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GENERALIZED LAGUERRE POLYNOMIALS AND THE POLYNOMIALS RELATED TO THEM, III

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1. INTRODUCTION.

In our recent papers [1, 2, 3], we studied the generalized Laguerre polynomials which are defined by

$$(1.1)\quad (1-t)^{-c}\ \exp.\left\{-\left(\frac{r}{1-t}\right)^{r} x\, t\right\}=\sum_{n=0}^{\infty} f_{n}^{c}(x,r)\, t^{n},$$

and also their related polynomials defined by

$$(1.2)\quad \sum_{k=0}^{n} A_{k}^{c}(x,r)\, f_{n-k}^{c+k}=0 \quad n\geqslant 1,$$

$$(1.3)\quad A_{0}^{c}(x,r)=1,$$

$$(1.4)\quad R_{n}^{(c,\,b)}(y,x,r)=\sum_{k=0}^{n} f_{k}^{c}(y,r)\, A_{n-k}^{b+k}(x,r),$$

and

$$(1.5)\quad N_{n}^{(c,\,b)}(x,y,r)=\sum_{k=0}^{n} A_{k}^{c}(x,r)\, f_{n-k}^{b+k}(y,r).$$

For a ready reference here, we write the following generating relations [1, 2], which will be used in our investigations :

$$(1.6)\quad (1+t)^{c-1} \exp.\{-x\, t\, r^r (1+t)^{r-1}\} = \sum_{n=0}^{\infty} f_n^{c-n}(x, r)\, t^n,$$

$$(1.7)\quad (1+t)^{-c} \quad \exp.\{r^r (1+t)^{r-1} x\, t\} = \sum_{n=0}^{\infty} A_n^c(x, r)\, t^n,$$

$$(1.8)\quad (1-t)^{c-1} \quad \exp.\{r^r x\, t\, (1-t)^{-r}\} = \sum_{n=0}^{\infty} A_n^{c-n}(x, r)\, t^n.$$

In this paper, we discuss the nature of $f_n^c(x, r)$ and $A_n^c(x, r)$, and establish some recurrence relations for them. We also obtain Rodrigues' formulae and some summation formulae for the above polynomials. Some of the summation formulae are the generalizations of [3, (4.4), (4.5)]. Finally, we derive two new identities associated with these polynomial systems.

2. NATURE OF $f_n^c(x, r)$ and $A_n^c(x, r)$.

In this section, we notice that the polonomials $f_n^c(x, r)$ are non-orthogonal except when $r=1$, and the polynomials $A_n^c(x, r)$ are non-orthogonal for all values of r.

Again by definitions (1.1) and (1.7), it is clear that $f_n^c(x, r)$ and $A_n^c(x, r)$ are both of Sheffer A-type zero [9]. All orthogonal polynomials which are of Sheffer A-type zero have been determined by several authors e. g. Meixner and Sheffer etc. Our polynomials $f_n^c(x, r)$ are not among them except for $r=1$, and similarly $A_n^c(x, r)$ are also not included in them for all values of r.

Using the above property of these polynomials, we establish the following recurrence relations which appear to be new :

$$(2.1)\quad \sum_{k=0}^{n-1} \left(c - \frac{x\,(k+1)\, r^r\, (r)_k}{k\,!}\right) f_{n-1-k}^c(x, r) = n\, f_n^c(x, r) \qquad n \geqslant 1,$$

$$(2.2)\quad x\frac{d}{dx}\, f_n^c(x,r)-n f_n^c(x,r)=\sum_{k=0}^{n-1}\left(\frac{x\, r^r\,(r)_{n-1-k}}{(n-k-2)\,!}-c\right) f_k^c(x,r)\quad n\geqslant 1,$$

$$(2.3)\quad \sum_{k=0}^{n-1}\left\{(-1)^{k+1}\, c+r^r\,(k+1)\binom{r-1}{k}\, x\right\} A_{n-1-k}^c(x,r)=n A_n^c(x,r)\quad n\geqslant 1,$$

$$(2.4)\quad \sum_{k=0}^{\min(n-1,\, r-1)} r^r\binom{r-1}{k} A_{n-1-k}^c(x,r)=\frac{d}{dx}\, A_n^c(x,r)\quad n\geq 1,$$

and

$$(2.5)\quad x\frac{d}{dx}\, A_n^c(x,r)-n\, A_n^c(x,r)=-\sum_{k=0}^{n-1}\left\{(-1)^{k+1}\, c+\binom{r-1}{k} r^r k\, x\right\} A_{n-1-k}^c(x,r)\qquad n\geqslant 1$$

3. SOME OTHER RECURRENCE RELATIONS.

Starting from the generating relation (1.1), we also obtain the following recurrence relations :

$$(3.1)\quad (n+1)f_{n+1}^c(x,r)+[r^r x-c-(r+1)\, n]f_n^c(x,r)+\left[\binom{r+1}{2}(n-1)-r^r x+r^{r+1} x+c\, r\right]\times f_{n-1}^c(x,r)$$
$$=\sum_{j=2}^{\min(r,\, n)}(-1)^j\left[\binom{r+1}{j+1}(n-j)+c\binom{r}{j}\right] f_{n-j}^c(x,r),$$

$$(3.2)\quad \left[1-c-rn+\binom{r+1}{2}(n-1)+r^{r+1} x+c\, r\right]\frac{d}{dx} f_n^c(x,r)-(n+1)\, r^r f_n^{c+r-1}(x,r)$$
$$+r^r\left[\binom{r+1}{2}(n-1)-r^r x+r^{r+1} x+c\, r\right] f_{n-1}^{c+r-1}(x,r)+r^r f_n^c(x,r)$$
$$+r^r\,(r-1)\, f_{n-1}^c(x,r)=\sum_{j=2}^{\min(r,\, n)}(-1)^j\left[\binom{r+1}{j+1}+c\binom{r}{j}\right]\frac{d}{dx} f_{n-j}^c(x,r),$$

and

$$(3.3)\quad (1-c-rn+r^r x)\frac{d}{dx} f_n^c(x,r)+\left[\binom{r+1}{2}(n-1)-r^r x+r^{r+1}x+cr\right]\frac{d}{dx} f_{n-1}^c(x,r)$$

$$+r^r\left[f_n^c(x,r)+(r-1)\, f_{n-1}^c(x,r)-(n+1)\, f_n^{c+r-1}(x,r)\right]$$

$$=\sum_{j=2}^{\min(r,n)}(-1)^j\left[\binom{r+1}{j+1}(n-j)+c\binom{r}{j}\right]\frac{d}{dx} f_{n-j}^c(x,r) \quad .$$

The relations (3.1), (3.2) and (3.3) are generalizations of [5, p. 297, (7), p. 299, (12), p. 300, (13)] respectively, and they reduce to them for $r=1$, $c=a+1$.

Now making an appeal to the relationship [1, (3.8)]

$$A_k^c(x,r)=f_k^{-(c+k-1)}(-x,r),$$

we derive the following results from (3.1), (3.2) and (3.3) respectively :

$$(3.4)\quad (n+1)\ A_{n+1}^{c-1}(x,r)+\left[-r^r x+c-1-rn\right]A_n^c(x,r)$$

$$+\left[\binom{r+1}{2}(n-1)+r^r x-r^{r+1}x-(c+n-1)\,r\right]A_{n-1}^{c+1}(x,r)$$

$$=\sum_{j=2}^{\min(r,\,n)}(-1)^j\left[\binom{r+1}{j+1}(n-j)-(c+n-1)\binom{r}{j}\right]A_{n-j}^{c+j}(x,r)\ ,$$

$$(3.5)\quad \left[(c+n)(1-r)-rn+\binom{r+1}{2}(n-1)-r^{r+1}x+r\right]\frac{d}{dx} A_n^c(x,r)$$

$$+(1+n)r^r A_n^{c-r+1}(x,r)$$

$$-r^r\left[\binom{r+1}{2}(n-1)+r^r x-r^{r+1}x-(c+n-1)r\right]A_{n-1}^{c-r+2}(x,r)$$

$$-r^r\left[A_n^c(x,r)+(r-1)\ A_{n-1}^{c+1}(x,r)\right]$$

$$= \sum_{j=2}^{min\,(r,n)} (-1)^j \left[\binom{r+1}{j+1} (n-j) - (c+n-1) \binom{r}{j} \right] \frac{d}{dx} A_{n-j}^{c+j} (x,r)$$

and

$$(3.6) \quad [c+n(1-r)-r^r x] \frac{d}{dx} A_n^c (x,r) + \left[\binom{r+1}{2} (n-1) + r^r x - r^{r+1} x - (c+n-1) r \right]$$

$$\frac{d}{dx} A_n^{c+1} (x,r) + r^r \left[(1+n) A_n^{c-r+1} (x,r) - A_n^c (x,r) - (r-1) A_{n-1}^{c+1} (x,r) \right]$$

$$= \sum_{j=2}^{min\,(r,n)} (-1)^j \left[\binom{r+1}{j+1} (n-j) - (c+n-1) \binom{r}{j} \right] \frac{d}{dx} A_{n-j}^{c+j} (x,r) .$$

Particularly for $r=1$, $c=\alpha+1$, the relation (3.4) reduces to [6, (2.10)].

4. SOME SUMMATION FORMULAS.

Starting with the generating relations (1.1), (1.6), (1.7) and (1.8), we derive the following summation formulas respectively :

$$(4.1) \quad f_n^{c_1+c_2+\ldots+c_m} (x_1+x_2+\ldots+x_m, r) = \sum_{n_1+n_2+\ldots+n_m=n} \prod_{j=1}^{m} f_{n_j}^{c_j} (x_j, r),$$

$$(4.2) \quad f_n^{c_1+c_2+\ldots+c_m-n+1} (x_1+x_2+\ldots+x_m, r) = \sum_{n_1+n_2+\ldots+n_m=n} \prod_{j=1}^{m} f_{n_j}^{c_j-n_j+1} (x_j, r),$$

$$(4.3) \quad A_n^{c_1+c_2+\ldots+c_m} (x_1+x_2+\ldots+x_m, r) = \sum_{n_1+n_2+\ldots+n_m=n} \prod_{j=1}^{m} A_{n_j}^{c_j} (x_j, r),$$

and

$$(4.4) \quad A_n^{c_1+c_2+\ldots+c_m-n+1}(x_1+x_2+\ldots+x_m, r) = \sum_{n_1+n_2+\ldots+n_m=n} \prod_{j=1}^{m} A_{n_j}^{c_j-n_j+1}(x_j, r) .$$

The relation (4.1) is a generalization of [3, (4.4). Indeed for $x_1=x_2=\ldots=x_m=kx$; $c_1=c_2=\ldots=c_m=kr+1$ and $m=c$, the relation (4.1) reduces to [3, (4.4)]. Similarly, (4.3) is a generalization of [3, (4.5)] and it can be obtained from (4.3) by taking $x_1=x_2=\ldots=x_m=kx$, $c_1=c_2=\ldots=c_m=s$ and $m=c$.

Again, for $x_1=x_2=\ldots=x_m=x$, the relation (4.1) reduces to

$$(4.5) \quad f_n^{c_1+c_2+\ldots+c_m}(mx, r) = \sum_{n_1+n_2+\ldots+n_m=n} \prod_{j=1}^{m} f_{n_j}^{c_j}(x, r) ,$$

which, on equating the coefficients of x^n on both sides, gives us the identity

$$(4.6) \quad \frac{m^n}{n!} = \sum_{n_1+n_2+\ldots+n_m=n} \prod_{j=1}^{m} \frac{1}{(n_j)!} .$$

The relations (4.5) and (4.6) further suggest that the relation [8, (3.3)]

$$\frac{d^{\lambda k}}{dx^{\lambda k}} \left\{ C_{n+k}^{\lambda}(x) \right\} = 2^{\lambda k} (\lambda)_{\lambda k} \sum_{i_1+i_2+\ldots+i_{k+1}=n} \prod_{j=1}^{k+1} C_{i_j}^{\lambda}(x)$$

will give us another identity :

$$(4.7) \quad \frac{(\lambda k)_n}{n!} = \sum_{i_1+i_2+\ldots\ldots i_k=n} \prod_{j=1}^{k} \frac{(\lambda)_{i_j}}{(i_j)!} .$$

Now making an application to the definitions of the polynomials, we can also derive the following results :

$$(4.8) \quad R_n^{(c,b)}(x,y,r) = R_n^{(-b,-c)}(x,y,r) = R_n^{(c,b)}(-y,-x,r) ,$$

$$(4.9)\quad N_n^{(c,\,b)}(x,y,r) = N_n^{(-b,\,-c)}(x,y,r) = R_n^{(c,\,b)}(-y,-x,r)$$

$$(4.10)\quad R_n^{(c+c',\,b+b')}(x+x',\,y+y',\,r)\ \sum_{m=0}^{n} R_m^{(c,\,b)}(x,y,r)\,R_{n-m}^{(c',\,b')}(x',y',r)$$

and

$$(4.11)\quad N_n^{(c+c',\,b+b')}(x+x',\,y+y',\,r)$$

$$= \sum_{m=0}^{n} N_m^{(c,\,b)}(x,y,r)\,N_{n-m}^{(c',\,b')}(x',y',r).$$

The last two results may further be generalized in the following way :

$$(4.12)\quad R_n^{(c_1+c_2+\ldots+c_m,\ b_1+b_2+\ldots+b_m)}(y_1+y_2+\ldots+y_m,\ x_1+x_2+\ldots+x_m,\ r)$$

$$= \sum_{k_1+k_2+\ldots+k_m=n}\ \prod_{j=1}^{m} R_{k_j}^{(c_j,\,b_j)}(y_j,x_j,r),$$

and

$$(4.13)\quad N_n^{(c_1+c_2+\ldots+c_m,\ b_1+b_2+\ldots+b_m)}(x_1+x_2+\ldots+x_m,\ y_1+y_2+\ldots+y_m,\ r)$$

$$= \sum_{k_1+k_1+\ldots+k_m=n}\ \prod_{j=1}^{m} N_{k_j}^{(c_j,\,b_j)}(x_j,y_j,r)\ .$$

Rainville [4] has obtained a series for Legendre polynomials and Singh [7] has obtained a series for Srivastava's polynomials [6]. Here we shall obtain a similar expression for $A_n^c(x,r)$.

From (1.7), we have

$$\sum_{n=0}^{\infty} A_n^c(\tan\beta,\,r)\,t^n = (1+t)^{-c}\exp.\ \{ r^r (1+t)^{r-1}\,t\tan\beta \},$$

and

$$\sum_{n=0}^{\infty} A_n^c(\tan\delta,\,r)\,t^n = (1+t)^{-c}\exp.\ \{ r^r (1+t)^{r-1}\,t\tan\delta \}$$

$$= (1+t)^{-c} \exp. \{r^r (1+t)^{r-1} t \tan\beta\} \exp. \{r^r (1+t)^{r-1} t (\tan\delta - \tan\beta)\}.$$

combining the above two results and equating the coefficients of t^n, on both sides, we get

$$(4.14) \quad A_n^c (\tan\delta, r) = \sum_{m=0}^{n} \frac{r^{rm}}{m!} \left[\frac{\text{Sin}(\delta-\beta)}{\text{Cos}\,\delta\ \text{Cos}\,\beta} \right]^m A_{n-m}^{c-(r-1)m} (\tan\beta, r)$$

As a particular case, the above result reduces to [7, (4.1)] for $c=\alpha+1$, $r=1$.

5. RODRIGUES' FORMULAE.

The Rodrigues' formula for $f_n^c(x, r)$ are given by

$$(5.1) \quad f_n^c (x, r) = \frac{x^{-c+1}}{n!} \frac{d^n}{dx^n} \left\{ x^{c+n-1} \ {}_{r+1}F_{r+1} \left[\begin{matrix} -n, c, \Delta(r-1, c+n); \\ c+n, \Delta(r, c); \end{matrix} (r-1)^{r-1} x \right] \right\},$$

and

$$(5.2) \quad f_n^c (x, r) = \frac{(c)_n}{(n!)^2} \frac{d^n}{dx^n} \left\{ x^n \ {}_{r+1}F_{r+1} \left[\begin{matrix} -n, 1, \Delta(r-1, c+n); \\ n+1, \Delta(r, c); \end{matrix} (r-1)^{r-1} x \right] \right\},$$

where $\Delta(r, c)$ stands for the set of r parameters

$$\frac{c}{r}, \frac{c+1}{r}, \ldots\ldots, \frac{c+r-1}{r}, r \geqq 1.$$

In particular, for $c=\alpha+1$, $r=1$, we derive the following results from (5.1) and 5.2) :

$$(5.3) \quad L_n^{(\alpha)}(x) = \frac{x^{-\alpha}}{n!} \frac{d^n}{dx^n} \left\{ x^{\alpha+n} \ {}_1F_1 \left[\begin{matrix} -n; \\ \alpha+n+1; \end{matrix} x \right] \right\},$$

$$(5.4) \quad L_n^{(\alpha-n)}(x) = \frac{x^{-\alpha+n}}{(1+\alpha)_n} \frac{d^n}{dx^n} \left\{ x^{\alpha} L_n^{(\alpha)}(x) \right\}$$

and

$$(5.5) \quad L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{(n!)^2} \frac{d^n}{dx^n} \left\{ x^n \ {}_2F_2 \left[\begin{matrix} -n, 1; \\ n+1, \alpha+1; \end{matrix} x \right] \right\}$$

Again, for $a=0$, (5.5) reduces to

$$(5.6)\qquad L_n(x)=\frac{1}{n!}\frac{d^n}{dx^n}\left\{x^n {}_1F_1\,(-n;\,n+1;\,x)\right\}\ .$$

6. SOME RELATIONS WITH WELL KNOWN POLYNOMIALS.

In this section, we obtain some relations of $A_n^c\ (x, r)$ with well known polynomials. In our previous paper [1, (3.5)], we have

$$(6.1)\qquad A_k^c\ (x,r) = \sum_{j=0}^{k} \frac{x^j\ r^{rj}}{j\,!} \binom{(r-1)\,j-c}{k-j}\ .$$

Also for Laguerre polynomials, we have the relation [4a, p. 207]

$$\frac{x^n}{n\,!} = \sum_{k=0}^{n} \frac{(-1)^k\ (1+a)_n\ L_k^{(a)}\ (x)}{(n-k)\ !\ (1+a)_k}$$

From the above results, we establish

$$(6.2)\qquad A_n^c\ (x,r) = \sum_{k=0}^{n}\ {}_{r+1}F_r \left[\begin{matrix}-n+k,\ \Delta\ (r,\ c-(r-1)\ n);\\ -a-n,\ \Delta\ (r-1,\ c-(r-1)n);\end{matrix}\ -(r-1)^{r-1}\right]\times$$

$$\times\frac{r^{rn}\ (-1)^k(1+a)_n\ L_k^{(a)}(x)}{(1+a)_k\ (n-k)\ !}\ .$$

Now using (6.1) and the relation [4a, p. 194]

$$\frac{x^n}{n!} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{H_{n-2k}(x)}{2^n k!(n-2k)!}\ ,$$

for Hermite polynomials, we derive

$$(6.3)\qquad A_n^c\ (x,r) =$$

$$=\sum_{m=0}^{n}\ {}_{2r}F_{2r}\left[\begin{matrix}-\frac{m}{2},\ \frac{-m+1}{2},\ \Delta(2r-2,\ -c+1+(r-1)\ (n-m));\\ \Delta\,(2r,\ -c+1-m+(r-1)\ (n-m));\end{matrix}\ \frac{(r-1)^{2r-2}}{4}\right]$$

$$\times \frac{\left((r-1)(n-m)-c-m+1\right)_m \, r^{r(n-m)}}{2^{n-m}\,(n-m)! \qquad m!} H_{n-m}(x).$$

Similarly, we can also express $A_n^c(x,r)$ in terms of $Z_n(x)$, $C_n^v(x)$, $H_n(s, p, x)$, $P_n(x)$ Sister Celine's polynomials and Bernoulli polynomials etc. Similar results for $f_n^c(x,r)$ can also be obtained.

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