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CN TWO MULTIPLE HYPERGEOMETRIC FUNCTIONS RELATED TO LAURICELLA'S F_D.

By

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1. INTRODUCTION AND DEFINITIONS : REGIONS OF CONVERGENCE.

In some recent investigations of quadruple hypergeometric functions, I encountered certain instances which are special cases of the two functions under consideration in this paper. These new functions are defined as follows:

$$\begin{matrix}(k)\\(1)\end{matrix} E_D^{(n)}(\alpha, \beta_1, .., \beta_n; \gamma, \gamma'; x_1, ..., x_n) =$$

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(\alpha)_{m_1+\ldots+m_n}(\beta_1)_{m_1}\ldots(\beta_n)_{m_n}}{(\gamma)_{m_1+\ldots+m_k}(\gamma')_{m_{k+1}+\ldots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \quad \ldots\ldots(1.1)$$

and

$$\begin{matrix}(k)\\(2)\end{matrix} E_D^{(n)}(\alpha, \alpha', \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) =$$

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(\alpha)_{m_1+\ldots+m_k}(\alpha')_{m_{k+1}+\ldots+m_n}(\beta_1)_{m_1}\ldots(\beta_n)_{m_n}}{(\gamma)_{m_1+\ldots+m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n} \quad \ldots\ldots(1.2)$$

and it is evident that they are closely related to Lauricella's function $F_D^{(n)}$ ([4], p. 113), defined by

$$F_D^{(n)}(\alpha, \beta_1,\dots, \beta_n; \gamma; x_1,\dots, x_n)=$$

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(\alpha)_{m_1+\dots+m_n}(\beta_1)_{m_1}\dots(\beta_n)_{m_n}}{(\gamma)_{m_1+\dots+m_n}} \frac{x_1^{m_n}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \quad \dots (1.3)$$

We note the following special cases* :

$$\begin{Bmatrix}(0)\\(1)\end{Bmatrix} E_D^{(n)}(\alpha, \beta_1,\dots, \beta_n; \gamma; x_1,\dots, x_n)=$$

$$\begin{Bmatrix}(0)\\(2)\end{Bmatrix} E_D^{(n)}(\alpha, \beta_1,\dots, \beta_n; \gamma; x_1,\dots, x_n)=$$

$$F_D^{(n)}(\alpha, \beta_1,\dots, \beta_n; \gamma; x_1,\dots, x_n)$$

$$\begin{Bmatrix}(1)\\(1)\end{Bmatrix} E_D^{(2)}(\alpha, \beta_1, \beta_2; \gamma, \gamma'; x_1, x_2)=F_2(\alpha, \beta_1, \beta_2; \gamma, \gamma'; x_1, x_2)$$

$$\begin{Bmatrix}(1)\\(1)\end{Bmatrix} E_D^{(3)}(\alpha, \beta_1,\beta_2,\beta_3; \gamma, \gamma'; x_1,x_2,x_3) = F_G(\alpha,\alpha,\alpha,\beta_1,\beta_2,\beta_3;\gamma,\gamma',\gamma';x_1,x_2,x_3)\,[6]$$

$$\begin{Bmatrix}(3)\\(1)\end{Bmatrix} E_D^{(4)}(\alpha,\beta_1,\beta_2,\beta_3,\beta_4;\gamma,\gamma':x_1,x_2,x_3,x_4)=$$
$$K_{11}(\alpha,\alpha,\alpha,\alpha, \beta_1,\beta_2,\beta_3,\beta_4; \gamma,\gamma,\gamma,\gamma'; x_1,x_2,x_3,x_4)\ [2]$$

$$\begin{Bmatrix}(2)\\(1)\end{Bmatrix} E_D^{(4)}(\alpha, \beta_1,\beta_2,\beta_3,\beta_4; \gamma,\gamma'; x_1,x_2,x_3,x_4)=$$
$$K_{12}(\alpha,\alpha,\alpha,\alpha, \beta_1,\beta_2,\beta_3,\beta_4; \gamma,\gamma,\gamma',\gamma'; x_1,x_2,x_3,x_4)\ [2]$$

$$\begin{Bmatrix}(1)\\(2)\end{Bmatrix} E_D^{(2)}(\alpha,\alpha'\ \beta_1,\beta_2;\gamma; x_1,x_2)=F_3(\alpha,\alpha', \beta_1,\beta_2;\gamma; x_1,x_2)$$

$$\begin{Bmatrix}(1)\\(2)\end{Bmatrix} E_D^{(3)}(\alpha,\alpha', \beta_1,\beta_2,\beta_3;\gamma; x_1,x_2,x_3)= F_S(\alpha,\alpha',\alpha', \beta_1,\beta_2,\beta_3; \gamma,\gamma,\gamma; x_1,x_2,x_3)\,[6]$$

$$\begin{Bmatrix}(3)\\(2)\end{Bmatrix} E_D^{(4)}(\alpha,\alpha'\ \beta_1,\beta_2,\beta_3,\beta_4;\gamma; x_1,x_2,x_3,x_4)=$$
$$K_{15}(\alpha,\alpha,\alpha,\alpha', \beta_1,\beta_2,\beta_3,\beta_4; \gamma,\gamma,\gamma,\gamma; x_1,x_2,x_3,x_4)\,[2]$$

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* *Note that F_G and F_S are only alternative notations for Lauricella's triple hypergeometric functions F_8 and F_7 relatively (cf. [4], p. 114).*

$${}^{(2)}_{(2)}E_D^{(4)}\ (\alpha,\alpha',\ \beta_1,\beta_2,\beta_3,\beta_4;\gamma;\ x_1,x_2,x_3,x_4)=$$

$$K_{20}\ (\alpha,\alpha,\ \beta_3,\beta_4,\beta_1,\beta_2,\ \alpha',\alpha';\ \gamma,\gamma,\gamma,\gamma;\ x_1,x_2,x_3,x_4)[2].$$

The regions of convergence of ${}^{(k)}_{(1)}E_D^{(n)}$ and ${}^{(k)}_{(2)}E_D^{(n)}$ are investigated using the technique employed by Horn [5] and Srivastava ([7], [8]).

Let r_i , $i=1,2,3,\ldots$, be called the associated radii of convergence of the multiple power series

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} A_{m_1,\ldots,m_n} x_1^{m_1}\ldots x_n^{m_n}$$

if it converges absolutely when $|x_i| < r_i$ and diverges when $|x_i| > r_i$.

It is found that for the convergence of ${}^{(k)}_{(1)}E_D^{(n)}$,

$$r_1=r_2=\ldots=r_k,$$

$$r_{k+1}=r_{k+2}=\ldots=r_n\ ,$$

$$r_k+r_n=1,$$

and in the case of ${}^{(k)}_{(2)}E_D^{(n)}$,

$$r_1=r_2=\ldots=r_k,$$

$$r_{k+1}=r_{k+2}=\ldots=r_n\ ,$$

$$r_k+r_n=r_k r_n\ .$$

I omit the details of the working out.

2. INTEGRALS OF EULER TYPE.

From (1.1), ${}^{(k)}_{(1)}E_D^{(n)}(\alpha,\ \beta_1,\ldots,\beta_n;\ \gamma,\gamma';\ x_1,\ldots,x_n) =$

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} \frac{(\alpha)_{m_1+\ldots+m_k}(\beta_1)_{m_1}\ldots(\beta_k)_{m_k}}{(\gamma)_{m_1+\ldots+m_k}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_k^{m_k}}{m_k!}\times$$

$$F_D^{(n-k)}\left(\alpha+m_1+\dots+m_k,\ \beta_{k+1},\dots,\beta_n;\ \gamma';\ x_{k+1},\dots,x_n\right) \tag{2.1}$$

Lauricella's formula, ([4], p. 147)

$$F_D^{(n)}\left(\alpha,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n\right)=\Gamma\left[\begin{matrix}\gamma\\ \beta_1,\dots,\beta_n,\gamma-\beta_1-\dots-\beta_n\end{matrix}\right]\times$$

$$\int\dots(n)\dots\int u_1^{\beta_1-1}\dots u_n^{\beta_n-1}\,(1-u_1-\dots-u_n)^{\gamma-\beta_1-\dots-\beta_n-1}$$
$$(1-u_1x_1-\dots-u_nx_n)^{-\alpha}\,du_1\dots du_n \tag{2.2}$$

$$u_1\geqslant 0,\dots,u_n\geqslant 0,\ u_1+\dots+u_n\leqslant 1,$$
$$Re(\gamma)>0,\ Re(\beta_1)>0,\ \ ,Re(\beta_n)>0, Re(\gamma-\beta_1-\dots-\beta_n)>0$$

is now applied to the inner $F_D^{(n-k)}$ series of the right-hand member of (2.1), which becomes

$$\Gamma\left[\begin{matrix}\gamma'\\ \beta_{k+1},\dots,\beta_n,\gamma'-\beta_{k+1}-\dots-\beta_n\end{matrix}\right]\times$$

$$\sum_{m_1=0}^{\infty}\dots\sum_{m_k=0}^{\infty}\int\dots(n-k)\dots\int u_{k+1}^{\beta_{k+1}-1}\dots u_n^{\beta_n-1}$$
$$(1-u_{k+1}-\dots-u_n)^{\gamma'-\beta_{k+1}-\dots-\beta_n-1}\times$$

$$(1-u_{k+1}x_{k+1}-\dots-u_nx_n)^{-\alpha}\ \frac{(\alpha)_{m_1+\dots+m_k}(\beta_1)_{m_1}\dots(\beta_k)_{m_k}}{(\alpha)_{m_1+\dots+m_k}\,m_1!\ \ m_k!}\times$$

$$\left\{\frac{x_1}{1-u_{k+1}x_{k+1}-\dots-u_nx_n}\right\}^{m_1}\dots\left\{\frac{x_k}{1-u_{k+1}x_{k+1}-\dots-u_nx_n}\right\}^{m_k}du_{k+1}\dots du_n \tag{2.3}$$

if $|x_i|\leqslant\xi_i,\ 1\leqslant i\leqslant k,$ then

$$\left|\frac{x_i}{1-u_{k+1}x_{k+1}-\dots-u_nx_n}\right|\leqslant\frac{\xi_i}{1-\xi_{k+1}-\dots-\xi_n},$$

which is less than unity if

$$\xi_i+\xi_{k+1}+\dots+\xi_n<1.$$

In this domain of $x_1,\dots,x_n$, the above series converges uniformly over the region of integration, so that the order of integration and summation may be reversed. Hence, (2.3) becomes

$$\Gamma\left[\begin{matrix} \gamma' \\ \beta_{k+1}, \ldots, \beta_n, \gamma'-\beta_{k+1}-\ldots-\beta_n \end{matrix}\right] \times$$

$$\int \ldots (n-k) \ldots \int u_{k+1}^{\beta_{k+1}-1} u_n^{\beta_n-1} (1-u_{k+1}-\ldots-u_n)^{\gamma'-\beta_{k+1}-\ldots-\beta_n-1} \times (1-u_{k+1}x_{k+1}-\ldots-u_n x_n)^{-a} \times$$

$$F_D^{(k)}\left(a, \beta_1, \ldots, \beta_k; \gamma; \frac{x_1}{1-u_{k+1}x_{k+1}-\ldots-u_n x_n}, \ldots, \frac{x_k}{1-u_{k+1}x_{k+1}-\ldots-u_n x_n}\right) du_{k+1} \ldots du_n \qquad (2.4)$$

(2.2) is now applied to the inner $F_D^{(k)}$ series of (2.4), when we have the result

$$\begin{matrix}(k)\\(1)\end{matrix} E_D^{(n)} (a, \beta_1, \ldots, \beta_n; \gamma, \gamma'; x_1, \ldots, x_n) =$$

$$\Gamma\left[\begin{matrix} \gamma, \quad \gamma' \\ \beta_1, \ldots \beta_n, \gamma-\beta_1-\ldots-\beta_k, \gamma'-\beta_{k+1}-\ldots-\beta_n \end{matrix}\right] \times$$

$$\int \ldots (n) \ldots \int u_1^{\beta_1-1} \ldots u_n^{\beta_n-1} (1-u_1-\ldots-u_k)^{\gamma-\beta_1-\ldots-\beta_k-1}$$

$$(1-u_{k+1}-\ldots-u_n)^{\gamma'-\beta_{k+1}-\ldots-\beta_n-1} \times (1-u_1 x_1-\ldots-u_n x_n)^{-a} du_1 \ldots du_n \qquad (2.5)$$

$u_1 \geqslant 0, \ldots, u_n \geqslant 0, \quad u_1+\ldots+u_k \leqslant 1, \quad u_{k+1}+\ldots+u_n \leqslant 1$

$Re(\gamma)>0,\ Re(\gamma')>0,\ Re(\beta_1)>0, \ldots, Re(\beta_n)>0,$

$Re(\gamma-\beta_1-\ldots-\beta_k)>0, \quad Re(\gamma'-\beta_{k+1}-\ldots-\beta_n)>0.$

$\xi_i+\xi_{k+1}+\ldots+\xi_n<1 \qquad 1 \leqslant i \leqslant k$

The above technique was used by Srivastava [7] .

The inner $F_D^{(k)}$ series of (2.4) may also be replaced by Lauricella's single integral representation of $F_D^{(n)}$ ([4], p. 146)

$$F_D^{(n)} (a, \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n) =$$

$$\Gamma\left[\begin{matrix} \gamma \\ a, \gamma-a \end{matrix}\right] \int_0^1 u^{a-1} (1-u)^{\gamma-a-1} (1-ux_1)^{-\beta_1} \ldots (1-ux_n)^{-\beta_n} du \qquad (2.6)$$

This gives the following result :

$$\overset{(k)}{\underset{(1)}{}} E_D^{(n)} (\alpha, \beta_1, \ldots, \beta_n; \gamma, \gamma'; x_1, \ldots, x_n) =$$

$$\Gamma\left[\begin{matrix} & & \gamma, & \gamma' \\ \alpha, & \beta_{k+1}, \ldots, \beta_n, & \gamma-\alpha, & \gamma'-\beta_{k+1}-\ldots-\beta_n \end{matrix}\right] \times$$

$$\int ..(n-k+1).. \int \frac{v^{\alpha-1} u_{k+1}^{\beta_{k+1}-1} \ldots u_n^{\beta_n-1} (1-v)^{\gamma-\alpha-1} \left(1-u_{k+1}-\ldots-u_n\right)^{\gamma'-\beta_{k+1}-\ldots-\beta_n-1}}{\left(1-vx_1-u_{k+1}x_{k+1}-\ldots-u_nx_n\right)^{\beta_1} \ldots \left(1-vx_k-u_{k+1}x_{k+1}-\ldots-u_nx_n\right)^{\beta_k}} \times$$

$$\left(1-u_{k+1}x_{k+1}-\ldots-u_nx_n\right)^{-\alpha-\beta_1-\ldots-\beta_k} dv\, du_{k+1} \ldots du_n \qquad (2.7)$$

$0 \leqslant v \leqslant 1, \quad u_{k+1} \geq 0, \ldots, u_n \geq 0,$

$u_{k+1} + \ldots + u_n \leqslant 1$

$Re(\gamma) > 0, \quad Re(\gamma') > 0, \quad Re(\alpha) > 0, \quad Re(\gamma-\alpha) > 0$

$Re\left(\beta_{k+1}\right) > 0, \ldots, Re\,(\beta_n) > 0, \quad Re\left(\gamma' - \beta_{k+1} - \ldots - \beta_n\right) > 0$

$\xi_i + \xi_{k+1} + \ldots + \xi_n < 1 \qquad\qquad 1 \leqslant i \leqslant k$.

The following integral representations of $\overset{(k)}{\underset{(2)}{}} E_D^{(n)}(\alpha, \alpha' \beta_1, \ldots, \beta_n; \gamma; x_1, \ldots, x_n)$ may be obtained by similar methods :

$$\Gamma\left[\begin{matrix} & \gamma \\ \alpha, \alpha', & \gamma-\alpha-\alpha' \end{matrix}\right] \times$$

$$\int_0^1 \int_0^1 u^{\alpha-1} v^{\alpha'-1} (1-u)^{\gamma-\alpha-\alpha'-1} (1-v)^{\gamma-\alpha'-1} \left\{1 - x_1^u (1-v)\right\}^{-\beta_1} \ldots$$

$$\left\{1 - x_k^u (1-v)\right\}^{-\beta_k} (1-x_{k+1}v)^{-\beta_{k+1}} \ldots (1-x_nv)^{-\beta_n} du\, dv \qquad (2.8)$$

$Re(\gamma) > 0, Re(\alpha) > 0, Re(\alpha') > 0, Re(\alpha - \alpha - \alpha') > 0,$

$| x_i | < 1, i = 1, 2, \ldots, n$

$$\Gamma\left[\begin{matrix} & \gamma \\ \beta_1, \ldots, \beta_n, & \gamma-\beta_1-\ldots-\beta_n \end{matrix}\right] \times$$

$$\int \ldots (n) \ldots \int u_1^{\beta_1 - 1} \ldots u_n^{\beta_n - 1} (1-u_1-\ldots-u_k)^{\gamma-\beta_1-\ldots-\beta_n-1} (1-u_{k+1}-\ldots-u_n)^{\gamma-\beta_{k+1}-\ldots-\beta_n-1} \times$$

$$(1-u_{k+1}x_{k+1}-\ldots-x_n u_n)^{-a'} \times$$

$$\left\{1-(u_1x_1+\ldots+u_kx_k)(1-u_{k+1}-\ldots-u_n)\right\}^{-a'} dn_1 \ldots du_n \tag{2.9}$$

$Re(\gamma)>0$, $Re(\beta_1)>0,\ldots,Re(\beta_n)>0$,

$Re(\gamma-\beta_1-\ldots-\beta_n)>0$

$u_1 \geqslant 0, \ldots, u_n \geqslant 0$, $u_1+\ldots+u_k \leqslant 1$, $u_{k+1}+\ldots+u_n \leqslant 1$

$|x_i|<1$, $i=1,2,\ldots,n$.

$$\Gamma\left[\begin{matrix}\gamma \\ a,\beta_{k+1},\ldots,\beta_n,\ \gamma-a-\beta_{k+1}-\ldots-\beta_n\end{matrix}\right] \times$$

$$\int \ldots (n-k+1) \ldots \int v^{a-1} u_{k+1}^{\beta_{k+1}-1} \ldots u_n^{\beta_n-1} (1-v)^{\gamma-a-\beta_{k+1}-\ldots-\beta_n-1} (1-u_{k+1}-\ldots-u_n)^{\gamma-\beta_{k+1}-\ldots-\beta_n-1} \times$$

$$\left\{1-vx_1(1-u_{k+1}-\ldots-u_n)\right\}^{-\beta_1} \ldots \left\{1-vx_k(1-u_{k+1}-\ldots-u_n)\right\}^{-\beta_k} \times$$

$$(1-u_{k+1}x_{k+1}-\ldots-u_nx_n)^{-a'} dv\, du_{k+1} \ldots du_n \tag{2.10}$$

$Re(\gamma)>0$, $Re(a)>0$, $Re(\beta_{k+1})>0,\ldots,Re(\beta_n)>0$

$Re(\gamma-a-\beta_{k+1}-\ldots-\beta_n)>0 \qquad 0 \leqslant v \leqslant 1$,

$u_{k+1} \geqslant 0, \ldots, u_n \geqslant 0$, $u_{k+1}+\ldots+u_n \leqslant 1$.

Various integral representation of similar type may also be obtained by the same methods.

3. POCHHAMMER INTEGRALS.

Consider the integral

$$I = \int_C \begin{matrix} (-t)^{-\rho}(t-1)^{-\rho'} F_D^{(k)}\left(a,\beta_1,\ldots,\beta_k;\gamma;\frac{x_1}{t},\ldots,\frac{x_k}{t}\right) \times \\ F_D^{(n-k)}\left(a',\beta_{k+1},\ldots,\beta_n;\gamma';\frac{x_{k+1}}{1-t},\ldots,\frac{x_n}{t}\right) dt \end{matrix} \tag{3.1}$$

where C is a pochhammer double-loop slung around the points O and 1.
If the integrand is expanded, supposing that integration and summation may be interchanged, we have

$$I=\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(\alpha)_{m_1+\ldots+m_k}(\alpha')_{m_{k+1}+\ldots+m_n}(\beta_1)_{m_1}\ldots(\beta_n)_{m_n}}{(\gamma)_{m_1+\ldots+m_k}(\gamma')_{m_{k+1}+\ldots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\times$$

$$(-1)^{m_1+\ldots+m_n}\int_C(-t)^{-\rho-m_1-\ldots-m_k}(t-1)^{-\rho'-m_{k+1}-\ldots-m_n}dt\ \ldots\quad(3.2)$$

The inner integral on the right-hand side of (3.2) may be evaluated by means of the formula

$$\int_C(-t)^{\alpha-1}(t-1)^{\beta-1}dt=\frac{(2\pi i)^2}{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(\alpha+\beta)}\quad(3.3)$$

([1], p. 378).

Hence, $I=\dfrac{(2\pi i)^2}{\Gamma(\rho)\Gamma(\rho')\Gamma(2-\rho-\rho')}\times$

$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(\rho+\rho'-1)_{m_1+\ldots+m_n}(\alpha)_{m_1+\ldots+m_k}(\alpha')_{m_{k+1}+\ldots+m_n}}{(\gamma)_{m_1+\ldots+m_k}(\gamma')_{m_{k+1}+\ldots+m_n}}\times$$

$$\frac{(\beta)_{m_1}\ldots(\beta_n)_{m_n}}{(\rho)_{m_1+\ldots+m_k}(\rho')_{m_{k+1}+\ldots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\quad(3.4)$$

so that if $\alpha=\rho$, $\alpha'=\rho'$, we have the result

$$\binom{k}{1}E_D^{(n)}(\rho+\rho'-1,\beta_1,\ldots,\beta_n;\gamma,\gamma';x_1,\ldots,x_n)=$$

$$\frac{\Gamma(\rho)\Gamma(\rho')\Gamma(2-\rho-\rho')}{(2\pi i)^2}\times$$

$$\int_C(-t)^{-\rho}(t-1)^{-\rho'}F_D^{(k)}\left(\rho,\beta_1,\ldots,\beta_k;\gamma;\frac{x_1}{t},\ldots,\frac{x_k}{t}\right)\times$$

$$F_D^{(n-k)}\left(\rho',\beta_{k+1}\ldots,\beta_n;\gamma';\frac{x_{k+1}}{1-t},\ldots,\frac{x_n}{1-t}\right)dt\quad(3.5)$$

Similarly, $\binom{k}{2}E_D^{(n)}(\alpha,\alpha',\beta_1,\ldots,\beta_n;\rho+\rho';x_1,\ldots,x_n)=$

$$\frac{\Gamma(1-\rho)\Gamma(1-\rho')\Gamma(\rho+\rho')}{(2\pi i)^2}\times$$

$$\int_C (-t)^{\rho-1}(t-1)^{\rho'-1} F_D^{(k)}(\alpha,\beta_1,\ldots,\beta_k;\rho; x_1 t,\ldots, x_k t) \times$$

$$F_D^{(n-k)}\Big(\alpha',\beta_{k+1},\ldots,\beta_n;\rho'; x_{k+1}(1-t),\ldots, x_n(1-t)\Big)\, dt \qquad (3.6)$$

4. SIMPLE TRANSFORMATIONS AND REDUCTIONS.

The following transformations are obtained by the elementary manipulation of series.

Lauricella's transformations ([4], p. 149)

$$F_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)$$

$$=(1-x_1)^{-\alpha} F_D^{(n)}\Big(\alpha,\gamma-\beta_1-\ldots-\beta_n,\beta_2,\ldots,\beta_n;\gamma;\frac{x_1}{x_1-1},\frac{x_1-x_2}{x_1-1},\ldots,\frac{x_1-x_n}{x_1-1}\Big)$$

... ...

$$=(1-x_n)^{-\alpha} F_D^{(n)}\Big(\alpha,\beta_1,\ldots,\beta_{n-1},\gamma-\beta_1-\ldots-\beta_n;\gamma;\frac{x_n-x_1}{x_n-1},\ldots,\frac{x_n-x_{n-1}}{x_n-1},\frac{x_n}{x_n-1}\Big) \quad (4.1)$$

may be applied to the inner $F_D^{(n-k)}$ series of (2.1) to give the transformations

$${}^{(k)}_{(1)}E_D^{(n)}(\alpha,\beta_1,\ldots,\beta_n;\gamma,\gamma';x_1,\ldots,x_n)$$

$$=(1-x_{k+1})^{-\alpha}\,{}^{(k)}_{(1)}E_D^{(n)}\Big(\alpha,\beta_1,\ldots,\beta_k,\gamma'-\beta_{k+1}-\ldots-\beta_n,\beta_{k+2},\ldots,\beta_n;\gamma,\gamma';$$
$$\frac{x_1}{1-x_{k+1}},\ldots,\frac{x_k}{1-x_{k+1}},\frac{x_{k+1}}{x_{k+1}-1},\frac{x_{k+1}-x_{k+2}}{x_{k+1}-1},\ldots,$$
$$\frac{x_{k+1}-x_n}{x_{k+1}-1}\Big)$$

... ...

$$=(1-x^n)^{-\alpha}\,{}^{(k)}_{(1)}E_D^{(n)}\Big(\alpha,\beta_1,\ldots,\beta_{n-1},\gamma'-\beta_{k+1}-\ldots-\beta_n;\gamma,\gamma';$$
$$\frac{x_1}{1-x_n},\ldots,\frac{x_k}{1-x_n},\frac{x_n-x_{k+1}}{x_n-1},\ldots,\frac{x_n-x_{n-1}}{x_n-1},$$
$$\frac{x_n}{x_n-1}\Big) \qquad (4.2)$$

Similarly, we also have

$$ {}^{(k)}_{(1)}E_D^{(n)}(\alpha, \beta_1,\dots,\beta_n; \gamma, \gamma'; x_1, \dots, x_n) $$

$$ =(1-x_1)^{-\alpha}\, {}^{(k)}_{(1)}E_D^{(n)}\Big(\alpha, \gamma-\beta_1-\dots-\beta_k, \beta_2,\dots,\beta_n; \gamma, \gamma'; \frac{x_1}{x_1-1}, \frac{x_1-x_2}{x_1-1},\dots,\frac{x_1-x_k}{x_1-1}, \frac{x_{k+1}}{1-x_1},\dots,\frac{x_n}{1-x_1}\Big) $$

... ...

$$ =(1-x_k)^{-\alpha}\, {}^{(k)}_{(1)}E_D^{(n)}\Big(\alpha, \beta_1,\dots,\beta_{k-1}, \gamma-\beta_1-\dots-\beta_k; \gamma, \gamma'; \frac{x_k-x_1}{x_k-1},\dots,\frac{x_k-x_{k-1}}{x_k-1}, \frac{x_k}{x_k-1}, \frac{x_{k+1}}{1-x_k},\dots,\frac{x_n}{1-x_k}\Big) $$

$$ =(1-x_1-x_{k+1})^{-\alpha}\, {}^{(k)}_{(1)}E_D^{(n)}\Big(\alpha, \gamma-\beta_1-\dots-\beta_k, \beta_2,\dots,\beta_k, \gamma'-\beta_{k+1}-\dots-\beta_n, \beta_{k+2},\dots,\beta_n; \gamma, \gamma'; \frac{x_1}{1-x_1-x_{k+1}}, \frac{x_1-x_2}{1-x_1-x_{k+1}},\dots,\frac{x_1-x_k}{1-x_1-x_{k+1}}, \frac{x_{k+1}}{1-x_1-x_{k+1}}, \frac{x_{k+2}-x_{k+2}}{1-x_1-x_{k+1}},\dots,\frac{x_{k+1}-x_n}{1-x_1-x_{k+1}}\Big) $$

... ...

$$ =(1-x_1-x_n)^{-\alpha}\, {}^{(k)}_{(1)}E_D^{(n)}\Big(\alpha, \gamma-\beta_1-\dots-\beta_k, \beta_2,\dots,\beta_k, \beta_{k+1},\dots,\beta_{n-1}, \gamma'-\beta_{k+1}-\dots-\beta_n; \gamma, \gamma'; \frac{x_1}{1-x_1-x_n}, \frac{x_1-x_2}{1-x_1-x_n},\dots,\frac{x_1-x_k}{1-x_1-x_n}, \frac{x_n-x_{k+1}}{1-x_1-x_n},\dots,\frac{x_n-x_{n-1}}{1-x_1-x_n}, \frac{x_n}{1-x_1-x_n}\Big) $$

$$ =(1-x_2-x_{k+1})^{-(\alpha)}\, {}^{(k)}_{(1)}E_D^{(n)}\Big(\alpha, \beta_1, \gamma-\beta_1-\dots-\beta_k, \beta_3,\dots,\beta_k, \gamma'-\beta_{k+1}-\dots-\beta_n, \beta_{k+2},\dots,\beta_n; \gamma, \gamma'; \frac{x_2-x_1}{1-x_2-x_{k+1}}, \frac{x_2}{1-x_2-x_{k+1}}, \frac{x_2-x_3}{1-x_2-x_{k+1}}, ,\; \frac{x_2-x_k}{1-x_2-x_{k+1}}, \frac{x_{k+1}}{1-x_2-x_{k+1}}, \frac{x_{k+1}-x_{k+2}}{1-x_2-x_{k+1}},\dots,\frac{x_{k+1}-x_n}{1-x_2-x_{k+1}} $$

... ...

$$ =(1-x_k-x_n)^{-\alpha}\, {}^{(k)}_{(1)}E_D^{(n)}(\alpha,\beta_1,\dots,\beta_{k-1},\gamma-\beta_1-\dots-\beta_k,\beta_{k+1},\dots,\beta_{n-1}, $$

$$\gamma'-\beta_{k+1}-\dots-\beta_n;\gamma,\gamma'; \qquad \frac{x_k-x_1}{1-x_k-x_n},\dots,$$

$$\frac{x_k-x_{k-1}}{1-x_k-x_n},\frac{x_k}{1-x_k-x_n},\frac{x_n-x_{k+1}}{1-x_k-x_n},\dots,\frac{x_n-x_{n-1}}{1-x_k-x_n},\frac{x_n}{1-x_k-x_n}\Big) \tag{4.3}$$

Simple transformations of the above type do not appear to exist for the function ${}^{(k)}_{(2)}E^{(n)}_{D}$

If Lauricella's reduction formula ([4] p. 150)

$$F^{(n)}_{D}(\alpha,\beta_1,\dots,\beta_n;\gamma;x,\dots,x)= {}_2F_1(\alpha,\beta_1+\dots+\beta_n;\gamma;x) \tag{4.4}$$

where all the variables are made equal, is employed, it may readily be established that

$$\begin{aligned}&{}^{(k)}_{(1)}E^{(n)}_{D}(\alpha,\beta_1,\dots,\beta_n;\gamma,\gamma';x_1,x_i,x_i,\dots,x_i,x_{k+1},\dots,x_j,x_j,\dots,x_j)\\ &={}^{(i)}_{(1)}E^{(j-k+i)}_{D}(\alpha,\beta_1,\dots,\beta_i-1,\beta_i+\dots+\beta_k,\beta_{k+1},\dots,\beta_j-1,\beta_j+\dots+\beta_n;\gamma,\gamma';\\ &\qquad x_1,\dots,x_i,x_{k+1},\dots,x_j)\\ &\qquad i\leqslant k,\ j>k,\end{aligned} \tag{4.5}$$

giving finally

$$\begin{aligned}&{}^{(k)}_{(1)}E^{(n)}_{D}(\alpha,\beta_1,\dots,\beta_n;\gamma,\gamma';x_1,x_1,\dots,x_1,x_{k+1},x_{k+1},\dots,x_{k+1})=\\ &F_2(\alpha\ \beta_1+\dots+\beta_k,\beta_{k+1}+\dots+\beta_n;\gamma,\gamma';x_1,x_{k+1})\end{aligned} \tag{4.6}$$

together with a similar reduction for ${}^{(k)}_{(2)}E^{(n)}_{D}$

This section is concluded with an interesting reduction of ${}^{(k)}_{(2)}E^{(n)}_{D}$.

From the definition of $F^{(n)}_{D}$, it is evident that

$$F^{(n)}_{D}(\alpha,\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n)=$$

$$\sum_{m_1=0}^{\infty}\dots\sum_{m_k=0}^{\infty}\frac{(\alpha)_{m_1+\dots+m_k}(\beta_1)_{m_1}\cdots(\beta_k)_{m_k}}{(\gamma)_{m_1+\dots+m_k}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_k^{m_k}}{m_k!}$$

$$\gamma'-\beta_{k+1}-\ldots-\beta_n;\gamma,\gamma'; \qquad \frac{x_k-x_1}{1-x_k-x_n},\ldots,$$

$$\frac{x_k-x_{k-1}}{1-x_k-x_n},\frac{x_k}{1-x_k-x_n},\frac{x_n-x_{k+1}}{1-x_k-x_n},\ldots,\frac{x_n-x_{n-1}}{1-x_k-x_n},\frac{x_n}{1-x_k-x_n}\Bigg) \tag{4.3}$$

Simple transformations of the above type do not appear to exist for the function ${}^{(k)}_{(2)}E^{(n)}_D$

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where all the variables are made equal, is employed, it may readily be established that

$${}^{(k)}_{(1)}E^{(n)}_D(\alpha,\beta_1,\ldots,\beta_n;\gamma,\gamma';x_1,x_i,x_i,\ldots,x_i,x_{k+1},\ldots,x_j,x_j,\ldots,x_j) \\ = {}^{(i)}_{(1)}E^{(j-k+i)}_D(\alpha,\beta_1,\ldots,\beta_i-1,\beta_i+\ldots+\beta_k,\beta_{k+1},\ldots,\beta_j-1,\beta_j+\ldots+\beta_n;\gamma,\gamma'; \\ x_1,\ldots,x_i,x_{k+1},\ldots,x_j) \tag{4.5}$$

$$i\leqslant k,\ j>k,$$

giving finally

$${}^{(k)}_{(1)}E^{(n)}_D(\alpha,\beta_1,\ldots,\beta_n;\gamma,\gamma';x_1,x_1,\ldots,x_1,x_{k+1},x_{k+1},\ldots,x_{k+1})= \\ F_2(\alpha\ \beta_1+\ldots+\beta_k,\beta_{k+1}+\ldots+\beta_n;\gamma,\gamma';x_1,x_{k+1}) \tag{4.6}$$

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This section is concluded with an interesting reduction of ${}^{(k)}_{(2)}E^{(n)}_D$.

From the definition of $F^{(n)}_D$, it is evident that

$$F^{(n)}_D(\alpha,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)= \\ \sum_{m_1=0}^{\infty}\ldots\sum_{m_k=0}^{\infty}\frac{(\alpha)_{m_1+\ldots+m_k}(\beta_1)_{m_1}\cdots(\beta_k)_{m_k}}{(\gamma)_{m_1+\ldots+m_k}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_k^{m_k}}{m_k!}$$

$$F_D^{(n-k)}(a+m_1+\ldots+m_k,\beta_{k+1},\ldots,\beta_n;\gamma+m_1+\ldots+m_k;x_{k+1},\ldots,x_n) \qquad (4.7)$$

The transformation due to Lauricella ([4], p. 148),

$$F_D^{(n)}(a,\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n)=$$

$$(1-x_1)^{-\beta_1}\ldots(1-x_n)^{-\beta_n}\, F_D^{(n)}\left(\gamma-a,\beta_1,\ldots,\beta_n;\gamma;\ \frac{x_1}{x_1-1},\ldots,\frac{x_n}{x_n-1}\right) \qquad (4.8)$$

is now applied to the inner $E_D^{(n-k)}$ series on the right of (4.7), which yields after slight re-arrangement, the result

$$\binom{l}{2}E_D^{(n)}(a,\ \gamma-a,\ \beta_1,\ldots\beta_n;\ \gamma;\ x_1,\ldots,x_n)=$$

$$\left(1-x_{k+1}\right)^{-\beta_{k+1}}\ldots\left(1-x_n\right)^{-\beta_n}\times$$

$$F_D^{(n)}\left(a,\ \beta_1,\ldots\beta_n;\ \gamma;\ x_1,\ldots,x_k,\frac{x_{k+1}}{x_{k+1}-1},\ldots,\frac{x_n}{x_n-1}\right) \qquad (4.9)$$

which generalises the well-known result

$$F_3\ (a,\ \gamma-a,\ \beta,\ \beta';\ \gamma;\ x,\ y)=$$

$$\left(1-y\right)^{-\beta'} F_1\left(a,\beta,\beta';\ \gamma;x,\ \frac{y}{y-1}\right).$$

5. TRANSFORMATION ARISING FROM THE POCHHAMMER INTEGRALS.

In the integral formula (3.5), let $\rho=\gamma$, $\rho'=\gamma'$ when we have

$$\binom{k}{1}E_D^{(n)}(\gamma+\gamma'-1,\ \beta_1,\ldots,\ \beta_n;\ \gamma,\ \gamma';\ x',\ldots,\ x_n)=$$

$$\frac{\Gamma(\gamma)\Gamma(\gamma')\Gamma(2-\gamma-\gamma')}{(2\pi i)^2}\times$$

$$\int_C (-t)^{-\gamma} (t-1)^{-\gamma'} \left(1-\frac{x_1}{t}\right)^{-\beta_1} \cdots \left(1-\frac{x_k}{t}\right)^{-\beta_k} \left(1-\frac{x_{k+1}}{1-t}\right)^{-\beta_{k+1}}$$

$$\cdots \left(1-\frac{x_n}{1-t}\right)^{-\beta_n} dt \quad (5.1)$$

The factors in the integrand may now be expanded using the formulae

$$\left(1-\frac{x}{1-t}\right)^{-\beta} = (1-x)^{-\beta} \sum_{m=0}^{\infty} \frac{(\beta)_m}{m!} \left(\frac{x}{1-x} \cdot \frac{1-t}{t}\right)^m \quad (5.2)$$

and

$$\left(1-\frac{x}{1-t}\right)^{-\beta} = (1-x)^{-\beta} \sum_{m=0}^{\infty} \frac{(\beta)_m}{m!} \left(\frac{x}{1-x} \cdot \frac{t}{1-t}\right)^m$$

whereby, various transformations involving genneralised functions of Horn type may be obtained by the application of (3.3). The most interesting is as follows :

$$\begin{matrix}(k)\\(1)\end{matrix} E_D^{(n)} (\gamma+\gamma'-1, \beta_1, \ldots, \beta_n; \gamma, \gamma'; x_1, \ldots, x_n) =$$

$$(1-x_1)^{-\beta_1} \cdots (1-x_n)^{-\beta_n} C_n^{(k)} \left(\beta_1, \ldots, \beta_n; 1-\gamma, 1-\gamma'; \frac{x_1}{1-x_1}, \ldots, \frac{x_n}{1-x_n}\right) \quad (5.3)$$

where

$$C_n^{(k)} (a_1, \ldots, a_n; \beta_1, \beta_2; x_1, \ldots, x_n) =$$

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a_1)_{m_1} \cdots (a_n)_{m_n} (\beta_1)_{m_{k+1}+\cdots+m_n-m_1-\cdots-m_k} (\beta_2)_{m_1+\cdots+m_k-m_{k+1}-\cdots-m_n}}{m_1! \cdots m_n!}$$

$$\times x_1^{m_1} \cdots x_n^{m_n} \quad (5.4)$$

[3] *p*.86.

If the formulae (4.1) and (4.8) are applied to the E_D series in the integrand of (6.5) and (6.6), various transformations involving new types of multiple hypergeometric series of more variables and of higher order are obtainable. For example, if (4.8) is applied to the $F_D^{(k)}$ series in the integrand of (6.6), we have

$$\begin{aligned}
{}^{(k)}_{(2)} E_D^{(n)}(\alpha, \alpha', \beta_1; \rho+\rho'; x_1, \ldots, x_n) &= \frac{\Gamma(1-\rho)\Gamma(1-\rho')\Gamma(\rho+\rho')}{(2\pi_i)^2} \times \\
&\int_C (-t)^{\rho-1} (t-1)^{\rho'-1} (1-x_1 t)^{-\beta_1} \ldots\ldots (1-x_k t)^{-\beta_k} \times \\
&F_D^{(k)}\left(\rho-\alpha, \beta_1, \ldots, \beta_k; \rho; \frac{x_1 t}{x_1 t-1}, \ldots, \frac{x_k t}{x_k t-1}\right) \times \\
F_D^{(n-k)}&\left(\alpha', \beta_{k+1}, \ldots, \beta_n; \rho'; x_{k+1}(1-t), \ldots, x_n(1-t)\right) dt
\end{aligned} \tag{5.5}$$

After considerable reduction, it may be shown that the right-hand member of (5.5) may be written

$$\sum_{m_1=0}^{\infty} \ldots \sum_{m_{n+k}=0}^{\infty} \frac{(\rho-\alpha)_{m_1+\ldots\ldots+m_k} (\alpha')_{m_{k+1}+\ldots\ldots+m_n} (\beta_1)_{m_1+\ldots+m_{n+1}} \cdot \ldots (\beta_k)_{m_k+\ldots+m_{n+k}}}{(\rho+\rho')_{m_1+\ldots+m_{n+k}}}$$

$$\times \frac{(\rho)_{m_1+\ldots+m_k+m_{n+1}+\ldots+m_{n+k}}}{(\rho)_{m_1+\ldots+m_k}} \frac{(-x_1)^{m_1}}{m_1!} \ldots \frac{(-x_k)^{m_k}}{m_k!} \frac{x_{k+1}^{m_{k+1}}}{m_{k+1}!} \ldots \frac{x_n^{m_n}}{m_n!} \times \frac{x_1^{m_{n+1}}}{m_{n+1}!} \ldots \frac{x_k^{m_{n+k}}}{m_{n+k}!} \tag{5.6}$$

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