to the weight factor $(1-x^2)^{\lambda-\frac{1}{2}}$, each set corresponding to a value of $\lambda > -\frac{1}{2}$, [4, p.276]

$$(1-2xt+t^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{\lambda}(x) t^n \qquad (1)$$

These polynomials on interval $(-A, A)$ are defined as the sets of polynomials orthogonal on this interval with respect to the weight factor $(1-x^2)/A^2)^{\lambda-\frac{1}{2}}$. This gives rise to the polynomials $C_n^{\lambda}(x/A)$.

Now for a function $f(x)$ expandable in these polynomials one gets,

$$f(x) = \sum_{n=0}^{\infty} a_n^{\lambda} \, P_n^{\lambda}(x/A) \tag{2}$$

where

$$a_n^{\lambda} = \frac{\int_{-1}^{1} f(Ax)\, P_n^{\lambda}(x)\, (1-x^2)^{\lambda-\frac{1}{2}}\, dx}{\int_{-1}^{1} \left[P_n^{\lambda}(x)\right]^2 (1-x^2)^{\lambda-\frac{1}{2}}\, dx} \tag{3}$$

The series in (2) may be terminated to get a linear or a cubic approximation according to the degree of accuracy desired and approximation for $f(x)$ takes the form,

$$\left[f(x)\right]_* = a_1^{\lambda} P_1^{\lambda}\left(\frac{x}{A}\right) + a_3^{\lambda} P_3^{\lambda}\left(\frac{x}{A}\right) \tag{4}$$

3. APPLICATION OF POLYNOMIAL LINEARISATION TO NONLINEAR DIFFERENTIAL EQUATION WITH DERIODIC COEFFICIENTS :

The differential equation,

$$y'' + (a - 2q \cos 2z)\, y + by^3 = 0, \quad (b>0,\ z=wt) \tag{5}$$

arises in Melde's experiment [1, p. 120]. It is noted there that (i) the motion is periodic, the main component having half the frequency of the driving reed, (ii) the displacement was symmetrical about the equilibrium position of the thread. Due to the existence of the subharmonic of the order 2, the solution is taken as

$$y = A_1 \cos z + A_3 \cos 3z \tag{6}$$

This implies that a be unity or nearly unity. This ignores the effect of the nonlinearity on the frequency which should be taken into account. If we employ the polynomial linearisation to get an equivalent differential equation the effect of the nonlinearity on the frequency becomes distinct. The results obtained are closer to the actual solution since these reduce to the results in [1] on neglecting higher powers of q involved in them. If A be the amplitude for the oscillations produced by (5) the linear ultraspherical polynomial approximatin to $ay+by^3$ is on the interval (—A, A)

$$[ay+by^3]_* = \left[a+\frac{3bA^2}{2(\lambda+2)}\right]y=n^2y \tag{7}$$

and (5) reduces to the Mathieu equation,

$$y''+(n^2-2q\cos 2z)y=0 \tag{8}$$

To fulfill the experimental observations we must have n^2 nearly unity. This would mean that if a is nearly one, b should be very small. We observe that if b and A are suitably taken a need not be nearly one. It is enough that n^2 be so. Now substituting (6) into (7) we get,

$$\begin{aligned}&-[A_1\cos z+9A_3\cos 3z]+n^2[A_1\cos z+A_3\cos 3z]\\&-q[A_1\cos 3z+A_1\cos z+A_3\cos 5z+A_3\cos z]=0\end{aligned} \tag{9}$$

Now we must have coefficients of cos z, cos $3z$, separately equal to zero.

$$\begin{aligned}&(n^2-1-q)A_1-qA_3=0\\&-qA_1+(n^2-9)A_3=0\end{aligned} \tag{10}$$

If A_1 and A_3 are non-zero, we get,

$$n^4-(10+q)n^2+9(1+q)-q^2=0 \tag{11}$$

which gives,

$$n^2=a+\frac{3bA^2}{2(\lambda+2)}=\frac{10+q\pm\sqrt{5q^2-16q+64}}{2} \tag{12}$$

A binomial expansion of (12) upto q^3 yields,

$$n^2=\left(9+\frac{q^2}{8}\right)\text{ or }\left(1+q-\frac{q^2}{8}-\frac{q^3}{64}\right) \tag{13}$$

the former being impossible sible since n^2 is nearly unity we accept the later. Now we get,

$$A^2 = \frac{2(\lambda+2)}{3b}\left(1+q-\frac{q^2}{8}-\frac{q^3}{64}-a\right) \tag{14}$$

Taking $\lambda=0$, and neglecting q^2 we get, (7) in [1, p. 124] which is obtained for $A_3=0$. Equation (10) gives the ratio

$$\left|\frac{A_3}{A_1}\right| = \left|\frac{n^2-1-q}{q}\right| = \left|\frac{q}{n^2-9}\right| = \frac{q}{8} \text{ nearly} \tag{15}$$

which gives, $A_1=\left(1+\frac{q}{8-q}\right)A$; $A_3=-qA/(8-q)$

Equation (15) gives same ratio for A_3/A_1 as in art. 2.17 in [5, p.20—21]. When $b<0$, the solution is taken as

$$y=B_1 \sin z+B_3 \sin 3z \tag{16}$$

Proceeding as before, pair (10) is reqlaced by

$$\left.\begin{aligned}(n_1{}^2-1+q)-qB_3=0\\ -qB_1+(n^2-9)B_3=0\end{aligned}\right\} \tag{17}$$

where $n_1{}^2=a-\frac{3bB^2}{2\lambda+4}$, and we have

$$n_1{}^4-(10-q)\,n_1{}^2+9(1-q)-q^2=0 \tag{18}$$

Roots for which are $9-q^2/8$, $1-q+q^2/8$. We accept the later being consistent with the experiment. The amplitude B is now given by

$$B^2=(B_1+B_3)^2=\frac{(4+2\lambda)}{3b}(-1+q+a-q^2/8) \tag{19}$$

We may compare with [1, p. 124], from where we have

$A_1=\pm 2\,(q/b)^{\frac{1}{2}}$. Now if we neglect q^2 and q^3 in (14) and let $a=1-2q$, $A=2\,(q/b)^{\frac{1}{2}}$ and for $a=1+2q$, $B=2\,(q/b)^{\frac{1}{2}}$ as in [1].

As shown in [1] art. 7. 231, the parametric point of (5) lies on the characteristic curve a_1 for Mathieu's equation (8). Since

$$a_1=n^2 = 1+q-\frac{q^2}{8}-\frac{q^3}{64}\dots \tag{20}$$

and parametric point (n^2, q) satisfies it. This is in agreement with a more accurate analysis made by taking larger number of terms of the Fourier series in (6) as a solution of (5). The second order approximation taken in [1, p.123] fails to bring out this property.

Since n^2 is nearly unity and the point (n^2, q) lies on the characteristic curve a_1 for (8) the solution becomes, in terms of Mathien functions,

$$y = C\, Ce_1\, (z, q) \quad \text{or} \quad C\, Se_1\, (z, q)$$

Constant C is determined from the initial values. It is clear that the initial value of the amplitude is to Satisfy (14) or (19) as the case be for the steady oscillations. Hence the initial values of y at $z=0$ are obtained from (14) or (19). For a more accurate value of the amplitudes one may use the property that $n^2=a_1$, hence

$$A^2 = \frac{2\ (\lambda+2)}{3b}\,(a_1-a)$$

a_1 being given from tables for values of q. A similar manipulation in case of B^2 also will hold. For small oscillations $\lambda=0$ and for large oscillations $\lambda=0\ 1$ yield good results.

4. PENDULUM WITH OSCILLATING SUPPORT :

Next we consider the pendulum with movable support. As shown in [1, p. 133], such a pendulum of length l, with a bob of mass m at the free end is suspended from a point on a long flexible cantilever with a large load m_1 at the free end. If the periodic time of the pendulum is twice that of the bar, the amplitude builds up and remains constant, apart from a slow decay due to damping. Introducing a damping term motion is given by

$$\theta''+2k\,\theta'+(a-2q\ \cos 2z)\sin\theta=0 \tag{21}$$

McLachlan [1] solves this by approximating $\sin\theta$ as

$$\sin\theta = \theta-\theta^3/6. \tag{22}$$

We shall see that a cublic approximation employing ultraspherical polynomial leads to a better approximation. Now on the interval $(-A, A)$ where A is amplitude [2]

$$[\sin\ \theta]_* = \left(\frac{2}{A}\right)^{\lambda+1} \Gamma(\lambda+2) \Big[\ J_{(\lambda+1)}(A) + (\lambda+3) J_{\lambda+3}(A)\ \Big] \theta$$

$$-\left(\frac{2}{A}\right)^{\lambda+3} \Big[\frac{\Gamma(\lambda+4)}{6}\ J_{\lambda+3}(A)\ \Big] \theta^3 \tag{23}$$

We expect to get better results from this approximation since (24) is closer to sin θ than (22). We put (23) as

$$(\sin\ \theta)_* = \alpha\theta - \beta\theta^3 \tag{24}$$

The equation (21) now becomes,

$$\theta'' + 2k\theta' + (a - 2q\ \cos\ 2z)\ (\alpha\theta - \beta\theta^3) = 0 \tag{25}$$

Now seeking a subharmonic of the order 2, we develop a first approximation,

$$\theta = A_1\ \sin z + B_1\ \cos z. \tag{26}$$

Substituting into (25) and equating to zero coefficients of sin z and cos z, one obtains

$$A_1[\alpha(a+q) - 1] - \frac{3a\ \beta\ A_1}{4} A^2 - q\beta A_1{}^3 - 2kB_1 = 0$$

$$B_1[\alpha(a-q) - 1] - \frac{3a\ \beta\ B_1}{4} A^2 + q\beta B_1{}^3 + 2kB_1 = 0 \tag{27}$$

Droping the cubic terms in A_1 and B_1, the pair becomes

$$\left.\begin{aligned} \Big[\Big\{\alpha(a+q) - 1\ \Big\} - \frac{3a\ \beta\ A^2}{4}\Big] A_1 - 2k\ B_1 = 0 \\ \Big[\Big\{\alpha(a-q) - 1\ \Big\} - \frac{3a\ \beta\ A^2}{4}\Big] B_1 + 2k\ A_1 = 0 \end{aligned}\right\} \tag{28}$$

which gives,

$$\Big[\alpha(a+q) - 1 - \frac{3a\ \beta\ A^2}{4}\Big] \Big[\alpha(a-q) - 1 - \frac{3a\ \beta\ A^2}{4}\Big] + 4k^2 = 0 \tag{29}$$

that is

$$\alpha a-1-\frac{3a\,\beta\,A^2}{4}=\pm\sqrt{\alpha^2q^2-4k^2} \tag{30}$$

which finally leads to,

$$A^2=\frac{3\,\alpha}{3a\,\beta}\left[\sqrt{q^2-\frac{4k^2}{\alpha}}+\left(a-\frac{1}{\alpha}\right)\right] \tag{31}$$

For A to be real, we must have $\alpha q^2>4k^2$. This becomes (8), [p. 134 in 1] for $\alpha=1$, and $\beta=\frac{1}{6}$. Getting A^2 from (31) is a matter of some difficulty since α and β both are functions of A^2. Their approximate forms being, for $\lambda=0$,

$$\alpha=\left(1-\frac{A^2}{384}\right) \text{ and } \beta=\tfrac{1}{6}\left(1-\frac{A^2}{16}\right) \tag{32}$$

First we get A^2 from (31) with $\alpha=1$. $\beta=\frac{1}{6}$ and find α and β from (32). Then use them at (31) to find A^2 more accurately. Now lctting $k=0$, with $\theta=A\sin z$, (25) gives,

$$\theta''+\left[a\alpha-\frac{\beta\,A^2}{2}(q+a)-2\left\{\alpha q-\frac{\beta\,A^2}{2}\left(q+\frac{a}{\alpha}\right)\right\}\cos 2z\right]\theta=0 \tag{33}$$

with $a=1$,

$$A^2=\frac{4\,\alpha}{3\,\beta}\left(q+1-\frac{1}{\alpha}\right) \tag{34}$$

So the parametric point, for (25) when k=0, becomes

$$a=\left[\frac{\alpha}{3}(1-4q)+\frac{2}{3}(q+1)\right],\; q=\left[\frac{2q}{3}+\frac{\alpha-1}{2}\right] \tag{35}$$

For $\alpha=1$, this is $\left(1-\frac{2q}{3},\frac{2q}{3}\right)$ as in [1, p.135]

which lies on the characteristic curve b_1 for (8). But the point (a, q) will now be in the stable region between curves a_0, b_1 in diagram 36 in [1, p. 114] since the curve b_1 is given by

$$b_1=1+q \tag{36}$$

and $$b-1-q<0 \tag{37}$$

This removes the ambiguity in [1, p. 135, art. 7·413].

5. STABILITY ASPECTS OF THE EQUIVALENT EQUATION :

The stability behaviour of a close equivalent equation forecasts the stability behaviour of the original nonlinear differential equation. First we consider the nonlinear differential equation.

$$\theta''+(a-2q\cos 2z)\ \theta+f(\theta)=0 \tag{38}$$

The polynomial linearisation of $f(\theta)$ will give

$$[f(\theta)]_* = P\theta \tag{39}$$

and equivalent linearisation of (38) becomes,

$$\theta''+(a+P-2q\cos 2z)\theta=0 \tag{40}$$

Now letting,

$$\theta=A(z)\cos z+B(z)\sin z \tag{41}$$

assuming A and B to be slow variables in $z=wt$,

such that A'' and B'' may be negligible. We get on substituting (41) A and B as variables into (40),

$$(2B'-A)\cos z-(2A'+B)\sin z+(a+P)(A\cos z+B\sin z)$$
$$-q\ (A\cos z-B\sin z+A\cos 3z+B\sin 3z)=0 \tag{42}$$

Equating to zero coefficients of $\cos z$ and $\sin z$,

$$\left.\begin{aligned} 2B'-A+A(a+P)-qA &= 0\\ -2A'-B+(a+P)B+qB &= 0\end{aligned}\right\} \tag{43}$$

which give

$$\frac{dA}{dB}=\frac{B\ (a+q-1+P)}{-A\ (a+P-1-q)} \tag{44}$$

Since (44) is an equation of the type

$$\frac{dy}{dv}=\frac{\alpha v+\beta y}{\lambda v+\delta y} \tag{45}$$

We apply criteria of stability at the singular points in 9.20 from [1, p. 189],

Now $\alpha = a+P+q-1$, $\beta=\gamma=0$, $\delta=-[a+P-1-q]$

We get $\alpha\delta-\beta\gamma = -(a+P-1)^2-q^2]$ (46)

and $D=4\,[q^2-(a+P-1)^2] = (\beta-\gamma)^2+4\alpha\delta$ (47)

Thus we observe,

(i) $D>O$, for $q>(a+P-1)$ or $q>1-a-P$, which also makes $\alpha\delta-\beta\gamma$ $+ve$. This gives that the singular point is a Col. Hence motion once started will be unstable. We may say that (40) will be unstable if

$$q+1-a>P \quad \text{or} \quad P>1-a-q. \tag{48}$$

(ii) $D<0$, $q<(a+P-1)$ and $q<-(a+P-1)$

which shows that for stable oscillations,

$$P>q+1-a \quad \text{or} \quad 1-q-a>P \tag{49}$$

In this case the motion dies out.

(iii) $D=0$, since $\beta+\gamma=0$, we have a neutral case, Oscillations have a period 2π in z. This happens when $P=q+1-a$ or $1-q-a$. (50)

Equations (43) yield the values of A and B as

$$A = A_1 e^{\mu z}+A_2\, e^{-\mu z} \text{ and} \tag{51}$$

$$B = B_1\, e^{\mu z}+B_2\, e^{-\mu z}$$

where $\mu = \frac{1}{2}\,[q^2-(a+P-1)^2]^{\frac{1}{2}}$

Application of (iii) above shows that the amplitude becomes a constant and steady state is obtained.

These conditions are amplitude dependent and explain Melde's experiment where if amplitude is less than a certain A_0 the oscillation dies out or if greater than A_0 the string breaks. While first order analysis and application of Poincare's criteria leads only to amplitude independent results [1, Chap. 9]

Example :

The determination of the stability of the subharmonic solution $y=\left[\frac{4f}{\beta}\right]^{1/3}\cos\omega t$, of the forced Duffing's equation, [1]

$$\ddot{y}+ay+\beta y^3=f\cos 3\,\omega t \tag{52}$$

where $a, \beta, f>0$ and $w^2=a+3\left(\frac{f^2\beta}{4}\right)^{1/3}$ leads to

Mathieu's equation (8) [1]. As a matter of fact it is of the type (38), but neglecting small quantities it reduces to (8). Instead of neglecting small terms completely we construct an equivalent linear equation as before. We substitute $y+v$ for y in (52), v being a small change in y. This leads to,

$$\ddot{v}+(a+3\beta y^2)\,v+\beta v^3+3\beta v^2 y=0 \tag{53}$$

Applying Chebyshev polynomials to approximate $\beta v^3+3\beta y v^2$ we get, A being the amplitude of v,

$$\ddot{v}\left[w^2+3\left(\frac{\beta f^2}{4}\right)^{1/3}+6\left(\frac{\beta f^2}{4}\right)^{1/3}\cos 2\,\omega t\right]v+\frac{3\,\beta A^2}{4}\,v=0$$

now letting $\omega t=z$ we have, (54)

$$v''+\left[1+\frac{3}{w^2}\left(\frac{\beta f^2}{4}\right)^{1/3}+\frac{3\beta A^2}{4w^2}+\frac{6}{w^2}\left(\frac{\beta f^2}{4}\right)^{1/3}\cos 2z\right]v=0 \tag{55}$$

with $q=\frac{3}{w^2}\left(\frac{\beta f^2}{4}\right)^{\frac{1}{3}}$. Comparing with (40)

$$a+P=1+q+\frac{w^4q^3A^2}{qf^2} \tag{56}$$

that is, $a+P>1+q$ which fulfills (49) for stability.

6. Having compared the polynomially approximated equation analytically with the Fourier series approximations we proceed to apply the procedure to any given case. The most general form being

$$y''+2ky'+(a-2q\cos 2z)\,y+f\,(y)=0 \tag{57}$$

becomes on approximation,

$$y''+2ky'+(a+P-2q\cos 2z)y=0 \tag{58}$$

and letting $.=e^{-kz}u(z)$ this becomes,

$$u''+(a+P-k^2-2q\cos 2z)u=0 \tag{59}$$

We take (41) as a solution for this where A and B are giuen by their values in (51) and $\mu=\frac{1}{2}[q^2-(a+P-k^2-1^2)]^{\frac{1}{2}}$.

This leads to

$$y=e^{-kz}\Big[(A_1e^{\mu z}+A_2e^{-\mu z})\cos z+(B_1e^{\mu z}+B_2e^{-\mu z})\sin z\Big] \tag{60}$$

For the system to acquire a steady state $\mu=k$ and coefficients of A_2 and $B_2\to 0$ as $z\to\infty$. If $\mu=k$, we have

$$P=\sqrt{q^2-4k^2}+1-a+k^2. \tag{61}$$

This gives the initial amplitude to start the system with. For the reality of this $q>4k^2$:

Example :

Now if we have $a=\cdot 9314$, $b=0\cdot 1$, $q=\cdot 16$, $k=\cdot 08$ and $f(y)=by^3$, we shall have, for $\lambda=0$,

$$P=\frac{0\cdot 3}{4}(1)^2=\cdot 075$$

If we expect that the initial amplitude, $A=1$, remains constant since $k=q/2$ we take the interval as $(-1, 1)$.

$$a+P-k^2=\cdot 9314+\cdot 075-\cdot 0064=1$$

and (59) becomes,

$$u''+(1-\cdot 32\cos 2z)u=0 \tag{62}$$

The parametric point (1, 0·16) lies in the unstable region of Fig. 38, p. 114 in [1]. From [5, p. 122], the unstable solution of (62) is

$$u=Ce^{\cdot 08z}(\cos z-0\cdot 021\cos 3z+0\cdot 94\sin z-\cdot 0175\sin 3z) \tag{63}$$

and $y=e^{-kz}u(z)$ is,

$$y=C(\cos z-\cdot 021\cos 3z+0\cdot 94\sin z-\cdot 0175\sin 3z) \tag{64}$$

with $y=A=1$ at $t=0$ and $\frac{dy}{dz}=0$, C is known $=\frac{1}{0.979}$

Discussion :

Ultraspherical polynomial approximations linearise the nonlinear $f(x)$ into a function of the amplitude A, giving equivalent linearised equation so that 'a' of the Mathieu's linearised equation is now replaced by $a+P$. This causes two things (i) it makes the period amplitude dependent which is verified by employing the perturbation method, (ii) it makes the parametric point amplitude dependent, so that the stability now depends on the amplitude also, as is clear from the stability behaviour of Melde's experiment [1. P. 120], If the amplitude of the prong is less than a certain value say A_0, the oscillations die away or when it is greater than A_0, the amplitude increases till the string breaks. In the discussions made above the amplitude was small and we have used $\lambda=0$ for linearising polynomial. We made comparisons of our results with those of [1] at various points and have shown that the later are only first approximations obtainable on dropping higher power terms involved. Since the analytical comparisons are good enough direct numerical applications are likely to be quite accurate. We feel it can be safely assumed that even if a were negative, we may choose b and A, so that $n^2=a+\frac{3bA^2}{2(\lambda+2)}$ may be positive so as to yield oscillatory motion.

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